Vertex-Neighbor-Integrity of Powers of Cycles

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Abstract. Let G be a graph. A vertex subversion strategy of G, S, is a set of vertices in G whose closed neighborhood is deleted from G. The survival-subgraph is denoted by G/S. The vertex-neighbor-integrity of G, VNI(G), is defined to be $\mathrm{VNI(G)} = \min_{S \subseteq V(G)} \{|S| + \omega(G/S)\}$, where S is any vertex subversion strategy of G, and $\omega(G/S)$ is the maximum order of the components of G/S. In this paper, we evaluate the vertex-neighbor-integrity of the powers of cycles, and show that among the powers of the n-cycle, the maximum vertex-neighbor-integrity is $\lceil 2\sqrt{n}\, \rceil - 3$ and the minimum vertex-neighbor-integrity is $\lceil n/(2\lfloor n/2 \rfloor + 1) \rceil$.

I. Introduction

In 1987 Barefoot, Entringer, and Swart introduced the integrity of a graph to measure the "vulnerability" of the graph. [1] [2] We have developed a graphic parameter, called "vertex-neighbor-integrity" [4], incorporating the concept of the integrity [1] [2] and the idea of the vertex-neighbor-connectivity [6].

Let G=(V,E) be a graph and u be a vertex in G. $N(u) = \{v \in V(G) | v \neq u$, v and u are adjacent$\}$ is the *open neighborhood* of u, and $N[u] = \{u\} \cup N(u)$ denotes the *closed neighborhood* of u. A vertex u in G is said to be *subverted* if the closed neighborhood $N[u]$ is deleted from G. A set of vertices $S = \{u_1, u_2, ..., u_m\}$ is called a *vertex subversion strategy* of G if each of the vertices in S has been subverted from G. Let G/S be the *survival-subgraph* left after each vertex of S has been subverted from G. (If $N[S] = V(G)$, where $N[S] = \cup_{u \in S} N[u]$, V(G/S) is empty.) The vertex-neighbor-integrity of a graph G, VNI(G), is defined to be

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$$\mathrm{VNI}(G) = \min_{S \subseteq V(G)} \{|S| + \omega(G/S)\},$$

where S is any vertex subversion strategy of G, and $\omega(G/S)$ is the maximum order of the components of G/S.

Example 1: K_n is a complete graph of oder n. $\mathrm{VNI}(K_n) = 1$. [4]

Example 2: $K_{n,m}$, where $n > 1$ and $m > 1$, is a complete bipartite graph with a bipartition (X,Y), where $|X| = n$ and $|Y| = m$. $\mathrm{VNI}(K_{n,m}) = 2$. [4]

In this paper, we evaluate the vertex-neighbor-integrity of a family of graphs — powers of cycles, and show that among the powers of the n-cycle, the maximum vertex-neighbor-integrity is $\lceil 2\sqrt{n}\,\rceil - 3$, and the minimum vertex-neighbor-integrity is $\lceil n/(2\lfloor n/2 \rfloor + 1)\rceil$. $\lceil x \rceil$ is the smallest integer greater than or equal to x. $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

II. Vertex-Neighbor-Integrity of Powers of Cycles

We list the following known results which will be used later in the paper.

Lemma 1 [2]: For positive integers, n and m, if n is fixed, then the function $g(m) = m + \lceil \frac{n}{m} \rceil$ has the minimum value $\lceil 2\sqrt{n}\,\rceil$ at $m = \lceil \sqrt{n}\,\rceil$.

Lemma 2 [4]: Let P_n be a path of order $n \geq 1$. Then

$$\mathrm{VNI}(P_n) = \begin{cases} \lceil 2\sqrt{n+3}\,\rceil - 4, & \text{if } n \geq 2; \\ 1, & \text{if } n = 1. \end{cases}$$

Let $C_n = (v_0, v_1, ..., v_{n-1})$ be an n-cycle. The k-th ($1 \leq k \leq \lfloor n/2 \rfloor$) power of the n-cycle, C_n^k, has the vertices v_0, v_1, ..., v_{n-1}, and two vertices v_i and v_j are adjacent if $i - k \leq j \leq i + k$ (where the addition is taken modulo n). The notation will be used throughout the paper.

We first discuss the vertex-neighbor-integrity of an n-cycle, a special case of the powers of n-cycle.

Theorem 3: Let C_n be an n-cycle, where $n \geq 3$. Then

$$\mathrm{VNI}(C_n) = \begin{cases} \lceil 2\sqrt{n}\,\rceil - 3, & \text{if } n > 4; \\ 2, & \text{if } n = 4; \\ 1, & \text{if } n = 3. \end{cases}$$

Proof: If $n = 3$, it is clear that $\mathrm{VNI}(C_n) = 1$.

If $n \geq 4$, let S be a subset of $V(C_n)$ for which $\mathrm{VNI}(C_n)=|S| + \omega(C_n/S)$, and v be a vertex in S. Then $C_n/\{v\} = P_{n-3}$, and the remaining $|S| - 1$ vertices of S must be chosen from $V(P_{n-3})$ to minimize

$$(|S| - 1)+\left\lceil \frac{n - 3|S|}{|S|} \right\rceil = |S|+\left\lceil \frac{n}{|S|} \right\rceil - 4,$$

and

$$\mathrm{VNI}(P_{n-3}) = \min_{S' \subseteq V(P_{n-3})} (|S'| + \omega(P_{n-3}/S'))$$

$$= \min_{m \geq 0} \left(m+\left\lceil \frac{(n-3) - 3m}{m+1} \right\rceil \right)$$

$$= \min_{m \geq 0} \left(m+\left\lceil \frac{n - 3(m+1)}{m+1} \right\rceil \right)$$

$$= \min_{m \geq 0} \left(m + 1+\left\lceil \frac{n}{m+1} \right\rceil \right) - 4.$$

Therefore

$$\text{VNI}(\text{C}_n) = \begin{cases} \text{VNI}(\text{P}_{n-3}) + 1, & \text{if } n \geq 4; \\ 1, & \text{if } n = 3. \end{cases}$$

$$= \begin{cases} \lceil 2\sqrt{(n-3)+3}\, \rceil - 4 + 1, & \text{if } n > 4; \\ 1 + 1, & \text{if } n = 4; \\ 1, & \text{if } n = 3. \end{cases} \qquad \text{(By Lemma 2.)}$$

$$= \begin{cases} \lceil 2\sqrt{n}\, \rceil - 3, & \text{if } n > 4; \\ 2, & \text{if } n = 4; \\ 1, & \text{if } n = 3. \end{cases} \qquad \text{QED.}$$

To evaluate the vertex-neighbor-integrity of the k-th power of n-cycle, we need to use the following lemmas.

Lemma 4: For any positive integer n, if $x \geq \sqrt{n}$, then

$$\left\lceil \frac{n}{x} \right\rceil \leq \left\lceil \frac{n}{x+1} \right\rceil + 1.$$

Proof: Since

$$x \geq \sqrt{n} \Longrightarrow x^2 \geq n,$$

we have

$$x^2 + x \geq n + \sqrt{n} > n.$$

Then

$$n < x^2 + x \Longleftrightarrow$$

$$nx + n - x^2 - x < nx \Longleftrightarrow$$

$$\frac{n-x}{x} < \frac{n}{x+1} \Longrightarrow$$

$$\left\lceil \frac{n}{x} - 1 \right\rceil \leq \left\lceil \frac{n}{x+1} \right\rceil \Longleftrightarrow$$

$$\left\lceil \frac{n}{x} \right\rceil \leq \left\lceil \frac{n}{x+1} \right\rceil + 1.$$

QED.

Lemma 5: For any positive integer n, if $k \geq (\sqrt{n}-1)/2$ and h is a nonnegative integer, then

$$\left\lceil \frac{n}{2k+1} \right\rceil = \min_{h \geq 0} \left(\left\lceil \frac{n}{2k+h+1} \right\rceil + h \right).$$

Proof:

$$k \geq \frac{\sqrt{n}-1}{2} \Longrightarrow 2k+1 \geq \sqrt{n}.$$

Let $x = 2k+h+1$, where h is any nonnegative integer. Thus $x \geq 2k+1$, and hence $x \geq \sqrt{n}$. By Lemma 4, we have

$$\left\lceil \frac{n}{2k+h+1} \right\rceil \leq \left\lceil \frac{n}{2k+h+2} \right\rceil + 1,$$

and then

$$\left\lceil \frac{n}{2k+h+1} \right\rceil + h \leq \left\lceil \frac{n}{2k+(h+2)} \right\rceil + (h+1),$$

where h is any nonnegative integer. Hence we know that if $k \geq (\sqrt{n}-1)/2$, then $f(h) = \lceil n/(2k+h+1) \rceil + h$ is an increasing function for all nonnegative integer h, and therefore $f(h)$ has the minimum value $\lceil n/(2k+1) \rceil$ at $h = 0$. QED.

Using the above result, we can show the vertex-neighbor-integrity of the powers of n-cycle as follows:

Theorem 6: Let C_n^k be the k-th power of the n-cycle, where $n \geq 3$ and $1 \leq k \leq \lfloor n/2 \rfloor$. Then

$$\mathrm{VNI}(C_n^k) = \begin{cases} \lceil 2\sqrt{n}\,\rceil - (2k+1), & \text{if } 1 \leq k < \frac{\sqrt{n}-1}{2}; \\ \left\lceil \frac{n}{2k+1} \right\rceil, & \text{if } \frac{\sqrt{n}-1}{2} \leq k \leq \lfloor \frac{n}{2} \rfloor. \end{cases}$$

Proof: By the definition,

$$\mathrm{VNI}(C_n^k) = \min_{S \subseteq V(C_n^k)} \{|S| + \omega(C_n^k/S)\}.$$

Let $S^* = \{v_{i_1}, v_{i_2}, \dots, v_{i_m}\}$, where $0 \leq i_1 \leq i_2 \leq \dots \leq i_m \leq n-1$, be a subset of $V(C_n^k)$ for which $\mathrm{VNI}(C_n^k) = |S^*| + \omega(C_n^k/S^*)$. If $n-(2k+1)m > 0$, then $i_1 + n - i_m > 2k+1$ or $i_j - i_{j-1} > 2k+1$, for some $j = 2, 3, \dots, m$, and

$$\omega(C_n^k/S^*) \geq \left\lceil \frac{n-(2k+1)m}{m} \right\rceil.$$

Hence

$$\mathrm{VNI}(C_n^k) \geq \min_{m\geq 0}\left\{m+\left\lceil \frac{n-(2k+1)m}{m} \right\rceil\right\}$$

$$= \min_{m\geq 0}\left\{m+\left\lceil \frac{n}{m} \right\rceil\right\} - (2k+1)$$

$$= \lceil 2\sqrt{n}\,\rceil - (2k+1), \qquad (1)$$

where $m = \lceil \sqrt{n}\,\rceil$. (By Lemma 1.)
Since

$$n-(2k+1)m = n-(2k+1)\lceil \sqrt{n}\,\rceil > 0,$$

we have

$$k < \frac{\sqrt{n}-1}{2}.$$

Let $\sqrt{n} = i + d$, where i is a positive integer and $0 \leq d < 1$.

(i) If $d = 0$, then $\sqrt{n} = i$ and $2\sqrt{n} = 2i = \lceil 2\sqrt{n}\,\rceil$. Let $S' = \{v_0, v_{\sqrt{n}}, v_{2\cdot\sqrt{n}}, \ldots, v_{(\sqrt{n}-1)\cdot\sqrt{n}}\}$. Then

$$\omega(G/S') = (\sqrt{n}-1) - 2k = \sqrt{n} - (2k+1)$$

and

$$|S'| + \omega(G/S') = (\sqrt{n}-1+1) + \sqrt{n} - (2k+1)$$
$$= 2\cdot\sqrt{n} - (2k+1) = \lceil 2\sqrt{n}\,\rceil - (2k+1).$$

(ii) If $0 < d \leq (1/2)$, then $2\cdot\lfloor\sqrt{n}\rfloor = 2i$ and $\lceil 2\sqrt{n}\,\rceil = \lceil 2i+2d\,\rceil = 2i+1$ $= 2\cdot\lfloor\sqrt{n}\rfloor + 1$. Let $S' = \{v_0, v_{\lfloor\sqrt{n}\rfloor}, v_{2\cdot\lfloor\sqrt{n}\rfloor}, \ldots, v_{\lfloor\sqrt{n}\rfloor\cdot\lfloor\sqrt{n}\rfloor}\}$. Then

$$\omega(G/S') = (\lfloor\sqrt{n}\rfloor - 1) - 2k = \lfloor\sqrt{n}\rfloor - (2k+1)$$

and

$$\begin{aligned}|S'| + \omega(G/S') &= (\lfloor\sqrt{n}\rfloor + 1) + (\lfloor\sqrt{n}\rfloor - (2k+1)) \\ &= 2 \cdot \lfloor\sqrt{n}\rfloor + 1 - (2k+1) = \lceil 2\sqrt{n}\,\rceil - (2k+1).\end{aligned}$$

(iii) If $(1/2) < d < 1$, then $\lfloor\sqrt{n}\rfloor = i$, $\lceil\sqrt{n}\,\rceil = i+1$, and $\lceil 2\sqrt{n}\,\rceil = \lceil 2i+2d\,\rceil$ $= 2i+2 = \lfloor\sqrt{n}\rfloor + \lceil\sqrt{n}\,\rceil + 1$. Let $S' = \{v_0, v_{\lceil\sqrt{n}\,\rceil}, v_{2\cdot\lceil\sqrt{n}\,\rceil}, \ldots, v_{\lfloor\sqrt{n}\rfloor\cdot\lceil\sqrt{n}\,\rceil}\}$. Then

$$\omega(G/S') = (\lceil\sqrt{n}\,\rceil - 1) - 2k = \lceil\sqrt{n}\,\rceil - (2k+1)$$

and

$$|S'| + \omega(G/S') = (\lfloor\sqrt{n}\rfloor + 1) + (\lceil\sqrt{n}\,\rceil - (2k+1)) = \lceil 2\sqrt{n}\,\rceil - (2k+1).$$

Therefore

$$\begin{aligned}VNI(C_n^k) &= \min_{S \subseteq V(C_n^k)} \{|S| + \omega(C_n^k/S)\} \\ &\leq |S'| + \omega(G/S') \\ &= \lceil 2\sqrt{n}\,\rceil - (2k+1). \qquad (2)\end{aligned}$$

Combining (1) and (2), therefore

$$VNI(C_n^k) = \lceil 2\sqrt{n}\,\rceil - (2k+1), \quad \text{if } 1 \leq k < \frac{\sqrt{n}-1}{2}.$$

If $k \geq (\sqrt{n}-1)/2$, let $|S| = m$ and $\omega(C_n^k/S) = h \geq 0$.

$$n \leq (2k+1)m + mh = m(2k+1+h),$$

hence

$$m \geq \left\lceil \frac{n}{2k+1+h} \right\rceil.$$

Thus

$$\mathrm{VNI}(C_n^k) = \min_{S \subseteq V(C_n^k)} \left(|S| + \omega(C_n^k/S)\right)$$

$$\geq \min_{h \geq 0} \left(\left\lceil \frac{n}{2k+1+h} \right\rceil + h \right)$$

$$= \left\lceil \frac{n}{2k+1} \right\rceil. \qquad \text{(By Lemma 5.)}$$

Let $S^* = \{v_0, v_{2k+1}, v_{2(2k+1)}, \ldots, v_{(\lfloor n/(2k+1) \rfloor)(2k+1)}\}$, then $G/S^* = \emptyset$, and

$$\mathrm{VNI}(C_n^k) \leq |S^*| + \omega(G/S^*) = \left\lceil \frac{n}{2k+1} \right\rceil.$$

Thus

$$\mathrm{VNI}(C_n^k) = \left\lceil \frac{n}{2k+1} \right\rceil, \text{ if } k \geq \frac{\sqrt{n}-1}{2}.$$

Therefore

$$\mathrm{VNI}(C_n^k) = \begin{cases} \lceil 2\sqrt{n} \rceil - (2k+1), & \text{if } 1 \leq k < \frac{\sqrt{n}-1}{2}; \\ \left\lceil \frac{n}{2k+1} \right\rceil. & \text{if } \frac{\sqrt{n}-1}{2} \leq k \leq \lfloor \frac{n}{2} \rfloor. \end{cases}$$

QED.

Let's consider the following examples of C_n^k, where $1 \leq k \leq \lfloor n/2 \rfloor$, $\sqrt{n} = i + d$, i is a positive integer, and $0 \leq d < 1$:

Example 3: The k-th power of the 64-cycle, C_{64}^k, where $1 \leq k \leq 32$. ($\sqrt{64} = 8$, so $d = 0$.)

Let S_k^* be a subset of $V(C_{64}^k)$ for which $\mathrm{VNI}(C_{64}^k) = |S_k^*| + \omega(C_{64}^k/S_k^*)$. Since $(\sqrt{n} - 1)/2 = (\sqrt{64} - 1)/2 = 3.5$, by Theorem 6,

$$\mathrm{VNI}(C_{64}^k) = \begin{cases} \lceil 2\sqrt{64} \rceil - (2k+1) = 15 - 2k, & \text{if } k = 1, 2, 3; \\ \left\lceil \frac{64}{2k+1} \right\rceil, & \text{if } 4 \leq k \leq 32. \end{cases}$$

By the proof of Theorem 6, we can find S_k^* and $\omega(C_{64}^k/S_k^*)$, where $1 \le k \le 32$, as follows:

$$S_1^* = S_2^* = S_3^* = \{v_0, v_8, v_{16}, v_{24}, v_{32}, v_{40}, v_{48}, v_{56}\},$$

and

$$\omega(C_{64}/S_1^*) = 5, \quad \omega(C_{64}^2/S_2^*) = 3, \quad \omega(C_{64}^3/S_3^*) = 1.$$

$$S_k^* = \{v_0, v_{2k+1}, v_{2(2k+1)}, \ldots, v_{\lfloor 64/(2k+1)\rfloor(2k+1)}\}, \quad \text{where } 4 \le k \le 32,$$

and

$$\text{for } 4 \le k \le 32, \quad \omega(C_{64}^k/S_k^*) = 0, \quad \text{since } C_{64}^k/S_k^* = \emptyset.$$

Example 4: The k-th power of the 50-cycle, C_{50}^k, where $1 \le k \le 25$. ($\sqrt{50} \approx 7.07$, so $0 < d \le (1/2)$.)

Let S_k^* be a subset of $V(C_{50}^k)$ for which $VNI(C_{50}^k) = |S_k^*| + \omega(C_{50}^k/S_k^*)$. Since $(\sqrt{n} - 1)/2 = (\sqrt{50} - 1)/2 \approx 3.04$, by Theorem 6,

$$VNI(C_{50}^k) = \begin{cases} \lceil 2\sqrt{50}\,\rceil - (2k+1) = 14 - 2k, & \text{if } k = 1, 2, 3; \\ \left\lceil \frac{50}{2k+1} \right\rceil, & \text{if } 4 \le k \le 25. \end{cases}$$

By the proof of Theorem 6, we can find S_k^* and $\omega(C_{50}^k/S_k^*)$, where $1 \le k \le 25$, as follows:

$$S_1^* = S_2^* = S_3^* = \{v_0, v_7, v_{14}, v_{21}, v_{28}, v_{35}, v_{42}, v_{49}\},$$

and

$$\omega(C_{50}/S_1^*) = 4, \quad \omega(C_{50}^2/S_2^*) = 2, \quad \omega(C_{50}^3/S_3^*) = 0.$$

$$S_k^* = \{v_0, v_{2k+1}, v_{2(2k+1)}, \ldots, v_{\lfloor 50/(2k+1)\rfloor(2k+1)}\}, \quad \text{where } 4 \le k \le 25,$$

and

$$\text{for } 4 \le k \le 25, \quad \omega(C_{50}^k/S_k^*) = 0, \quad \text{since } C_{50}^k/S_k^* = \emptyset.$$

Example 5: The k-th power of the 95-cycle, C_{95}^k, where $1 \le k \le 47$. ($\sqrt{95} \approx 9.75$, so $(1/2) < d < 1$.)

Let S_k^* be a subset of $V(C_{95}^k)$ for which $VNI(C_{95}^k) = |S_k^*| + \omega(C_{95}^k/S_k^*)$. Since $(\sqrt{n}-1)/2 = (\sqrt{95}-1)/2 \approx 4.37$, by Theorem 6,

$$VNI(C_{95}^k) = \begin{cases} \lceil 2\sqrt{95}\,\rceil - (2k+1) = 19 - 2k, & \text{if } k = 1,2,3,4; \\ \left\lceil \frac{95}{2k+1} \right\rceil, & \text{if } 5 \le k \le 47. \end{cases}$$

By the proof of Theorem 6, we can find S_k^* and $\omega(C_{95}^k/S_k^*)$, where $1 \le k \le 47$, as follows:

$$S_1^* = S_2^* = S_3^* = S_4^* = \{v_0, v_{10}, v_{20}, v_{30}, v_{40}, v_{50}, v_{60}, v_{70}, v_{80}, v_{90}\},$$

and

$$\omega(C_{95}/S_1^*) = 7, \quad \omega(C_{95}^2/S_2^*) = 5, \quad \omega(C_{95}^3/S_3^*) = 3, \text{ and } \omega(C_{95}^4/S_4^*) = 1.$$

$$S_k^* = \{v_0, v_{2k+1}, v_{2(2k+1)}, \ldots, v_{\lfloor 95/(2k+1)\rfloor(2k+1)}\}, \quad \text{where } 5 \le k \le 47,$$

and

$$\text{for } 5 \le k \le 47, \quad \omega(C_{95}^k/S_k^*) = 0, \quad \text{since } C_{95}^k/S_k^* = \emptyset.$$

Next, we find the maximum and minimum VNI among the powers of the n-cycle.

Lemma 7: Let n and k be two positive integers. If $k = \lfloor(\sqrt{n}-1)/2\rfloor$, then

$$\lceil 2\sqrt{n}\,\rceil - (2k+1) \ge \left\lceil \frac{n}{2k+3} \right\rceil.$$

Proof: If $(\sqrt{n}-1)/2$ is an integer, then $\sqrt{n} = 2k+1$,

$$\lceil 2\sqrt{n}\,\rceil - (2k+1) = (4k+2) - (2k+1) = 2k+1,$$

and

$$\left\lceil \frac{n}{2k+3} \right\rceil = \left\lceil \frac{4k^2+4k+1}{2k+3} \right\rceil = \left\lceil 2k-1+\frac{4}{2k+3} \right\rceil = 2k.$$

Therefore

$$\lceil 2\sqrt{n}\,\rceil - (2k+1) > \left\lceil \frac{n}{2k+3} \right\rceil.$$

If $(\sqrt{n}-1)/2$ is not an integer, then let $\sqrt{n} = i+d$, where i is a positive integer and $0 \le d < 1$. Thus

$$k = \left\lfloor \frac{\sqrt{n}-1}{2} \right\rfloor = \left\lfloor \frac{(i-1)+d}{2} \right\rfloor = \begin{cases} \frac{i-2}{2}, & \text{if } i \text{ is even;} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd,} \end{cases}$$

and

$$2k+1 = \begin{cases} i-1, & \text{if } i \text{ is even;} \\ i. & \text{if } i \text{ is odd.} \end{cases}$$

$k = \lfloor(\sqrt{n}-1)/2\rfloor$, so $k+1 = \lceil(\sqrt{n}-1)/2\rceil > (\sqrt{n}-1)/2$, and $2k+3 > \sqrt{n}$. This implies that

$$\left\lceil \frac{n}{2k+3} \right\rceil \le \left\lceil \frac{n}{\sqrt{n}} \right\rceil = \lceil \sqrt{n}\,\rceil.$$

$$\lceil 2\sqrt{n}\,\rceil - (2k+1) - \left\lceil \frac{n}{2k+3} \right\rceil$$

$$\ge \begin{cases} \lceil 2\sqrt{n}\,\rceil - (i-1) - \lceil \sqrt{n}\,\rceil, & \text{if } i \text{ is even;} \\ \lceil 2\sqrt{n}\,\rceil - i - \lceil \sqrt{n}\,\rceil, & \text{if } i \text{ is odd} \end{cases}$$

$$\ge \begin{cases} 2i-(i-1)-i, & \text{if } d=0 \text{ and then } \sqrt{n}=i \text{ is even;} \\ (2i+1)-i-(i+1), & \text{if } d \ne 0 \end{cases}$$

$$\ge 0.$$

Therefore

$$\lceil 2\sqrt{n}\,\rceil - (2k+1) \ge \left\lceil \frac{n}{2k+3} \right\rceil.$$

QED.

Corollary 8: Among the powers of the n-cycle (for any fixed positive integer $n \geq 3$), the n-cycle, C_n, has the maximum vertex-neighbor-integrity $\lceil 2\sqrt{n}\,\rceil - 3$, and the $\lfloor n/2 \rfloor$-th power of the n-cycle, $C_n^{\lfloor n/2 \rfloor}$, has the minimum vertex-neighbor-integrity $\lceil n/(2\lfloor n/2 \rfloor + 1) \rceil$.

Proof: By Theorem 6,

$$\mathrm{VNI}(C_n^k) = \begin{cases} \lceil 2\sqrt{n}\,\rceil - (2k+1), & \text{if } 1 \leq k < \frac{\sqrt{n}-1}{2}; \\ \left\lceil \frac{n}{2k+1} \right\rceil, & \text{if } \frac{\sqrt{n}-1}{2} \leq k \leq \lfloor \frac{n}{2} \rfloor. \end{cases}$$

n is a fixed positive integer, so VNI can be considered to be a function of k. When $1 \leq k < (\sqrt{n} - 1)/2$, the function $\mathrm{VNI}(C_n^k) = \lceil 2\sqrt{n}\,\rceil - (2k+1)$ is decreasing with respect to k, and when $(\sqrt{n} - 1)/2 \leq k \leq \lfloor n/2 \rfloor$, the function $\mathrm{VNI}(C_n^k) = \lceil n/(2k+1) \rceil$ is also decreasing with respect to k.

If $k = \lfloor (\sqrt{n} - 1)/2 \rfloor$, then by Lemma 7,

$$\mathrm{VNI}(C_n^k) = \lceil 2\sqrt{n}\,\rceil - (2k+1) \geq \left\lceil \frac{n}{2(k+1)+1} \right\rceil = \mathrm{VNI}(C_n^{k+1}).$$

Therefore the function VNI is decreasing with respect to k, where $1 \leq k \leq \lfloor n/2 \rfloor$, and hence the function VNI has the maximum value $\lceil 2\sqrt{n}\,\rceil - 3$ at $k = 1$ and the minimum value $\lceil n/(2\lfloor n/2 \rfloor + 1) \rceil$ at $k = \lfloor n/2 \rfloor$. That is, among the powers of the n-cycle, the n-cycle, C_n, has the maximum vertex-neighbor-integrity and the $\lfloor n/2 \rfloor$-th power of the n-cycle, $C_n^{\lfloor n/2 \rfloor}$, has the minimum vertex-neighbor-integrity. QED.

III. Discussion and Open Questions

A spy network can be modeled by a graph whose vertices represent the stations and whose edges represent the lines of communication. If a station is destroyed, the adjacent stations will be betrayed so that the betrayed stations become useless to the network as a whole. The vertex-neighbor-integrity is to measure vulnerability of the representing graph of a spy network. It seems reasonable that for a connected representing graph, the more edges it has, the more jeopardy the spy network is in. Hence we present a criterion as follows:

Criterion (*) — Let G be a connected graph. If H is a connected spanning subgraph of G, then $\mathrm{VNI}(H) \geq \mathrm{VNI}(G)$.

The family of powers of an n-cycle, C_n^k, satisfies the criterion (*), since for any fixed integer $n \geq 3$, C_n^k is a connected spanning subgraph of C_n^{k+1}, and $VNI(C_n^k) \geq VNI(C_n^{k+1})$, where $1 \leq k \leq \lfloor n/2 \rfloor - 1$. However, not all of graphs satisfy this criterion, see the following example:

Example 6: The graphs H and G are shown in Figure 1 and Figure 2. $VNI(H) = \{u_1\} + \omega(H/\{u_1\}) = 2$, and $VNI(G) = \{u_1, w_2\} + \omega(G/\{u_1, w_2\})$ $= 2 + 1 = 3$. H is a connected spanning subgraph of G, but $VNI(H) < VNI(G)$.

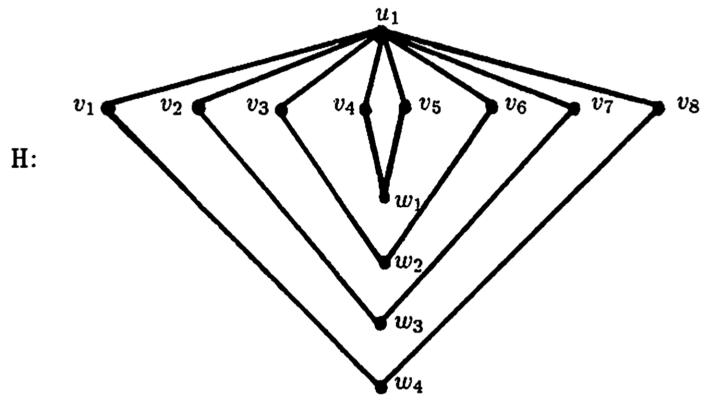

Figure 1

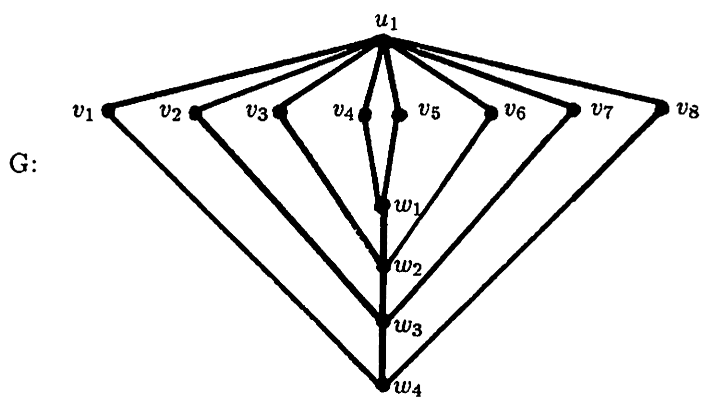

Figure 2

Therefore some interesting questions are raised: For a fixed number of vertices,

1. What graphs satisfy the criterion (*) of the model of a spy network?
2. What are the minimum and maximum numbers of edges of graphs with prescribed order and prescribed vertex-neighbor-integrity?

It is clear that if VNI = 1 and the number of vertices = n, then the complete graph with n vertices, K_n, has the maximum number of edges $= n(n-1)/2$, and the null graph with n vertices, $\bar{K}_n$, has the minimum number of edges = 0.

3. What are the maximum and minimum VNI's among a family of connected graphs with prescribed order?

We have shown that among the trees of order $n \geq 1$, the maximum VNI $= \lceil 2\sqrt{n+3}\, \rceil - 4$ and the minimum VNI = 1 [4], and among the powers of the n-cycle, the maximum VNI $= \lceil 2\sqrt{n}\, \rceil - 3$ and the minimum VNI $= \lceil n/(2\lfloor n/2 \rfloor + 1) \rceil$.

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