# The Balanced Properties of Bipartite Graphs with Applications

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ABSTRACT. In this paper we give some properties of balanced labeling, prove that graph $(m^2+1)C_4$ is balanced, and also solve balanceness of snakes $C_m(n)$.

## 1 Introduction

Let $G = (V, E)$ be a simple graph and $g: V \to \{0, 1, \ldots, |E|\}$ be an injection. Define an induced map $g^*: E \to \{1, 2, \ldots, |E|\}$ by $g^*(uv) = |g(u) - g(v)|$ for $uv \in E$. If $g^*$ is a bijection, then $g$ is said to be a graceful labeling of $G$, and the graph $G$ is said to be graceful.

Graph $G$ is said to be balanced if $G$ is graceful ($g$ is its graceful labeling) and there exists a number $c$ (is called character of $g$), such that $g(u) \leq c$, $g(v) > c$ for all $uv \in E$. $g$ is called balanced labeling ($\alpha$-valuation) of $G$.

Rosa [1] has defined a triangular cactus (let $n$ be a number of triangular blocks). He also conjectured that they are graceful for all $n \equiv 0, 1 \pmod 4$. D. Moulton [2] proved that every $\Delta_n$-snake (triangular snake) for $n$ congruent to 0 or 1 modulo 4 is graceful. Triangular snakes are the particular cases of snakes $C_m(n)$. $C_m(n)$ is the connected graph which $n$ blocks are the cycles $C_m$ and blocks cutpoint graph is a path.

Graph $nC_m$ is the disjoint union of $n$ cycles $C_m$. J. Abrham and A. Kotzig [4] proved $m^2C_4$ and $(m^2+m)C_4$ are balanced. In this paper we give some properties of balanced labeling, and prove that, for all positive integer $m$ and $n$, $C_{4m}(n)$ and $((m+1)^2+1)C_4$ are balanced, and $C_{4m+3}(n)$ and $C_{4m+1}(n)$ are not.

It is convenient to use the following notation

$$[a,b] = \{x \in Z: a \le x \le b\}$$
$$[a,b]_0 = \{x \in Z: x \in [a,b],\ x \equiv a \equiv 0 \pmod 2\}$$
$$[a,b]_1 = \{x \in Z: x \in [a,b],\ x \equiv a \equiv 1 \pmod 2\}$$

Where $a, b \in Z$ and $Z$ is the set of all integers.

## 2 The properties of balanced graphs

It is evident that character $c$ of balanced labeling $g$ is $\min\{g(u), g(v)\}$ where $g^*(uv) = 1$. If $g$ is balanced labeling of graph $G$ and $c$ is its character, let $X = \{u: u \in V(G), g(u) \le c\}$, $Y = \{u: u \in V(G), g(u) > c\}$, then $G$ is bipartite and $(X, Y)$ is a bipartition of vertex set.

**Lemma 1.** *Let $h$ be a balanced labeling of graph $G$, its character be $c$. If $f$ is a graceful labeling of graph $H$ and $|V(G \cap H)| = 1$, then $G \cup H$ is graceful.*

**Proof:** Let $X = \{v: h(v) \le c, v \in V(G)\}$, $Y = \{v: h(v) > c, v \in V(G)\}$, $h(v_0) = \max\{h(v): v \in X\}$, $V(G \cap H) = \{v_0\}$, $f(v_0) = 0$. Let

$$g(v) = \begin{cases} h(v), & \text{if } v \in X \\ h(v) + |E(H)|, & \text{if } v \in Y \\ h(v_0) + f(v), & \text{if } v \in V(H) \setminus \{v_0\} \end{cases}$$

then $g$ is a graceful labeling of graph $G \cup H$.

Indeed, since $h(y) + |E(H)| > f(v) + h(v_0) > h(x)$, for every $x \in X$, $y \in Y$, $v \in V(H) \setminus \{v_0\}$, and both $f$ and $h$ are injections, it follows that $g$ is an injection.

If $x \in X$, $y \in Y$, by definition, we obtain

$$g^*(xy) = |h(y) + |E(H)| - h(x)| = |E(H)| + |h(y) - h(x)|$$

and

$$|E(H)| < |E(H)| + |h(y) - h(x)| \le |E(H)| + |E(G)|.$$

When $xy \in E(H)$, $g^*(xy) = |(f(x) + h(v_0)) - (f(y) + h(v_0))| = |f(x) - f(y)| \le |E(H)|$.

By count, we can obtain $|g^*(E(G \cup H))| = |E(G) + |E(H)|$, hence $g^*$ map $E(G \cup H)$ onto $[1, |E(G)| + |E(H)|]$. □

**Theorem 1.** *Let $G$ and $H$ be two balanced bipartite graphs. If $|V(G \cap H)| = 1$, then $G \cup H$ is balanced.*

**Proof:** Suppose that $f$ is a balanced labeling of $H$, its character is $c'$. Let $X' = \{u: f(u) \le c', u \in V(H)\}$, $Y' = \{u: f(u) > c', u \in V(H)\}$. In the proof of lemma 1, let $X'' = X \cup X'$, $Y'' = Y \cup Y'$, then $(X'', Y'')$ is bipartition of $V(G \cup H)$, $g$ is a balanced labeling of $G \cup H$. The character of $g$ is $c + c'$. □

**Lemma 2.** *Cycle $C_n$ is graceful if and only if $n \equiv 0$ or $3 \pmod 4$. (see [3])*

Let the vertices of $C_n$ be denoted by $x_1, x_2, \ldots, x_n$ successively. The graceful labeling of $C_{4m}$ is as follows:

$$g(x_i) = \begin{cases} (i-1)/2, & i \in [1, 2m-1]_1 \\ (i+1)/2, & i \in [2m+1, 4m-1]_1 \\ 4m+1-i/2, & i \in [2, 4m]_0 \end{cases} \qquad (*)$$

Graceful labeling of $C_{4m+3}$ is as follows:

$$g(x_i) = \begin{cases} (i-1)/2, & i \in [1, 4m+3]_1 \\ 4m+4-i/2, & i \in [2, 2m+2]_0 \\ 4m+3-i/2, & i \in [2m+4, 4m+2]_0 \end{cases}$$

**Theorem 2.** *Cycle $C_{4m}$ is balanced while $C_{4m+3}$ is not.*

**Proof:** $(*)$ is a balanced labeling of $C_{4m}$, with a character of $g$ is $2m$. By contradiction. Suppose that $g$ is balanced labeling of $C_{4m+3}$, $c$ is character of $g$. There exists $u \in V(C_{4m+3})$, such that $g(u) = 0$. Without loss of generality, let $g(x_1) = 0$, then $g(x_2) > c, g(x_3) \le c, \ldots, g(x_{4m+2}) > c, g(x_{4m+3}) \le c, g(x_1) > c$. This contradicts $g(x_1) = 0 < c$. □

**Theorem 3.** *If $g$ is a balanced labeling of $C_{4m}$ and $[0, |E(C_{4m})|] - \{g(u)\colon u \in V(C_{4m})\} = a$, then $a = m$ or $3m$.*

**Proof:** Let the character of $g$ be $c$, $X = \{u\colon g(u) \le c, u \in V(C_{4m})\}$, $Y = \{u\colon g(u) > c, u \in V(C_{4m})\}$. If $g(x_1) = 0$, then $X = \{x_i : i \in [1, 4m-1]_1\}$, $Y = \{x_i : i \in [2, 4m]_0\}$. We may easily see $c = 2m-1$ or $2m$.

When $c = 2m-1$, there is $a > c$. The sum of all edge-labels is $1 + 2 + 2 \cdots + 4m = 2m(4m+1)$. On the other hand,

$$\begin{aligned} 2m(4m+1) &= |g(x_1) - g(x_n)| + |g(x_2) - g(x_1)| + \cdots + |g(x_n) - g(x_{n-1})| \\ &= 2\{[2m + (2m+1) + \cdots + 4m - a] - [1 + 2 \cdots + (2m-1)]\} \\ &= 2\{4m^2 + 4m - a\}, \end{aligned}$$

hence $a = 3m$.

When $c = 2m$, there is $a \le c$,

$$\begin{aligned} &|g(x_1) - g(x_n)| + |g(x_2) - g(x_1)| + \cdots + |g(x_n) - g(x_{n-1})| \\ &= 2\{[(2m+1) + \cdots + 4m] - (1 + 2 + \cdots + 2m - a)\} = 2\{4m^2 + a\} \end{aligned}$$

hence $a = m$. □

**Corollary.** *If $g$ is a balanced labeling of $nC_{4m}$, then* $[0, |E(nC_{4m})|] - \{g(v): v \in V(nC_{4m})\} = mn$ *or* $3mn$.

The same method of theorem 3 can be used to prove the corollary.

## 3 Applications

In what follows, we will denote the $2n-1$ vertices of snake $C_n(2)$ by $x_1, x_2, \ldots, x_n, \ldots, x_{2n-1}$ successively. The vertices on the path are $x_1$, $x_n, x_{2n-1}$.

**Lemma 3.** *Graph $C_{4m}(2)$ is balanced.*

**Proof:** The balanced labeling of $C_{4m}(2)$ is as follows:

$$g(x_{2i-1}) = \begin{cases} i-1, & i \in [1, 2m] \\ 6m-i, & i \in [2m+1, 3m-1] \text{ and } n \geq 2 \\ 6m-1-i, & i \in [3m, 4m-1] \\ 4m, & i = 4m \end{cases}$$

$$g(x_{2i}) = \begin{cases} 8m+1-i, & i \in [1, 2m-1] \\ 2m+2+i, & i \in [2m, 3m-1] \text{ and } n \geq 2 \\ 2m+3+i, & i \in [3m, 4m-2] \\ 4m+1, & i = 4m-1 \end{cases}$$

$$g^*(x_{2i}x_{2i-1}) = \begin{cases} 8m+2-2i, & i \in [1, 2m-1] \\ 2m+3, & i = 2m \\ 2-4m+2i, & i \in [2m+1, 3m-1] \\ 4-4m+2i, & i \in [3m, 4m-2] \\ 2m+1, & i = 4m-1 \end{cases}$$

$$g^*(x_{2i}x_{2i+1}) = \begin{cases} 8m+1-2i, & i \in [1, 2m-1] \\ 3-4m+2i, & i \in [2m, 3m-2] \\ 2m+2, & i = 3m-1 \\ 5-4m+2i, & i \in [3m, 4m-2] \\ 1, & i = 4m-1 \end{cases}$$

$g^*(x_1x_{4m}) = 4m+2$, $g^*(x_{4m}x_{8m-1}) = 2$. Let $X = \{x_{2i-1}: i \in [1, 4m]\}$, $Y = \{x_{2i}: i \in [1, 4m-1]\}$, then the character $c = \max g(X) = 4m$.

The fundamental idea to label $C_{4m}(2)$ is as follows: Vertex-label is greater than $4m$ and vertex-label is not greater than $4m$ are alternate. The set of edge-labels consists of a few continual integer sections. □

We can get that $C_{4m}(4)$ is balanced by theorem 1. If $g$ is a balanced labeling of $C_{4m}(2)$, then a balanced labeling of $C_{4m}(4)$ is as follows:

$$f(x) = \begin{cases} g(x), & x \in X \\ 8m + g(x), & x \in Y \\ 4m + g(x), & x \in V(C_{4m}(2)) \end{cases}$$

Graph $C_{4m}(2n)$ is partitioned $n$ graphs $C_{4m}(2)$. The $j$th $C_{4m}(2)$ is noted by $C_{4m,j}(2)$. Let $V(C_{4m,j}(2)) = \{x_{ij} : i \in [1, 8m-1]\}$, where $x_{ij}$ correspond to $x_i \in V(C_{4m}(2))$. By induction, we can obtain balanced labeling of $C_{4m}(2n)$ is as follows:

$$f(x_{ij}) = \begin{cases} 4m(j-1) + g(x_i), & x_i \in X \\ 4m(2n-j-1) + g(x_i), & x_i \in Y \quad j = 1, 2, \ldots, n. \end{cases}$$

Let

$$X' = \{x_{ij} : i \in [1, 8m-1], j \in [1, n], x_i \in X\}$$
$$Y' = \{x_{ij} : i \in [1, 8m-1], j \in [1, n], x_i \in Y\}$$

then $(X', Y')$ is a bipartition of $V(C_{4m}(2n))$.

By the above conclusion and theorem 2, we obtain a balanced labeling $h$ of $C_{4m}(2n+1)$ is as follows:

$$h(v) = \begin{cases} f(v), & v \in X' \\ f(v) + 4m, & v \in Y' \\ 4mn + s(v), & v \in V(C_{4m}) \end{cases}$$

where $s(v)$ is a balanced labeling of $C_{4m}$.

To sum up,we obtain this result:

**Theorem 4.** *Graphs $C_{4m}(n)$ are balanced for all positive integer $m$ and $n$.*

**Theorem 5.** *For all positive integer $n$ and $m$, $C_{4m+3}(n)$ and $C_{4m+1}(n)$ are not balanced.*

By contradiction and theorem 2, we can obtain the conclusion. □

Now, the balanceness of snake $C_m(n)$ has been solved completely.

**Lemma 4.** *$kC_4$ has a $\alpha$-valuation for $1 \leq k \leq 10$, $k \neq 3$. Let $k$ be a positive integer, then $k^2C_4$, $(k^2+k)C_4$ are balanced, and if $kC_4$ has an $\alpha$-valuation, graphs $(4k+1)C_4$, $(5k+1)C_4$, $(9k+2)C_4$ also have an $\alpha$-valuation.* (see [4])

**Theorem 6.** *When $n = (m+1)^2 + 1$ ($m$ is arbitrary positive integer), every $nC_4$ is balanced.*

**Proof:** Vertices of $i$th $C_4$ are denoted by $x_{ij}$ ($j = 1, 2, 3, 4$). Let $t = m(m+1)/2$. We express $(m+1)^2+1$ as $2t+m+2$. The balanced labeling of $nC_4$ is as follows: $g(x_{11}) = 4n$, $g(x_{12}) = 0$, $g(x_{13}) = 4n-1$, $g(x_{14}) = 2$, $g(x_{t+1,1}) = 2n+2$, $g(x_{t+1,2}) = 2n-2$, $g(x_{t+1,3}) = 2n+1$, $g(x_{t+1,4}) = 2n$,

$$g(x_{ij}) = \begin{cases} 4n - k(k+1)/2 + 1 - i, & j = 1 \\ 4n - k(k+3)/2 - i, & j = 3 \\ k(2k+3) + 2 - 2i, & j = 2 \\ k(2k+5) + 4 - 2i, & j = 4 \end{cases}$$

$$g(x_{t+i,j}) = \begin{cases} 2n + k(k+3)/2 + 1 + i, & j = 1 \\ 2n + k(k+1)/2 + i, & j = 3 \\ 2n - k(2k+5) - 4 + 2i, & j = 2 \\ 2n - k(2k+3) - 2 + 2i, & j = 4 \end{cases}$$

where $k(k+1)/2 + 1 \leq i \leq (k+1)(k+2)/2$, $1 \leq k \leq m-1$.

$$g(x_{n-i,j}) = \begin{cases} 3n + m + 2 - i, & \text{if } j = 1 \text{ and } 0 \leq i \leq m+1 \\ 3n - i, & \text{if } j = 3 \text{ and } 0 \leq i \leq m+1 \\ n + 2m + 4 - 2i, & \text{if } j = 4 \text{ and } 1 \leq i \leq m+1 \\ n - 1, & \text{if } j = 4 \text{ and } i = 0 \\ n + 1, & \text{if } j = 2 \text{ and } i = m+1 \\ n - 2 - 2i, & \text{if } j = 2 \text{ and } 0 \leq i \leq m \end{cases}$$

$g(V(nC_4)) = [0, 4n] - \{n\}$, character of $g$ is $2n$.

## References


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