Construction and Enumeration of Pandiagonal magic squares of order n from Step method

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Abstract

A Pandiagonal magic square (PMS) of order n is a square matrix which is an arrangement of integers $0,1,\ldots,n^2-1$ such that the sums of each row, each column and each extended diagonal are the same. In this paper we use the Step method to construct a PMS of order n for each $n > 3$ and n is not singly-even. We discuss how to enumerate the number of PMSs of order n constructed by the Step method, and we get the number of nonequivalent PMSs of order 8 with the top left cell 0 is 4,176,000 and the number of nonequivalent PMSs of order 9 with the top left cell 0 is 1,492,992.

1. Introduction

The Natural square $N = [n(i,j)]$ of order n is a square matrix, such that $n(i,j) = n \cdot i + j$, for each $0 \leq i,j \leq n-1$. A magic square of order n is an arrangement of N such that the sums of each row, each column and each of the main diagonal are the same. If also the sum of each extended diagonal (diagonal of the square mappped onto the surface of a torus) is the same, the magic square is called a Pandiagonal magic sqrare (PMS) [4]. An Auxiliary square $A = [a(i,j)]$ of order n is a square matrix which is an arrangement of n^2 consecutive integers such that $a(i,j) = a(i,0) + a(0,j)$, for each $0 \leq i,j \leq n-1$. The Natural square can be viewed as a basic Auxiliary square. Two squares $A = [a(i,j)]$ and $B = [b(i,j)]$ are orthogonal (orthogonal mates) if every ordered pair of symbols occurs exactly once among the n^2 pairs $\langle a(i,j),b(i,j)\rangle$ [3, p.154]. Two orthogonal doubly diagonal latin squares can be used to construct a magic square of the same order by a simple juxtaposition [4, p.206]. A Pandiagonal constant sum (PCS) matrix is a square matrix with n^2 entries of n consecutive integers, each appearing exactly n times, and such that the sums of each row, each column and each extended diagonal are the same [4].

Kraitchik showed that there is essentially only one magic square of order 3 and this is not pandiagonal, so the order of a PMS must exceed 3. Moreover, there are no PMS of singly-even order [1,3,4]. In [1] and [4], Ball and Kraitchik have proved that there is a PMS of order n where n is odd and n is not a multiple of 3 by using the generalized De la Loubére's method which is the Step method.

Step Method:

Let a, b and c, d be two pairs of distinct integers and $1 \le a,b,c,d \le n-1$, and N = [n(i,j)] the Natural square of order n. If M = [m(i,j)] is constructed as follows: m(i,j) = n(r,s), where $i \equiv a \cdot r + c \cdot s \pmod{n}$ and $j \equiv b \cdot r + d \cdot s \pmod{n}$, for each $0 \le r,s \le n-1$. Then M is a PMS provided that a, b, a+b, a-b, c, d, c+d, c-d and ad-bc are prime to n, respectively.

From the construction of M, M can be viewed as a rearrangement of N where n(0,0) is put in the cell (0,0) of M and if n(r,s) is in the cell (i,j) of M then n(r,s+1) is in the cell (i+c,j+d), which is down c steps and right d steps of n(r,s), and n(r+1,s) is in the cell (i+a,j+b), which is down a steps and right b steps of n(r,s) in M. Thus we call <a,b> the Column step, <c,d> the Row step. Candy [2] constructed PMS of some special composite order by using this Step method, and counted the number of PMSs of order 8 and 9 which can be constructed by this method, but he didn't give a systematic method and there is some error in the result of order 9. In this paper we will generalize the Step method to construct all feasible orders of PMS and give dettails of their enumeration. We obtain the number of nonequivalent PMSs of order 8 with the top left cell 0 is 4,176,000, and the number of nonequivalent PMS of order 9 with the top left cell 0 is 1,492,992.

The square matrix considered in this paper have subscripts in the range 0, 1, 2, ..., n-1.

2. Existence of PMSs

Let N be the Natural square of order n. Then N can be expressed as $N = n \cdot R + C$ where R = [r(i,j)] and C = [c(i,j)], in which r(i,j) = i, c(i,j) = j, for each $0 \le i,j \le n-1$, are two orthogonal square matrices. W. Proskurowski showed the following results in [4].

Lemma 2.1.[4] A square matrix of order n, S, is a PMS if there exist two orthogonal PCS matrices B and B′ of order n such that $S = n \cdot B + B'$.

Lemma 2.2.[4] Let m and n be two positive integers, , $1 < m < n-1$, and $q = \gcd(m,n)$. There exists a permutation p of the integers 0,1,2,...,n-1 such that for all values of j, $0 \le j < n$,

$$\sum_{0 \le i < n/q} p[(im+j) \bmod n] = n(n-1)/(2q).$$

By Lemma 2.1, if we can rearrange the entries of R and C to be PCS matrices R′ and C′ respectively, then we can construct a PMS $n \cdot R' + C'$. For convenience, we define a step <a,b> to be <u>effective</u> if a, b, a+b, and a-b are prime to n.

Lemma 2.3. Let <a,b> be an effective Columnn step, and d, c+d, c-d, and ad-bc all prime to n. Let R′ be the square matrix constructed by the Column step <a,b> and Row step <c,d> corresponding to R. If $\gcd(c,n) = t$, then each row of R′ contains t entries of R which are in the same row. Furthermore, if $\gcd(d,n) = m$, then each column of R′ contains m entries of R which are in the same row.

Proof. By the Step method, we know that if $r'(i,j) = r(u,v)$, then $i \equiv au+cv \pmod n$ and $j \equiv bu+dv \pmod n$. Thus for each i,j, we have $(ad-bc)u \equiv di-cj \pmod n$ and $(ad-bc)v \equiv aj-bi \pmod n$. Since ad-bc is prime to n, $\gcd(c,n) = t$, and $\gcd(d,n) = 1$, then there are n/t distinct integers in {0,1,2,...,n-1} for u to satisfy the equation $(ad-bc)u \equiv di-cj \pmod n$, for fixed i. Therefore, each row of R(contains t entries of R which are in the same row. Similarly, we can obtain that if $\gcd(d,n) = m$, then each column of R′ contains m entries of R which are in the same row. □

By Lemma 2.3, we know if <a,b> and <c,d> are effective steps then each integer will appear exactly once in each row and each column of R′, i.e. R′ is a Latin square. Then R′ is a PCS matrix. Similarly C′ is a PCS matrix. Thus $n \cdot R' + C'$ is a PMS.

Lemma 2.4. Let <a,b> be an effective step and ad-bc be prime to n. If $\gcd(c,n) = g_c$, $\gcd(d,n) = g_d$ and both c+d and c-d are prime to n. Then there is a permutation p such that if R′ is the square constructed by the Column step <a,b> and Row step <c,d> corresponding to p(R), then R′ is a PCS matrix.

Proof. Let $q = lcm\{g_c, g_d\}$. By Lemma 2.2, there is a permutation p such that

$$\sum_{0 \le i < n/q} p[(iq+j) \bmod n] = n(n-1)/(2q), \text{ for } 0 \le j < n.$$

Thus if $q = g_c \cdot m_c$, then for each j, $0 \le j < n$,

$$\sum_{0 \le i < n/g_c} p[(ig_c+j) \bmod n] = \sum_{0 \le k < m_c} \sum_{0 \le i < q} p[(iq+kg_c+j) \bmod n]$$

$$= m_c(n(n-1)/(2q)) = n(n-1)/(2g_c).$$

By Lemma 2.3, we know that each integer in the rows of R′ appears g_c times, thus each row sum of R′ will equal $n(n-1)/2$. Similarly, each column sum of R′ is $n(n-1)/2$. Since c+d and c-d are prime to n, each extended diagonal sum is n(n-1)/2, too. Therefore R′ is a PCS matrix. □

Theorem 2.5. Let n > 3 and n is not singly-even. Let a, b, c, and d be positive integers less than n. If ad-bc is prime to n, then there is a PMS of order n constructed by the Column step <a,b> and Row step <c,d> from the Step method corresponding to the Natural square N.

Proof. If g_1 = gcd(a,n), g_2 = gcd(b,n), g_3 = gcd(a+b,n), g_4 = gcd(a-b,n), g_5 = gcd(c,n), g_6 = gcd(d,n), g_7 = gcd(c+d,n), and g_8 = gcd(c-d,n). Let q = *lcm*{g_i | i = 1, 2, ..., 8}. By Lemma 2.2, we know there is a permutation p of integers 0,1,...,n-1such that for all values of j, $0 \le j < n$,

$$\sum_{0\le i<n/q} p[(iq+j) \bmod n] = n(n-1)/(2q).$$

Let A = [a(i,j)] be a square matrix and A = n·p(R) + p(C) where p(R) = [p[r(i,j)]] and p(C) = [p[c(i,j)]]. Let R′ and C′ be two square matrices constructed by the Column step <a,b> and Row step <c,d> corresponding to p(R) and p(C) respectively. By Lemma 2.4, we get R′ and C′ are two PCS matrices. Let A′ = n·R′ + C′. Since R is orthogonal to C, R′ is orthogonal to C′. By Lemma 2.1, we obtain that A′ is a PMS of order n. □

Example. Let n = 8, a = 1, b = 2, c = 3, d = 1.
Since gcd(b,n) = 2, gcd(c+d,n) = 4, we have q = 4. By the proof of Lemma 2.2, we get p = (0 1 2 3 7 6 5 4).

$$R = \begin{matrix} 0&0&0&0&0&0&0&0\\ 1&1&1&1&1&1&1&1\\ 2&2&2&2&2&2&2&2\\ 3&3&3&3&3&3&3&3\\ 4&4&4&4&4&4&4&4\\ 5&5&5&5&5&5&5&5\\ 6&6&6&6&6&6&6&6\\ 7&7&7&7&7&7&7&7 \end{matrix} \qquad C = \begin{matrix} 0&1&2&3&4&5&6&7\\ 0&1&2&3&4&5&6&7\\ 0&1&2&3&4&5&6&7\\ 0&1&2&3&4&5&6&7\\ 0&1&2&3&4&5&6&7\\ 0&1&2&3&4&5&6&7\\ 0&1&2&3&4&5&6&7\\ 0&1&2&3&4&5&6&7 \end{matrix}$$

$$p(R) = \begin{matrix} 0&0&0&0&0&0&0&0\\ 1&1&1&1&1&1&1&1\\ 2&2&2&2&2&2&2&2\\ 3&3&3&3&3&3&3&3\\ 7&7&7&7&7&7&7&7\\ 6&6&6&6&6&6&6&6\\ 5&5&5&5&5&5&5&5\\ 4&4&4&4&4&4&4&4 \end{matrix} \qquad p(C) = \begin{matrix} 0&1&2&3&7&6&5&4\\ 0&1&2&3&7&6&5&4\\ 0&1&2&3&7&6&5&4\\ 0&1&2&3&7&6&5&4\\ 0&1&2&3&7&6&5&4\\ 0&1&2&3&7&6&5&4\\ 0&1&2&3&7&6&5&4\\ 0&1&2&3&7&6&5&4 \end{matrix}$$

R′ and C′ are constructed by the Column step <a,b> and Row step <c,d> corresponding to p(R) and p(C), respectively.

$$R' = \begin{matrix} 0 & 4 & 5 & 6 & 7 & 3 & 2 & 1 \\ 3 & 2 & 1 & 0 & 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 3 & 2 & 1 & 0 & 4 \\ 1 & 0 & 4 & 5 & 6 & 7 & 3 & 2 \\ 7 & 3 & 2 & 1 & 0 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 & 3 & 2 & 1 & 0 \\ 2 & 1 & 0 & 4 & 5 & 6 & 7 & 3 \\ 6 & 7 & 3 & 2 & 1 & 0 & 4 & 5 \end{matrix} \qquad C' = \begin{matrix} 0 & 3 & 5 & 1 & 7 & 4 & 2 & 6 \\ 2 & 6 & 0 & 3 & 5 & 1 & 7 & 4 \\ 7 & 4 & 2 & 6 & 0 & 3 & 5 & 1 \\ 5 & 1 & 7 & 4 & 2 & 6 & 0 & 3 \\ 0 & 3 & 5 & 1 & 7 & 4 & 2 & 6 \\ 2 & 6 & 0 & 3 & 5 & 1 & 7 & 4 \\ 7 & 4 & 2 & 6 & 0 & 3 & 5 & 1 \\ 5 & 1 & 7 & 4 & 2 & 6 & 0 & 3 \end{matrix}$$

We can see that R′ is orthogonal to C′, and R′ and C′ are PCS matrices.

0	1	2	3	7	6	5	4
8	9	10	11	15	14	13	12
16	17	18	19	23	22	21	20
24	25	26	27	31	30	29	28
56	57	58	59	63	62	61	60
48	49	50	51	55	54	53	52
40	41	42	43	47	46	45	44
32	33	34	35	39	38	37	36

A

0	35	45	49	63	28	18	14
26	22	8	3	37	41	55	60
47	52	58	30	16	11	5	33
13	1	39	44	50	62	24	19
56	27	21	9	7	36	42	54
34	46	48	59	29	17	15	4
23	12	2	38	40	51	61	25
53	57	31	20	10	6	32	43

A′

Then A is an Auxiliary square, and A′ is a PMS of order 8.

3. Enumeration

Since the properties of pandiagonal magic square mapped onto the surface of a torus are invariant under rotation, transposition, and cyclic translation (shift of rows and columns). We define that two PMSs are equivalent if there is any transformation (rotation, transposition or cyclic translation) between them. The PMS can be written with any element in the top left corner without losing its pandiagonal magic properties. In this paper we will study the sets of nonequivalent squares in choosing 0 as lying in the top left cell.

Let #(PMS) be the total number of PMSs of order n constructed from Step method and consisting of the integers 0 to n^2-1 which have 0 in the top left cell. Since each PMS has 8 equivalent PMSs, the total number of nonequivalent PMSs of order n with 0 in the top left cell is #(PMS)/8.

From Section 2, we can construct PMSs by finding permutations corresponding to Column steps, Row steps and the Natural square. Thus a PMS constructed by the Step method depends upon three things. First, an Auxiliary square A. Second, the steps. Third, a permutation p, corresponding to steps, partitions integers that are in the first row and the first column of A into subsets, each with the same sum.

(1) Auxiliary squares

From the definition of an Auxiliary square, we know that any row permutation or column permutation of an Auxiliary square is an Auxiliary square. If n is a prime number then there is only one basic Auxiliary square, that is the Natural square. If n is not prime then we can partition an n by n square into subrectangles T = [t(i,j)], where t(i,j) = t(i,0) + t(0,j) for each i,j. For convenience, we call such T a basic Auxiliary rectangle if t(0,j) = j and t(i,0) = i, for each i,j. Let P = [p(i,j)] and Q be rectangle matrices of size r×s and u×v respectively. Then the Kronecker product P⊗Q is defined to be the ru × sv matrix:

$$\begin{bmatrix} p(0,0)*Q & p(0,1)*Q & p(0,2)*Q & \dots & p(0,s)*Q \\ p(1,0)*Q & p(1,1)*Q & p(1,2)*Q & \dots & p(1,s)*Q \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ p(r-1,0)*Q & p(r-1,1)*Q & p(r-1,2)*Q & \dots & p(r-1,s)*Q \end{bmatrix}$$

where p(i,j)*Q = [p(i,j)*q(k,*l*)] whenever Q = [q(k,*l*)].

Lemma 3.1. Let P and Q be two Auxiliary rectangles of size r×s and u×v respectively. If we define p(i,j)*q(k,*l*) = uvp(i,j) + q(k,*l*), for each $0 \le i < r$, $0 \le j < s$, $0 \le k < u$, and $0 \le l < v$. Then the Kronecker product P⊗Q is an Auxiliary rectangle of size ru × sv.

Proof. Let C = P⊗Q. Then each entry in C can be written by p(i,j) * q(k,*l*), for some i,j,k,*l*. Since p(i,j) = p(i,0) + p(0,j) and q(k,*l*) = q(k,0) + q(0,*l*), we have p(i,j) * q(k,*l*) = uvp(i,j) + q(k,*l*) = uv[p(i,0) + p(0,j)] + [q(k,0) + q(0,*l*)] = uv[p(i,0) + q(k,0)] + uv[p(0,j) + q(0,*l*)] = [p(i,0) * q(k,0)] + [p(0,j) * q(0,*l*)]. Thus each element in C can be written as the sum of two elements one of which is in the first column of C and the other one is in the first row of C. Hence C is an Auxiliary rectangle of size ru × sv. □

Corollary 3.2. If P and Q are two basic Auxiliary rectangles of size r×s and u×v respectively. Then the Kronecker product P⊗Q is an Auxiliary rectangle of size ru × sv.

From Collary 3.2, we can obtain that if n is a composite number then an Auxiliary square of order n can be represented as the Kronecker product of two basic Auxiliary rectangles. Thus the number of Auxiliary squares of order n depends on the number of different ways to represent it to be the Kronecker product of basic Auxiliary rectangles. We have the recurrsive formula to count the number of Auxiliary squares of order n in the following theorem.

Theorem 3.3. Let N(s×t) be the number of Auxiliary rectangles of size s×t. Then N(1×n) = 1; N(m×1) = 0; N(m×p) = 1, if p is prime; and

$$N(m\times n) = N(1\times n) + \sum_{p|m,p>1} \Big[\sum_{q \text{ is prime}, q<n, q|n} N(p\times (n/q)) - \sum_{t} N(p\times t)\Big]$$

where t = gcd{n/q1, n/q2}, q1, q2 are two distinct prime factors of n.

Proof. It is easy to see that N(1×n) = 1 and N(m×p) = 1, when p is prime. By Lemma 3.1, we know each Auxiliary rectangle can be written as the Kronecker product of two Auxiliary rectangles. Let A be an Auxiliary rectangle of size m×n. If $m = p\cdot u > 1$, $n = q\cdot v > 1$, then there exists two Auxiliary rectangles P and Q of size p×q and u×v respectively such that $A = P\otimes Q$. If we let Q be a basic Auxiliary rectangle, then the total number of Auxiliary rectangles A of size m×n is equal to the number of P. Therefore we use the Principle of Inclusion and Exclusion, we conclude the proof. □

For some special sizes, we can obtain the total number of Auxiliary rectangles by recurrsively using Theorem 3.3.

Corollary 3.4. Let p, q and t be prime numbers.
(i) $N(p^2\times p^2) = 3$.
(ii) If n is a positive integer, then $N(p^n\times p^n) = (2n-1)!/[n!(n-1)!]$.
(iii) If $q \neq t$, and $n = t^u q^v$, where u and v are integers, then
  $N(p\times n) = (u+1)(v+1)-1$.
(iv) $N(m\times q^2) = f(m)$, where f(m) is the number of factors of m.
(v) If p and q are distinct prime, then N(pq×pq) = 7.

Example. How many Auxiliary squares of order 12 are there?
By Theorem 3.3, we can get

$$\begin{aligned} N(12\times12) &= N(1\times12) + [N(2\times6)+N(2\times4)-N(2\times2)] + [N(3\times6)+N(3\times4)-N(3\times2)] \\ &\quad + [N(4\times6)+N(4\times4)-N(4\times2)] + [N(6\times6)+N(6\times4)-N(6\times2)] + \\ &\quad [N(12\times6)+N(12\times4)-N(12\times2)] \\ &= 1+(3+2-1)+(3+2-1) + [N(4\times6)+3-1] + [N(6\times6)+N(6\times4)-1] + \\ &\quad [N(12\times6)+ N(12\times4)-1] \\ &= 9+N(4\times6)+N(6\times6)+N(6\times4)+N(12\times6)+N(12\times4). \end{aligned}$$

$N(4\times6) = N(1\times6) + N(2\times2) + N(2\times3) + N(4\times2) + N(4\times3) = 5.$

$N(6\times6) = 7.$ $N(6\times4) = N(1\times4) + N(2\times2) + N(3\times2) + N(6\times2) = 4.$

$$\begin{aligned} N(12\times6) &= N(1\times6)+N(2\times2)+N(2\times3) +N(3\times2)+N(3\times3)+N(4\times2)+N(4\times3)+N(6\times2) \\ &\quad + N(6\times3)+ N(12\times2) + N(12\times3) = 11. \end{aligned}$$

$N(12\times4) = N(1\times4) + N(2\times2) + N(3\times2) + N(4\times2) + N(6\times2) + N(12\times2) = 6$.
Thus we conclude that $N(12\times12) = 9+5+7+4+11+6 = 42$. That is, there are 42 Auxiliary squares of order 12.

(2) Steps

Let a, b be two distinct integers, $0 < a, b < n$. We define $R_i = \{<a,b> \mid 0 < a,b < n, \gcd(a,n) = i, \gcd(i,b) = 1\}$, $C_i = \{<a,b> \mid 0 < a,b < n, \gcd(b,n) = i, \gcd(i,a) = 1\}$, $S_i = \{<a,b> \mid 0 < a,b < n, \gcd(a+b,n) = i\}$, and $D_i = \{<a,b> \mid 0 < a,b < n, \gcd(a-b,n) = i\}$, for each factor i of n, $1 < i < n$. If the step $<a,b>$ is not effective then $<a,b>$ belongs to one of the types R_i, C_i, S_i, and D_i or the combination of any two types, such as R_iC_j, R_iS_j, ..., S_iD_j. Since ad-bc should be prime to n, Column step $<a,b>$ and Row step $<c,d>$ at least can not belong to the same type. If one of a, b is prime to n then the step $<a,b>$ can be written by $a \cdot <1,y>$, a is prime to n, or $b \cdot <x,1>$, b is prime to n. Therefore we obtain

Lemma 3.5. The number of steps $<a,b>$ which can be used to construct a PMS equals $\varphi(n)$ times the number of the set $\{<1,y>, <x,1> \mid 1 < x,y < n-1\} \cup \{<x,y> \mid$ x,y are two distinct prime factors of n$\}$, where $\varphi(n)$ is the number of positive integers which are prime to n and less than n.

(3) Permutations

After we determine the steps, $<a,b>$ and $<c,d>$, we know there are several types.
(i) If $<a,b>$ and $<c,d>$ are both effective, then we can permute the first row or the first column of the Natural square to get an Valid Auxiliary square. Thus there are $((n-1)!)^2$ Valid Auxiliary squares.
(ii) If $<a,b>$ or $<c,d>$ is in one of the types R_m, C_m, S_m, and D_m, then we need to partition the first row or the first column of the Auxiliary square into m subsets, each with the same sum, T_m. For each T_m, we can get a set of sequences $P_m = \{(p[0],p[1],\dots,p[n-1]) \mid p[0] = 0$, and for each j, $0 \le j < m-1$, $\Sigma_{0\le i<n/m}$ $[(im+j) \bmod n]$ is constant$\}$. After we get a sequence p from T_m, we can permute terms in p to generate another sequences which still in P_m. Thus we obtain that $\#(P_m) = (m-1)! \cdot ((n/m)!)^{m-1} \cdot (n/m-1)! \cdot \#(T_m)$, where $\#(T_m)$ is the number of ways to partition n integers into m subsets, each with the same sum. Therefore we have $(\#(P_m))^2$ Valid Auxiliary squares.
(iii) If Column step $<a,b>$ and Row step $<c,d>$ are in different types, say $<a,b>$ in R_u and $<c,d>$ in C_v, then we need to partition the first column of the Auxiliary square into u subsets, each with the same sum, and partition the first row of the Auxiliary square into v subsets, each with the same sum. For each partition we can obtain sequences to form a Valid Auxiliary square such that the first column $r[0],r[1],\dots,r[n-1]$, and the first row $c[0],c[1],\dots, c[n-1]$ satisfy that $r[0] = c[0] = 0$,

$\Sigma_{0\leq i<n/u}$ r[(iu+j) mod n] is constant for each j, $0 \leq j < u-1$ and
$\Sigma_{0\leq i<n/v}$ c[(iv+j) mod n] is constant for each j, $0 \leq j < v-1$. Thus we can generate $\#(P_u) \cdot \#(P_v)$ Valid Auxiliary squares.

(4) Conclusion

The total number of PMSs of order n constructed from Step method depends on Auxiliary squares, steps and permutations to generate Valid Auxiliary squares. We can formulate it as follows:

#(PMSs of order n) = $\Sigma_{\text{Auxiliary square}}$ #(Column steps) · #(suitable Row steps) · #(Valid Auxiliary squares) / $(\varphi(n))^2$.

#(nonequivalent PMSs of order n) = #(PMSs of order n) / 8.

From (1) we can count how many Auxiliary squares of order n. From (2) we can count how many steps can be Column steps, but for the Row steps we need to check which one can not satisfy the condition "ad-bc is prime to n". The number of permutations depends on the property of the first row and the first column of the Auxiliary square in (1) and the steps in (2). Since we doubly count the steps and the permutations for the row and column $\varphi(n)$ times, we need to divide twice of it. From the above discussion we can get the following theorem.

Theorem 3.6. If n is prime, then there are $((n-1)!)^2(n-3)(n-4)$ PMSs of order n constructed from Step method.

If n is not prime, then the total number of PMSs of order n can not get the general formula to calculate it, it should calculate case by case except we can solve the following problem:

Problem. Let p and q be two integers and $p,q \geq 2$. If $n = pq$, how many different ways to partition a sequence of integers containing p copies of the set $\{0,1,2,\ldots,q-1\}$ into t parts, each with equal sum, where t is a factor of n.

If $n = p^2$, $t = p$ and p is prime, then the answer for the above problem is equal to the number of latin squares of order p.

4. Small cases

In this section, we will count how many nonequivalent PMSs of order 4, 8, and 9 can be constructed from Step method.

4.1 The number of PMSs of order 4

(1) By Corollary 3.4, $N(4\times4) = 3$. Thus there are three Auxiliary squares.

0	1	2	3		0	1	4	5		0	1	8	9
4	5	6	7		2	3	6	7		2	3	10	11
8	9	10	11		8	9	12	13		4	5	12	13
12	13	14	15		10	11	14	15		6	7	14	15

(2) There are only two types of steps: R_2 and C_2. And $|R_2| = 2 = |C_2|$. If <a,b> is in R_2, then <c,d> is in C_2. Thus #(Column steps) = 4, and #(suitable Row steps) = 2.
(3) Since $\#(T_2) = 1$ for the first row and the first column of each Auxiliary square in (1). $\#(P_2) = 2 \cdot \#(T_2) = 2$.
The total number of PMSs of order 4 is $(3 \cdot (4 \cdot 2) \cdot 2^2)/ 2^2 = 24$, and the total number of nonequivalent PMSs of order 4 is 3.

4.2 the number of PMSs of order 8

(1) By Corollary 3.4, we get that $N(8\times8) = 10$. There are 10 basic Auxiliary squares of order 8.
(2) For any two distinct integers a and b, the step <a,b> belongs to one of the sets R_2, C_2, R_4, and C_4, or one of the sets S_2D_4 and S_4D_2, where $S_iD_j = \{<a,b>|0 < a,b < n, \gcd(a+b,n) = i, \gcd(a-b,n) = j\}$. Since $|R_2| = |C_2| = 8$, $|R_4| = |C_4| = 4$, and $|S_2D_4| = |S_4D_2| = 4$. According to ad-bc is prime to n, we obtain the number of steps <a,b> and <c,d>
(i) if <a,b> and <c,d> are in R_2 or C_2,
#(Column steps) · #(suitable Row steps) = $16\cdot8 = 128$.
(ii) if <a,b> and <c,d> are in R_4, C_4, S_2D_4, or S_4D_2.
Since if <a,b> is in S_2D_4, <c,d> can not be in S_4D_2.
#(Column steps) · #(suitable Row steps) = $8\cdot12 + 2\cdot4\cdot8 = 160$.
(iii) if <a,b> is in R_2 or C_2, <c,d> is in R_4, C_4, S_2D_4, or S_4D_2. Or conversely.
#(Column steps) · #(suitable Row steps) = $2\cdot16\cdot12 = 384$.
(3) Corresponding to the steps discussed in (2), we need to get T_4 and T_2. We obtain that $\#(T_4) = 1$, and $\#(T_2) = 4$ for each Auxiliary square in (1). Thus $\#(P_4) = 48$, and $\#(P_2) = 576$.
The total number of PMSs of order 8 is $\{10 \cdot [128 \cdot (576)^2 + 160 \cdot (48)^2 + 384 \cdot (48 \cdot 576)]\} / 4^2 = 33{,}408{,}000$. The total number of nonequivalent PMSs of order 8 is $33{,}408{,}000/8 = 4{,}176{,}000$.

4.3 the number of PMSs of order 9

(1) By Corollary 3.4, $N(9\times9) = 3$. There are 3 basic Auxiliary squares of order 9:

0	1	2	3	4	5	6	7	8
9	10	11	12	13	14	15	16	17
18	19	20	21	22	23	24	25	26
27	28	29	30	31	32	33	34	35
36	37	38	39	40	41	42	43	44
45	46	47	48	49	50	51	52	53
54	55	56	57	58	59	60	61	62
63	64	65	66	67	68	69	70	71
72	73	74	75	76	77	78	79	80

0	1	2	27	28	29	54	55	56
3	4	5	30	31	32	57	58	59
6	7	8	33	34	35	60	61	62
9	10	11	36	37	38	63	64	65
12	13	14	39	40	41	66	67	68
15	16	17	42	43	44	69	70	71
18	19	20	45	46	47	72	73	74
21	22	23	48	49	50	75	76	77
24	25	26	51	52	53	78	79	80

0	1	2	9	10	11	18	19	20
3	4	5	12	13	14	21	22	23
6	7	8	15	16	17	24	25	26
27	28	29	36	37	38	45	46	47
30	31	32	39	40	41	48	49	50
33	34	35	42	43	44	51	52	53
54	55	56	63	64	65	72	73	74
57	58	59	66	67	68	75	76	77
60	61	62	69	70	71	78	79	80

(In [1], Candy gave 4 Basic Auxiliary squares of order 9. That is wrong.)
(2) For any two distinct integers a,b, the step <a,b> belongs to any one of four types R_3, C_3, S_3 and D_3. And $|R_3| = |C_3| = |S_3| = |D_3| = 12$. Thus #(Column steps) · #(suitable Row steps) = $(4 \cdot 12) \cdot (3 \cdot 12) = 1728$
(3) Corresponding to any choice of the steps, we need to get T_3. Since $\#(T_3) = 2$ for each Auxiliary square in (1). Thus $\#(P_3) = 2 \cdot (3!)^2(2!) \cdot \#(T_3) = 288$.
By using P_3, we can generate $(288)^2$ Valid Auxiliary squares of order 9. (In [1], Candy generated $(1296)^2$ Auxiliary squares. Since he over counted.)
Therefore the total number of PMSs of order 9 is $3 \cdot 1728 \cdot (288)^2 / 6^2 = 11{,}943{,}936$. The total number of nonequivalent PMSs of order 9 is 1,492,992.

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