The 2-Packing Number of 3-Dimensional Grids

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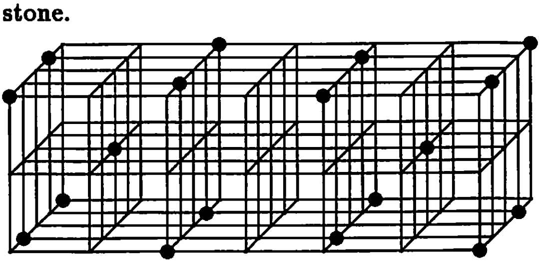
What is the 2-packing number of the $l \times m \times n$ complete grid graph? Fisher solved this for $1 \times m \times n$ grids for all m and n. We answer this for $2 \times m \times n$ grids for all m and n, and for $3 \times 3 \times n$, $3 \times 4 \times n$, $3 \times 7 \times n$, $4 \times 4 \times n$ and $5 \times 5 \times n$ grids for all n. Partial results are given for other sizes.

Given a graph G, a *2-packing* is a subset of its nodes with disjoint closed neighborhoods. Following Chang and Clark [1], nodes of a 2-packing will be called "stones" (as in the game of Go). Let $P_2(G)$ (the *2-packing number* of G) be the maximum cardinality of a 2-packing of G. Let P_n (note the unfortunate notational clash) be a path on nodes 1, 2, ..., n with node i adjacent to node j if $|i-j| = 1$. Then $P_m \times P_n$ is an $m \times n$ grid of nodes with node (g,h) adjacent to node (j,k) if $|g-j|+|h-k| = 1$. Fisher [2] found $P_2(P_m \times P_n)$: he showed that for all $m \le n$,

$$P_2(P_m \times P_n) = \begin{cases} \lceil (m+1)n/6 \rceil & \text{if } m \le 3 \\ \lceil 6n/7 \rceil & \text{if } m = 4 \text{ and } n \not\equiv 1 \pmod 7 \\ \lceil 6n/7 \rceil + 1 & \text{if } m = 4 \text{ and } n \equiv 1 \pmod 7 \\ 10 & \text{if } (m,n) = (7,7) \\ \lceil (mn+2)/5 \rceil & \text{if } 5 \le m \le 7 \text{ and } (m,n) \ne (7,7) \\ 17 & \text{if } (m,n) = (8,10) \\ \lceil mn/5 \rceil & \text{if } m \ge 8 \text{ and } (m,n) \ne (8,10). \end{cases} \quad (1)$$

Here we try to extend these results to a third dimension. Let $P_l \times P_m \times P_n$ (the $l \times m \times n$ *complete grid graph* or just the $l \times m \times n$ *grid*) be the graph on an $l \times m \times n$ grid of nodes with node (f,g,h) adjacent to node (i,j,k) if $|f-i|+|g-j|+|h-k| = 1$. Let $a_{l,m,n} = P_2(P_l \times P_m \times P_n)$. "Layers", "columns" and "rows" refer to $1 \times m \times n$, $l \times m \times 1$ and $l \times 1 \times n$ subgrids, respectively. Figure 1 shows a 2-dimensional diagram of a 2-packing of a 3-dimensional grid. An i in square (j,k) indicates that node (i,j,k) is a stone.



3		1		3		1
1		3		1		3
	2				2	
3		1		3		1
1		3		1		3

Figure 1. Above is a 2-packing of the $3 \times 5 \times 7$ grid with 18 stones and its 2-dimensional representation. Thus $a_{3,5,7} \ge 18$.

Lemma 1 is implicitly used throughout this paper.

Lemma 1. *Let k, l, m, n be positive integers with $k < n$. Then $a_{l,m,n} \leq a_{l,m,n-k} + a_{l,m,k}$.*

Proof. A maximal 2-packing of an $l \times m \times n$ grid has at most $a_{l,m,k}$ stones in the first k columns and $a_{l,m,n-k}$ stones in the remaining columns. ■

$2 \times m \times n$ Grids

In proving (1), Fisher relied on a computer to find the 2-packing number for numerous 2-dimensional grids. Surprisingly, the 2-packing number of any $2 \times m \times n$ grid can be found without citing computational results.

Theorem 2. *For all $2 \leq m \leq n$,*

$$a_{2,m,n} = \begin{cases} n+1 & \text{if } m = 3 \text{ and } n \text{ is odd} \\ \lceil mn/3 \rceil & \text{otherwise.} \end{cases}$$

Proof. Figure 2 shows $a_{2,m,n} \geq \lceil mn/3 \rceil$. Figure 3 shows $a_{2,3,n} \geq n+1$ when n is odd. Thus the above formula is a lower bound for $a_{2,m,n}$.

1			1			1		
	2			2			2	
		1			1			1
2			2			2		
	1			1			1	
		2			2			2
1			1			1		
	2			2			2	
		1			1			1

Figure 2. A 2-packing of $2 \times m \times n$ grids. If we start in the top left corner, it is easy to verify that the 2-packing has $\lceil mn/3 \rceil$ stones.

2		1		2		1		2		1	
1		2		1		2		1		2	

Figure 3. A 2-packing of $2 \times 3 \times n$ grids with $n+1$ stones for odd n.

To show equality, we need a number of cases. First let $m = 2$. From (1), we have $a_{2,2,1} = 1$ and hence $a_{2,2,2} \leq 2a_{2,2,1} = 2$. Suppose the $2 \times 2 \times 3$ grid can have 3 stones. Then one layer has at least 2 stones which must be in opposite corners. This precludes stones in the other layer, a contradiction. So $a_{2,2,3} = 2$ and hence $a_{2,2,n} = \lceil 2n/3 \rceil$ for $n \leq 3$. For $n > 3$, induction gives $a_{2,2,n} \leq a_{2,2,3} + a_{2,2,n-3} = 2 + \lceil 2(n-3)/3 \rceil = \lceil 2n/3 \rceil$.

Next let $m = 3$. We already have $a_{2,3,1} = 2$ and $a_{2,3,2} = 2$. For $n > 2$, induction gives $a_{2,3,n} \leq a_{2,3,2} + a_{2,3,n-2} = 2 + n - 2 = n$ if n is even, and $a_{2,3,n} \leq a_{2,3,2} + a_{2,3,n-2} = 2 + n - 1 = n + 1$ if n is odd.

Now let $m = 4$. We already have $a_{2,4,1} = 2$, $a_{2,4,2} = 3$ and $a_{2,4,3} = 4$. So $a_{2,4,n} = \lceil 4n/3 \rceil$ for $n \leq 3$. For $n > 3$, induction gives $a_{2,4,n} \leq a_{2,3,4} + a_{2,4,n-3} = 4 + \lceil 4(n-3)/3 \rceil = \lceil 4n/3 \rceil$.

The $m = 5$ case is more difficult since $a_{2,5,n} > \lceil 5n/3 \rceil$ for $n = 1$ or 3. We already have $a_{2,5,4} = 7$. For $n = 5$, suppose the $2 \times 5 \times 5$ grid can have 10 stones. Since $a_{2,5,1} = 3$ and $a_{2,5,4} = 7$, Column 1 has 3 stones. Also since $a_{2,5,2} = 4$ and $a_{2,5,3} = 6$, Columns 1 and 2 together have 4 stones implying Column 2 has 1 stone. However, 3 stones in Column 1 precludes stones in Column 2 (see Figure 4), a contradiction. So $a_{2,5,5} = 9$. For $n = 6$, we have $a_{2,5,6} \leq a_{2,2,6} + a_{2,3,6} = 4 + 6 = 10$. For $n = 7$, an argument similar to the $n = 5$ case shows that $a_{2,5,7} = 12$. For $n = 8$, we have $a_{2,5,8} \leq a_{2,2,8} + a_{2,3,8} = 6 + 8 = 14$. For $n = 9$, suppose the $2 \times 5 \times 9$ grid can have 16 stones. Since $a_{2,5,2} = 4$ and $a_{2,5,7} = 12$, Columns 1 and 2 together have 4 stones. Since $a_{2,5,3} = 6$ and $a_{2,5,6} = 10$, Columns 1 to 3 have a total of 6 stones. So Column 3 has 2 stones. Since no column has 4 stones, and 3 stones in one column and 1 stone in the adjacent column is impossible, Columns 1 and 2 also each have 2 stones. Since $a_{2,3,1} = 2$ and $a_{2,3,4} = 4$, Rows 1 and 5 each have 2 stones in the first 3 columns. However, this allows only one stone in Column 2 (see Figure 5), a contradiction. Thus $a_{2,5,9} = 14$ and hence $a_{2,5,n} = \lceil 5n/3 \rceil$ for $4 \leq n \leq 9$. For $n > 9$, induction gives $a_{2,5,n} \leq a_{2,5,6} + a_{2,5,n-6} = 10 + \lceil 5(n-6)/3 \rceil = \lceil 5n/3 \rceil$.

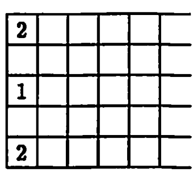


Figure 4. Placing 3 stones in Column 1 of a $2 \times 5 \times n$ grid precludes stones in Column 2.

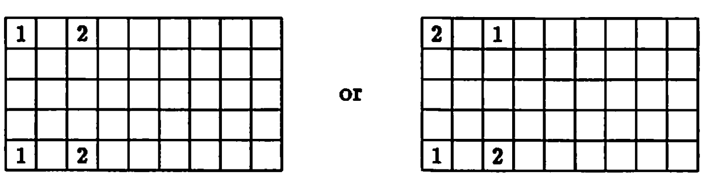


Figure 5. Two stones each in the first 3 columns of the $2 \times 5 \times 9$ grid is impossible. If Columns 1, 2 and 3 each have 2 stones, there are 2 stones in the first 3 columns of Row 1 and 2 stones in the first 3 columns of Row 5. But then Column 2 can have only one stone.

For $m = 6$, we already have $a_{2,6,2} = 4$, $a_{2,6,3} = 6$ and $a_{2,6,4} = 8$. Thus $a_{2,6,n} = 2n$ for $2 \le n \le 4$. For $n > 4$, induction gives $a_{2,6,n} \le a_{2,6,3} + a_{2,6,n-3} = 6 + 2(n-3) = 2n$.

For $m = 7$, we already have $a_{2,7,4} = 10$, $a_{2,7,5} = 12$ and $a_{2,7,6} = 14$. For $n = 7$, we have $a_{2,7,7} \le a_{2,2,7} + a_{2,5,7} = 5 + 12 = 17$. For $n = 8$, we have $a_{2,7,8} \le a_{2,3,8} + a_{2,4,8} = 8 + 11 = 19$. For $n = 9$, we have $a_{2,7,9} \le a_{2,2,9} + a_{2,5,9} = 6 + 15 = 21$. Thus $a_{2,7,n} = \lceil 7n/3 \rceil$ for $4 \le n \le 9$. For $n > 9$, induction gives $a_{2,7,n} \le a_{2,7,6} + a_{2,7,n-6} = 14 + \lceil 7(n-6)/3 \rceil = \lceil 7n/3 \rceil$.

For $m = 8$, we already have $a_{2,8,2} = 6$, $a_{2,8,3} = 8$ and $a_{2,8,4} = 11$. Thus $a_{2,8,n} = \lceil 8n/3 \rceil$ for $2 \le n \le 4$. For $n > 4$, induction gives $a_{2,8,n} \le a_{2,8,3} + a_{2,8,n-3} = 8 + \lceil 8(n-3)/3 \rceil = \lceil 8n/3 \rceil$.

For $m = 9$, we already have $a_{2,9,4} = 12$, $a_{2,9,5} = 15$, $a_{2,9,6} = 18$ and $a_{2,9,7} = 21$. Thus $a_{2,9,n} = 3n$ for $4 \le n \le 7$. For $n > 7$, induction gives $a_{2,9,n} \le a_{2,9,4} + a_{2,9,n-4} = 12 + 3(n-4) = 3n$.

We have shown $a_{2,m,n} = \lceil mn/3 \rceil$ for $4 \le m \le 9$ and $n \ge 4$. For $m > 9$ and $n \ge 4$, induction gives $a_{2,m,n} \le a_{2,6,n} + a_{2,m-6,n} = 2n + \lceil (m-6)n/3 \rceil = \lceil mn/3 \rceil$. ■

$3 \times m \times n$ Grids

We could not find general formulas for the 2-packing number of $3 \times m \times n$ grids. Previous sections found $a_{3,1,n}$ and $a_{3,2,n}$. We find $a_{3,3,n}$, $a_{3,4,n}$ and $a_{3,7,n}$, and partial results for other values of m. For $m > 4$, these results use a computer to implement an exhaustive branch-and-bound search.

Theorem 3. *For all $n > 0$, we have $a_{3,3,n} = \lceil 5n/3 \rceil$.*

Proof. Figure 6 shows $a_{3,3,n} \ge \lceil 5n/3 \rceil$. Equation (1) shows $a_{3,3,1} = 2$ and hence $a_{3,3,2} \le 2a_{3,3,1} = 4$. Suppose the $3 \times 3 \times 3$ grid can have 6 stones. Since $a_{3,3,1} = 2$, each column has 2 stones. The first two columns of Figure 6 shows, up to symmetry, the only way to put 2 stones each in Columns 1 and 2. However, this only allows 1 stone in Column 3, a contradiction. Thus $a_{3,3,3} = 5$ and hence $a_{3,3,n} = \lceil 5n/3 \rceil$ for $n \le 3$. For $n > 3$, induction gives $a_{3,3,n} \le a_{3,3,3} + a_{3,3,n-3} = 5 + \lceil 5(n-3)/3 \rceil = \lceil 5n/3 \rceil$. ■

1	3		1	3		1	3		1	3	
		2			2			2			2
3	1		3	1		3	1		3	1	

Figure 6. A 2-packing of $3 \times 3 \times n$ grids with $\lceil 5n/3 \rceil$ stones.

Theorem 4. *For $n > 3$, we have $a_{3,4,n} = 2n$.*

Proof. Figure 7 shows $a_{3,4,n} \ge 2n$. Theorem 2 gives $a_{3,4,2} = 4$ and $a_{3,4,4} \le 2a_{3,4,2} = 8$. Suppose the $3 \times 4 \times 5$ grid can have 11 stones. Since $a_{3,4,2} = 4$ and $a_{3,4,1} = 3$, Columns 1 to 5 have 3, 1, 3, 1 and 3 stones, respectively. Figure 8 shows the only patterns (up to symmetry) for Column 3. One pattern precludes Columns 2 and 4 from each having a stone. The other pattern allows this. Up to symmetry, the right side of Figure 8 shows the only way for this to occur. But this precludes 3 stones in Column 5, a contradiction. Thus $a_{3,4,5} = 10$. For $n > 5$, induction gives $a_{3,4,n} \le a_{3,4,2} + a_{3,4,n-2} = 4 + 2(n-2) = 2n$. ∎

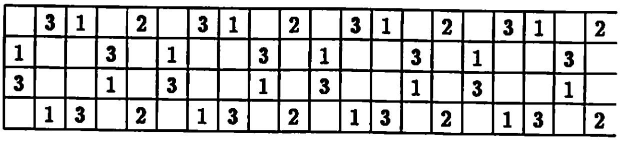


Figure 7. A 2-packing of $3 \times 4 \times n$ grids with $2n$ stones.

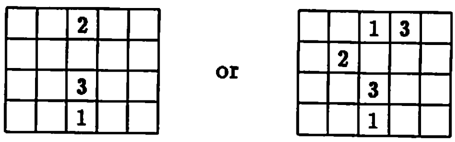


Figure 8. The only ways, up to symmetry, of placing 3 stones in Column 3 of the $3 \times 4 \times 5$ grid.

Trying to find $a_{3,5,n}$ proved frustating. Theorem 5 summarizes the computer results. Figure 1 gives the only known $3 \times 5 \times n$ grid with more than $\lceil (33n+6)/14 \rceil$ stones. Figure 9 shows a maximal 2-packing typical of $3 \times 5 \times n$ grids. We conjecture $a_{3,5,n} = \lceil (33n+6)/14 \rceil$ for all $n > 19$. Note that so far, upper bound arguments have been simple because for some k and n_0, we had $a_{l,m,n} = a_{l,m,k} + a_{l,m,n-k}$ for all $n \ge n_0$. However if the conjecture is true, this does not happen for $3 \times 5 \times n$ grids.

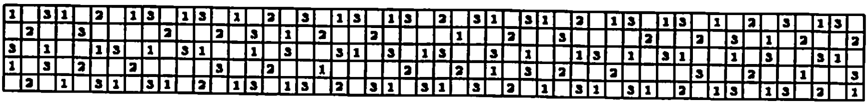


Figure 9. A maximal 2-packing of the $3 \times 5 \times 50$ grid. Note the alternating 14 column patterns, one an inversion of the other.

Theorem 5. *For* $5 \le n \le 500$, *we have*

$$a_{3,5,n} = \begin{cases} \lceil (33n+6)/14 \rceil - 1 & \text{if } n = 16 \text{ or } 19 \\ \lceil (33n+6)/14 \rceil + 1 & \text{if } n = 7 \\ \lceil (33n+6)/14 \rceil & \text{otherwise.} \end{cases}$$

We found what appears to be an exact lower bound for $a_{3,6,n}$ (the 2-packing in Figure 10 proves the first part of Theorem 6). Figure 11 shows the only known $3 \times 6 \times n$ grid with $n \geq 6$ has more than $\lceil (14n+2)/5 \rceil$ stones.

Theorem 6. *For all $n \geq 6$, we have $a_{3,6,n} \geq \lceil (14n+2)/5 \rceil$. Further for all $n \leq 500$ except $n = 8$, the inequality is an equality. Also $a_{3,6,8} = 24$.*

3	1			3	1		3	1		3	1			3	1		3	1		3	1			3	1		3	1	
		3	1			2			2			3	1			2			2			3	1			2			2
1					3		1	3		1					3		1	3		1					3		1	3	
3		1	3		1					3		1	3		1					3		1	3		1				
	2			2			3	1			2			2			3	1			2			2			3	1	
1		3	1		3	1			3	1		3	1		3		3	1			3	1			3	1			3

...

3	1		2		3	1		2	
		3		1			3		1
1					2		1		3
3		1	3			3		2	
	2			2		1			1
1		3	1		3		2		3

Figure 10. A 2-packing of $3 \times 6 \times n$ grids with $\lceil (14n+2)/5 \rceil$ stones. When $n \equiv 0 \pmod{10}$, repeat the first 10 column as many times as needed and then add the last 10 columns. When $n \equiv 5 \pmod{10}$, skip the first 5 columns, repeat the next 10 columns as often as needed and then add the last 10 columns. Otherwise, repeat the first 10 columns and truncate.

1		3	1		3		2
	2			2		1	
3		1	3				3
1				1	3		1
	3		2			2	
2		1		3	1		3

Figure 11. A maximal 2-packing of the $3 \times 6 \times 8$ grid with 24 stones.

Unlike the previous two cases, we found $a_{3,7,n}$.

Theorem 7. *For all $n > 3$, we have $a_{3,7,n} = 2 \lceil 5n/3 \rceil$.*

Proof. Figure 12 shows $a_{3,7,n} \geq 2 \lceil 5n/3 \rceil$ for all n. The computer showed $a_{3,7,n} = 2 \lceil 5n/3 \rceil$ for $4 \leq n \leq 9$. For $n > 9$, we have $a_{3,7,n} \leq a_{3,7,6} + a_{3,7,n-6} = 20 + 2 \lceil 5(n-6)/3 \rceil = 2 \lceil 5n/3 \rceil$. ■

3	1		3	1		3	1		3	1		3	1	
		2			2			2			2			2
1	3		1	3		1	3		1	3		1	3	
3	1		3	1		3	1		3	1		3	1	
		2			2			2			2			2
1	3		1	3		1	3		1	3		1	3	

Figure 12. A 2-packing of $3 \times 7 \times n$ grids with $\lceil 5n/3 \rceil$ stones.

The status for $3 \times 8 \times n$ grids is the reverse of that for $3 \times 6 \times n$ grids in that we appear to have an exact upper bound. To show equality, we must find a 2-packing of $3 \times 8 \times n$ grids with $\lceil 70n/19 \rceil$ stones for all $n > 57$.

Theorem 8. *For* $8 \leq n \leq 500$, *we have*

$$a_{3,8,n} = \begin{cases} \lceil 70n/19 \rceil + 1 & \text{if } n = 9,\ 10,\ 11,\ 12,\ 13,\ 14,\ 16,\ 17 \\ & 19,\ 20,\ 21,\ 23,\ 24,\ 26,\ 27,\ 29,\ 32 \\ & 35,\ 36,\ 38,\ 39,\ 42,\ 51,\ 54 \text{ or } 57 \\ \lceil 70n/19 \rceil & \text{otherwise.} \end{cases}$$

Further for $n > 57$, *we have* $a_{3,8,n} \leq \lceil 70n/19 \rceil$.

Proof. A computer gives the result for $n \leq 500$. For $n > 133$, induction gives $a_{3,8,n} \leq a_{3,8,n-76} + a_{3,8,76} = \lceil 70(n-76)/19 \rceil + 280 = \lceil 70n/19 \rceil$. ■

Theorems 9 and 10 summarize the computer results for $3 \times 9 \times n$ and $3 \times 10 \times n$ grids.

Theorem 9. *For* $9 \leq n \leq 100$, *we have* $a_{3,9,n} = \lceil (41n + 11)/10 \rceil$.

Theorem 10. *For* $10 \leq n \leq 26$, *we have* $a_{3,10,n} = \lceil 49n/11 \rceil + 2$.

For $3 \times 11 \times n$ grids, the computer gave the following. The 2-packing in Figure 13 matches the formula for $n \geq 14$.

3	1		3	1		3	1		3	1		3	1		1	3		2		1
		2			2			2			2			2			1			3
1	3		1	3		1	3		1	3		1	3		3			3	1	
															1		2			2
3	1		3	1		3	1		3	1		3	1			3		1	3	
		2			2			2			2			2		1				1
1	3		1	3		1	3		1	3		1	3		3		2			3
															1			3	1	
3	1		3	1		3	1		3	1		3	1			3	1			2
		2			2			2			2			2					2	
1	3		1	3		1	3		1	3		1	3			1	3		1	3

Figure 13. A maximal 2-packing of the $3 \times 11 \times 21$ grid. Except for the last 6 columns, this repeats every 3 columns. A 2-packing of $3 \times 11 \times n$ grids with $5n+2$ stones (if $n \equiv 2 \pmod 3$) or $5n+1$ stones can be formed by repeating the 3 columns as many times as needed and if $n \equiv 0 \pmod 3$ adding the last 6 columns.

Theorem 11. *For* $11 \leq n \leq 500$, *we have*

$$a_{3,11,n} = \begin{cases} 5n + 2 & \text{if } n \equiv 2 \pmod 3, \text{ or } n = 12 \text{ or } 13 \\ 5n + 1 & \text{otherwise.} \end{cases}$$

Further for all n, *the above formula is a lower bound for* $a_{3,11,n}$.

We may ask: *What is the asymptotic density of a maximal 2-packing on a* $3 \times m \times n$ *grid?* In other words, what is $\alpha_3 = \lim_{m,n \to \infty} a_{3,m,n}/(3mn)$? The 2-packing in Figure 14 proves Theorem 12. It is maximal for $3 \times 3 \times n$ and $3 \times 7 \times n$ grids, and appears almost maximal for $3 \times 11 \times n$ grids.

3	1		3	1		3	1		3	1		3	1		3	1		3	1	
		2			2			2			2			2			2			2
1	3		1	3		1	3		1	3		1	3		1	3		1	3	
3	1		3	1		3	1		3	1		3	1		3	1		3	1	
		2			2			2			2			2			2			2
1	3		1	3		1	3		1	3		1	3		1	3		1	3	
3	1		3	1		3	1		3	1		3	1		3	1		3	1	
		2			2			2			2			2			2			2
1	3		1	3		1	3		1	3		1	3		1	3		1	3	
3	1		3	1		3	1		3	1		3	1		3	1		3	1	
		2			2			2			2			2			2			2
1	3		1	3		1	3		1	3		1	3		1	3		1	3	

Figure 14. A 2-packing of $3 \times m \times n$ grids.

Theorem 12. *If* $m \equiv 3 \pmod 4$*, then* $a_{3,m,n} \geq \frac{m+1}{4}\left\lceil\frac{5n}{3}\right\rceil$.

Theorem 12 show $\alpha_3 \geq 5/36 \approx 0.138889$. However, Figure 15 shows an asymptoticly denser 2-packing. It and its 90° rotation give Theorem 13.

1	3			2			1	3			2			1	3			2		
		1	3			2			1	3			2			1	3			2
	2			1	3			2			1	3			2			1	3	
3			2			1	3			2			1	3			2			1
	1	3			2			1	3			2			1	3			2	
2			1	3			2			1	3			2			1	3		
		2			1	3			2			1	3			2			1	3
1	3			2			1	3			2			1	3			2		
		1	3			2			1	3			2			1	3			2
	2			1	3			2			1	3			2			1	3	
3			2			1	3			2			1	3			2			1
	1	3			2			1	3			2			1	3			2	
2			1	3			2			1	3			2			1	3		
		2			1	3			2			1	3			2			1	3

Figure 15. A denser 2-packing of $3 \times m \times n$ grids.

Theorem 13. *For all* m *and* n, $a_{3,m,n} \geq \begin{cases} \left\lceil\frac{3mn}{7}\right\rceil + 1 & \textit{if } \{m,n\} = \{1,2\} \\ \left\lceil\frac{3mn}{7}\right\rceil & \textit{otherwise.} \end{cases}$

Theorem 13 shows $\alpha_3 \geq 1/7 \approx 0.142857$. Though the 2-packing in Figure 15 is asymptotically denser, Figure 14 has more stones for rather large grids. For example on the $3 \times 47 \times 47$ grid, the pattern in Figure 14 has 948 stones, while Figure 15 has 947 stones.

We can also finds upper bounds on α_3. By dividing the $3 \times m \times n$ grid into $3 \times j \times k$ subgrids, Lemma 1 shows that for any j and k, we have $\alpha_3 \leq a_{3,j,k}/(3jk)$. Among the results in this paper, the best upper bound is found by using $a_{3,10,22} = 99$ giving $\alpha_3 \leq 3/20 = 0.15$.

$4 \times m \times n$ Grids

We also could not find a formula for the 2-packing number of $4 \times m \times n$ grids. However, we found $a_{4,4,n}$ and partial results for $m = 5$, 6 and 7.

4	1		3
2		4	
	3		1
14		2	4

14		2	4	1
	3			3
2		4	2	
4	1			14

	14		2	4	
2		3		1	3
4		1	4		
1	3		2		14

14		2	4		2	4	1
	3		1	3			3
2		4			4	2	
4	1		3	1			14

	14		2	4		2		4	1
2		3		1		4	1		3
4		1	4		2			2	
1	3		2		4	1	3		14

	14		2	4		1	4	2		4	2
2		3		1	3				3	1	
4		1	4			2		14			3
1	3		2		14		3		2	4	1

Figure 16. Maximal 2-packings of $4 \times 4 \times n$ grids with $\lceil 5n/2 \rceil + 1$ stones for $n = 4$, 5, 6, 8, 10 and 12.

3	1	4		1	4	2		3	1	4		1	4	2		3	1	4		1	4	2	
		2		3			14			2		3			14			2		3			14
2			14			3		2			14			3		2			14			3	
4	1	3		2	4	1		4	1	3		2	4	1		4	1	3		2	4	1	

Figure 17. A 2-packing of $4 \times 4 \times n$ grids with $\lceil 5n/2 \rceil$ stones.

Theorem 14. *For all $n \geq 4$, we have*

$$a_{4,4,n} = \begin{cases} \lceil 5n/2 \rceil + 1 & \text{if } n = 4, 5, 6, 8, 10 \text{ or } 12 \\ \lceil 5n/2 \rceil & \text{otherwise.} \end{cases}$$

Proof. Figure 17 shows that $a_{4,4,n} \geq \lceil 5n/2 \rceil$ for all n. The computer proves the result for $n \leq 26$ (see Figure 16). For $n > 26$, induction gives $a_{4,4,n} \leq a_{4,4,14} + a_{4,4,n-14} = 35 + \lceil 5(n-14)/2 \rceil = \lceil 5n/2 \rceil$. ■

Theorem 15, 16 and 17 summarizes the computer results for $4 \times 5 \times n$, $4 \times 6 \times n$ and $4 \times 7 \times n$ grids. Figure 18-21 gives examples of these 2-packings.

Theorem 15. *For $5 \leq n \leq 500$, we have*

$$a_{4,5,n} = \begin{cases} 3n + 1 & \text{if } n = 5, 6 \text{ or } 10 \\ 3n + 2 & \text{otherwise.} \end{cases}$$

	2	4	1		4	1		4	2		14		3	1		4	1		4	1		4	2		14		3	1	4
14				2			3	1		3		2		4	2			2			3	1		3		2			2
	3	1	4		1	4			4	1		4	1			1	4		1	4			4	1		4	1	3	
2			2		3		1	3			2			2	4		2		3		1	3			2				14
4	1	3		14		2	4		1	4		1	4		1	3		14		2	4		1	4		1	4	2	

Figure 18. A maximal 2-packing of the $4 \times 5 \times 30$ grid.

Theorem 16. *For $6 \leq n \leq 100$, we have*

$$a_{4,6,n} = \begin{cases} \lceil (46n+23)/13 \rceil - 1 & \text{if } n = 8,\ 9,\ 10,\ 12 \text{ or } 16 \\ \lceil (46n+23)/13 \rceil + 1 & \text{if } n \equiv 2 \pmod{13} \\ \lceil (46n+23)/13 \rceil & \text{otherwise.} \end{cases}$$

1	4	2		14		3	1		3		1	4		1	3		14		2	4		2		4	1		4	2		14		3	1		3		2	4		2
3			3		2		4		1	4		2		4		2		3		1		4	1		3		1		3		2		4		1	4		1	3	
	2	4	1		4	1		3		2			3		1	4		1	4		2		3			2		4	1		4	1		3		2				14
14				2		3		1	4		1	4		2			3		2		4	1		4	1		3			2		3		1	4		1	4	2	
	3	1		4	1		4		2		3		1		4		1	4		1		3		2		4		1		4	1		4		2		3			3
2		4	2		3		1	3		14		2	4		1	3		2		4	2		14		3	1		4	2		3		1	3		14		2	4	1

Figure 19. A maximal 2-packing of the $4 \times 6 \times 41$ grid.

Theorem 17. *For $7 \leq n \leq 100$, we have*

$$a_{4,7,n} = \begin{cases} \lceil (33n+18)/8 \rceil - 1 & \text{if } n = 7,\ 15 \text{ or } 16 \\ \lceil (33n+18)/8 \rceil + 1 & \text{if } n = 9 \text{ or } 20 \\ \lceil (33n+18)/8 \rceil & \text{otherwise.} \end{cases}$$

4	1	3		14		2	4		2	4		1	3		14		2	4	1
2			2		3		1	3			2	4		2		3			3
	3	1	4		1	4			1	3			1	4		1	4	2	
14				2			2	4			4	2			2				14
	2	4	1		4	1			3	1			4	1		4	1	3	
3			3		2		4	2			3	1		3		2			2
1	4	2		14		3	1		4	2		4	2		14		3	1	4

Figure 20. A maximal 2-packing of the $4 \times 7 \times 20$ grid.

4	1	3		14		2	4		2	4		1	3		14		2	4		2	4		1	3		14		2	4		2	4		1	3		14		2	4	1
2			2		3		1	3			2	4		2		3		1	3			2	4		2		3		1	3			2	4		2		3			3
	3	1	4		1	4			1	3			1	4		1	4			1	3			1	4		1	4			1	3			1	4		1	4	2	
14				2			2	4			4	2			2			2	4			4	2			2			2	4			4	2			2				14
	2	4	1		4	1			3	1			4	1		4	1			3	1			4	1		4	1			3	1			4	1		4	1	3	
3			3		2		4	2			3	1		3		2		4	2			3	1		3		2		4	2			3	1		3		2			2
1	4	2		14		3	1		4	2		4	2		14		3	1		4	2		4	2		14		3	1		4	2		4	2		14		3	1	4

Figure 21. A maximal 2-packing of the $4 \times 7 \times 42$ grid.

We may also ask: *What is $\alpha_4 = \lim_{m,n \to \infty} a_{4,m,n}/(4mn)$?* The 2-packing in Figure 22 and its 90° rotation gives the following result.

Theorem 18. *For all m and n,*

$$a_{4,m,n} \geq \begin{cases} \lceil \frac{4mn}{7} \rceil + 1 & \text{if } \{m,n\} \equiv \{1,2\} \text{ or } \{2,6\} \pmod 7 \\ \lceil \frac{4mn}{7} \rceil & \text{otherwise.} \end{cases}$$

Theorem 18 then shows $\alpha_4 \geq 1/7 \approx 0.142857$. An upper limit can be found by dividing a $4 \times m \times n$ grid into $4 \times j \times k$ subgrids. Then for any j and k, Lemma 1 shows that $\alpha_4 \leq a_{4,j,k}/(4jk)$. Among the results in this paper, the best limit uses $a_{4,7,94} = 390$ to give $\alpha_4 \leq 145/1316 \approx 0.148176$.

1	3			2	4		1	3			2	4		1	3			2	4	
4		1	3			2	4		1	3			2	4		1	3			2
	2	4		1	3			2	4		1	3			2	4		1	3	
3			2	4		1	3			2	4		1	3			2	4		1
	1	3			2	4		1	3			2	4		1	3			2	4
2	4		1	3			2	4		1	3			2	4		1	3		
		2	4		1	3			2	4		1	3			2	4		1	3
1	3			2	4		1	3			2	4		1	3			2	4	
4		1	3			2	4		1	3			2	4		1	3			2
	2	4		1	3			2	4		1	3			2	4		1	3	
3			2	4		1	3			2	4		1	3			2	4		1
	1	3			2	4		1	3			2	4		1	3			2	4
2	4		1	3			2	4		1	3			2	4		1	3		
		2	4		1	3			2	4		1	3			2	4		1	3

Figure 22. A 2-packing of $4 \times m \times n$ grids.

Larger Grids

We found the 2-packing number of $5 \times 5 \times n$ grids.

Theorem 19. *For all $n \geq 4$, we have $a_{5,5,n} = 4n$.*

Proof. Figure 23 shows $a_{5,5,n} \geq 4n$. The computer showed that $a_{5,5,4} = 16$, $a_{5,5,5} = 20$, $a_{5,5,6} = 24$ and $a_{5,5,7} = 28$. For $n > 7$, induction gives $a_{5,5,n} \leq a_{5,5,4} + a_{5,5,n-4} = 16 + 4(n-4) = 4n$. ∎

5	1	3	5	1	3	5	1	3	5	1	3
2	4		2	4		2	4		2	4	
		15			15			15			15
4	2		4	2		4	2		4	2	
1	5	3	1	5	3	1	5	3	1	5	3

Figure 23. A 2-packing of $5 \times 5 \times n$ grids with $4n$ stones.

Theorem 20 gives asymptotically exact bounds on $a_{l,m,n}$.

Theorem 20. *For all l, m and n, we have*

$$\frac{lmn}{7} \leq a_{l,m,n} \leq \frac{lmn + 2(a_{l,m,1} + a_{l,1,n} + a_{1,m,n})}{7}.$$

Proof. We can 2-pack an $l \times m \times n$ grid with $\lceil lmn/7 \rceil$ stones by repeating the $7 \times 7 \times 7$ subgrid in Figure 24 and truncating. This gives the lower bound. The cardinality of the closed neighborhood of a node is 7 minus the number of faces on which the node lies (*e.g.*, corner stones lie on three faces and have 4 nodes in their closed neighborhood). Since at most $a_{1,p,q}$ stones are on a $p \times q$ face, the result follows. ■

5	2	6	3	7	4	1
3	7	4	1	5	2	6
1	5	2	6	3	7	4
6	3	7	4	1	5	2
4	1	5	2	6	3	7
2	6	3	7	4	1	5
7	4	1	5	2	6	3

Figure 24. A 2-packing of the $7 \times 7 \times 7$ grid. Repeated use of this block forms a 2-packing of $l \times m \times n$ grids with at least $lmn/7$ stones.

Equation (1) gives $a_{1,m,n} \leq \lceil (mn+2)/5 \rceil \leq (mn+6)/5$ for all $m, n \geq 5$. Thus $a_{l,m,n} \in [lmn/7, (lmn + 0.4(lm + ln + mn + 18))/7]$ for all $l, m, n \geq 5$. Let $\alpha_l = \lim_{m,n\to\infty} a_{l,m,n}/(lmn)$. Then $1/7 \leq \alpha_l \leq 1/7 + 2/(35l)$ for all $l \geq 5$. Also the asymptotic density of a maximal 2-packing of 3-dimensional grids is $\alpha = \lim_{l\to\infty} \alpha_l = 1/7$.

Final Note

Two-dimensional complete grid graphs have been widely studied. The interest is understandable: it is a natural formulation that is surprising resistant to easy answers. This paper, as far as we know, is the first "expedition" of its type into 3-dimensional complete grid graphs. Much of the work on 2-dimensional grids could be extended (undoubtably some with more success than others) into three dimensions.

References

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