Sign-nonsingular matrices and orthogonal sign-patterns

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Abstract

We establish some basic facts about sign-patterns of orthogonal matrices, and use these facts to characterize the sign-nonsingular matrices which are sign-patterns of orthogonal matrices.

1 Introduction

Let A be an m by n matrix (with real entries). The *sign-pattern* of A is the m by n $(0, 1, -1)$-matrix obtained from A by replacing each positive entry by $+1$, and each negative entry by -1. The set of all matrices which have the same sign-pattern as A is denoted by $\mathcal{Q}(A)$.

There has been some recent interest in studying the sign-patterns of orthogonal matrices [BBS]. Determining combinatorial properties which characterize the sign-patterns of orthogonal matrices remains a fundamental open problem. Since this general problem appears to be quite difficult, it is perhaps useful, as suggested by C. Johnson [J], to place additional restrictions on the sign-patterns. The additional constraint we consider is that of sign-nonsingularity. A square sign-pattern B is *sign-nonsingular* provided every matrix with sign-pattern B is invertible. Sign-nonsingular matrices have been studied extensively (see [BS] for references).

In this note we characterize the sign-nonsingular matrices which are also sign-patterns of orthogonal matrices. Using the characterization, we establish some connections between sign-nonsingular matrices and sign-patterns of orthogonal matrices.

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2 Results

Let B be a square $(0, 1, -1)$-matrix. We say that B *allows orthogonality* provided $\mathcal{Q}(B)$ contains an orthogonal matrix, that is, provided B is the sign-pattern of an orthogonal matrix.

We now define a class of matrices slightly more general than the orthogonal matrices. An n by n matrix A is *symplectic* provided A is nonsingular and the sign-patterns of A^{-1} and A^T are the same. Symplectic matrices were introduced in [L] where their relationship with n-body problems in celestial mechanics is discussed (see also [LS]). Note that if A is symplectic then so is each matrix obtained from A by permuting rows and columns and by negating arbitrary rows and columns. We say that B *allows symplecticity* provided $\mathcal{Q}(B)$ contains a symplectic matrix. Clearly, an orthogonal matrix is symplectic, and thus if B allows orthogonality then B allows symplecticity.

Our first result shows that in studying symplecticity and orthogonality, one can restrict to fully indecomposable matrices. We recall the following standard facts from combinatorial matrix theory (see [BR]). Since symplectic matrices are invertible, if B is an n by n matrix which allows symplecticity then B does not contain a zero submatrix whose dimensions sum to $n+1$. Let A be an n by n matrix which does not contain a zero submatrix whose dimensions sum to $n+1$. We say that A is *partly decomposable*, if there exist permutation matrices P and Q such that

$$PAQ = \begin{bmatrix} B & O \\ C & D \end{bmatrix},$$

where B and D are square (nonvacuous) matrices. Equivalently, A is partly decomposable if and only if there exists a p by q zero submatrix of A for some positive integers p and q with $p+q \geq n$. We say that A is *fully indecomposable*, if A is not partly decomposable. The rows and columns of A can be permuted to obtain a matrix of the form

$$\begin{bmatrix} A_1 & O & \cdots & O \\ A_{21} & A_2 & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ A_{k1} & A_{k2} & \cdots & A_k \end{bmatrix} \tag{1}$$

for some integer $k \geq 1$, where each A_i is a fully indecomposable (square) matrix.

Proposition 2.1 *Let A be a partly decomposable symplectic matrix. Then there exist permutation matrices P and Q such that PAQ is the direct sum of fully indecomposable, symplectic matrices.*

Proof. Without loss of generality we may assume that A has the form given in (1). Thus in particular A and A^{-1} are block lower triangular. Since A is symplectic, it follows that each $A_{ij} = O$, and hence that A is the direct sum of matrices. It is now easy to verify that each of the A_i is symplectic. □

Note that if in Proposition 2.1, A is orthogonal then PAQ is a direct sum of orthogonal matrices. Clearly, a direct sum of matrices is symplectic (respectively orthogonal) if and only if each of the direct summands is symplectic (respectively, orthogonal). Therefore, in light of Proposition 2.1 we henceforth restrict our study to fully indecomposable matrices.

Our next result gives a necessary condition for a sign-pattern to allow symplecticity. The vectors $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$ *allow orthogonality* provided either (i) $u_i v_i = 0$ for $i = 1, 2, \ldots, n$ or (ii) there exist i and j such that $u_i v_i > 0$ and $u_j v_j < 0$. It is easy to verify that the vectors u and v allow orthogonality if and only if there exist vectors $\widetilde{u} \in \mathcal{Q}(u)$ and $\widetilde{v} \in \mathcal{Q}(v)$ which are orthogonal.

Lemma 2.2 *Let B be a square $(0,1,-1)$-matrix which allows symplecticity. Then any two row vectors and any two column vectors of B allow orthogonality.*

Proof. Let A be a symplectic matrix in $\mathcal{Q}(B)$. Since $A^{-1} \in \mathcal{Q}(B^T)$, the sign-pattern of the ith row u_i^T of A and the sign-pattern of the transpose of the ith column v_i of A^{-1} equal the ith row w_i of B. Since $AA^{-1} = I$, it follows that for $i \neq j$, u_i and v_j are orthogonal vectors which belong to $\mathcal{Q}(w_i)$ and $\mathcal{Q}(w_j)$, respectively. Therefore any two row vectors of B allow orthogonality. A similar argument works for columns. □

Let A be an n by n matrix with $n \geq 2$. The matrix A is *doubly indecomposable* [BS] provided there do not exist integers p and q and a p by q zero submatrix of A with $p+q \geq n-2$. Clearly, if A is doubly indecomposable, then A is fully indecomposable. In the next theorem we show that there are essentially only two doubly indecomposable, sign-nonsingular matrices which allow symplecticity. We use the following results concerning sign-nonsingular matrices. A fully indecomposable, sign-nonsingular matrix has a row or a column with 3 or fewer nonzero entries [T]. Every 2 by 3 and every 3 by 2 submatrix of a doubly indecomposable, sign-nonsingular matrix has at least one entry equal to 0 [BS].

A *signature matrix* is a diagonal matrix each of whose diagonal entries is either $+1$ or -1.

Theorem 2.3 *Let $B = [b_{ij}]$ be an n by n doubly indecomposable, sign-nonsingular $(0,1,-1)$-matrix. Then B allows symplecticity if and only if there exist permutation matrices P and Q and signature matrices D and E such that $DPBQE$ is either*

$$\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & 0 & -1 & -1 \\ 0 & 1 & -1 & 1 \end{bmatrix} \tag{2}$$

Proof. It is easy to verify that the matrices in (2) are doubly indecomposable, sign-nonsingular matrices. Since the rows of the matrices in (2) are orthogonal, it follows that each of these matrices allows orthogonality, and hence symplecticity.

Suppose that B allows symplecticity and $n > 2$. As previously mentioned, B has a row or column with 3 or fewer nonzero entries. Since B is doubly-indecomposable, and $n > 2$, each row and column of B has at least 3 nonzero entries. Thus, some row or column of B has exactly 3 nonzero entries. We may assume without loss of generality that row 1 of B has exactly 3 nonzero entries and that these are in columns 1, 2 and 3. Consider another row of B, say row i where $i \geq 2$. Since every 2 by 3 submatrix of B has a 0, Lemma 2.2 implies that either $a_{i1} = a_{i2} = a_{i3} = 0$ or exactly two of a_{i1}, a_{i2}, a_{i3} are nonzero. Since every 3 by 2 submatrix of B has a 0, it follows that there are at most four rows of B which have at least one nonzero entry in their first three columns. Hence the first 3 columns of B contain an $n-4$ by 3 zero submatrix. Since B is doubly indecomposable, this zero submatrix must be vacuous, and hence $n \leq 4$. Up to row and column permutation and multiplication of rows and columns by -1, the only doubly indecomposable, sign-nonsingular matrices with 4 or fewer rows are those in (2). The theorem now follows. □

We now study the structure of fully indecomposable matrices which allow symplecticity and which are not doubly indecomposable. Let A be a matrix, and let α be a subset of its rows and β a subset of its columns. Then $A(\alpha, \beta)$ denotes the submatrix whose row indices are not in α and whose column indices are not in β, and $A[\alpha, \beta]$ denotes the submatrix whose row indices are in α and whose column indices are in β.

Lemma 2.4 *Let*

$$A = \begin{bmatrix} B & O \\ D & C \end{bmatrix}$$

be an n by n symplectic matrix where B is k by $k+1$ and C is $\ell+1$ by ℓ, $k+\ell = n-1$, and $k, \ell \geq 1$. Then there exist nonzero vectors x and y such that $D = xy^T$. Moreover, the matrices

$$B' = \begin{bmatrix} B \\ y^T \end{bmatrix} \quad \text{and} \quad C' = \begin{bmatrix} x & C \end{bmatrix} \tag{3}$$

are symplectic. If A is orthogonal, then x and y may be chosen so that both B' and C' are orthogonal matrices.

Proof. Since A is symplectic, A^{-1} has the form

$$A^{-1} = \begin{bmatrix} E & G \\ O & F \end{bmatrix}$$

where E is $k+1$ by k, and F is ℓ by $\ell+1$. For all j_1 and j_2 with $1 \leq j_1 < j_2 \leq k+1$ and all i_1 and i_2 with $k+1 \leq i_1 < i_2 \leq n$, the $n-2$ by $n-2$ matrix $A^{-1}(\{j_1, j_2\}, \{i_1, i_2\})$ contains an ℓ by k zero submatrix. Since $k+\ell = n-1$, it follows that $\det A^{-1}(\{j_1, j_2\}, \{i_1, i_2\}) = 0$. By Jacobi's identity (see [HJ]p. 21), $\det A[\{i_1, i_2\}, \{j_1, j_2\}] = 0$. This implies that D has rank at most one. Since A is nonsingular, $D \neq O$, and thus D has rank equal to one. Hence there exist nonzero vectors x and y such that $D = xy^T$. The matrix A^{-1} is symplectic, since A is symplectic. Thus, by a similar argument, there exist nonzero vectors u and v such that $G = uv^T$. Since A is symplectic, we may assume that u and v are chosen so that u and y have the same sign-pattern and v and x have the same sign-pattern. Since $AA^{-1} = I$, we see that

$$BE = I, Buv^T = O, \text{ and } xy^TE = O. \tag{4}$$

Since v is nonzero and x is nonzero, we conclude that

$$Bu = O \text{ and } y^TE = O. \tag{5}$$

It is easy to verify that since A is invertible, the matrix B' in (3) is invertible, and since A^{-1} is invertible the matrix $E' = [E, u]$ is invertible. By (4) and (5) we see that

$$B'E' = \begin{bmatrix} I & O \\ O & y^Tu \end{bmatrix}.$$

Since both B' and E' are invertible, this implies that $y^Tu \neq 0$ and thus

$$(B')^{-1} = \begin{bmatrix} E & \frac{u}{y^Tu} \end{bmatrix}.$$

Since A is symplectic the sign-patterns of E and B^T are the same, and since y and u have the same sign-pattern so do y and $\frac{u}{y^Tu}$. Hence it follows that B' is symplectic. A similar argument shows that C' is symplectic.

Now suppose that A is orthogonal. Then we may take $u = y$ and $v = x$. The sum of the squares of the entries in the first $k+1$ columns of A equals $k+1$, and the sum of the squares of the entries in the first k rows of A equals k. Thus, the sum of the squares of the entries in $D = xy^T$ equals 1. It follows that $(x^Tx)(y^Ty) = 1$. By replacing x by $\frac{x}{\sqrt{x^Tx}}$ and y by $\sqrt{x^Tx}y$, we see that both B' and C' are orthogonal matrices. □

Let M be a k by k matrix, and let N be an ℓ by ℓ matrix. We define $M \star N$ to be the $k+\ell-1$ by $k+\ell-1$ matrix given by

$$M \star N = \begin{bmatrix} M' & O \\ xy^T & N' \end{bmatrix} \tag{6}$$

where M' is the $k-1$ by k submatrix obtained from M by deleting its last row, y^T is the last row of M, N' is the ℓ by $\ell-1$ submatrix obtained from N by deleting its first column, and x is the first column of N. Thus by Lemma 2.4, if A is a fully indecomposable symplectic matrix which is not doubly indecomposable, then (after row and column permutations) $A = M \star N$ for some symplectic matrices M and N.

columns of A we may further assume that O is in the upper righthand corner of A. Thus A has the form

$$\begin{bmatrix} B & O \\ D & C \end{bmatrix},$$

where B is p by $p+1$, and C is $q+1$ by q. By Lemma 2.4, there exist nonzero vectors x and y such that $D = xy^T$ and the matrices

$$M = \begin{bmatrix} B \\ y^T \end{bmatrix} \text{ and } N = \begin{bmatrix} x & C \end{bmatrix}$$

allow symplecticity. Thus $A = M \star N$. It is easy to verify that since A is fully indecomposable and sign-nonsingular, so are M and N. Furthermore, the choice of p implies that M is doubly indecomposable. Thus, by Theorem 2.3, up to row and column permutations and multiplication of rows and columns by -1, M is one of the matrices in (2). Arguing by induction on the number of rows, it follows that a sequence of matrices satisfying the desired properties exists.

Since the rows of each of the matrices in (2) are orthogonal, these matrices allow symplecticity. (Indeed, they allow orthogonality.) Thus, by Lemma 2.5 and induction on n, any matrix obtained by starting with one of the matrices in (2) and sequentially $\star$-ing with one of the matrices in (2) allows symplecticity. That such matrices are fully indecomposable, and sign-nonsingular is a result in [LMV]. □

As we noted each of the matrices in (2) allow orthogonality. Hence we have the following.

Corollary 2.7 *A sign-nonsingular matrix allows symplecticity if and only if it allows orthogonality.*

A sign-nonsingular matrix M is *maximal* [LM] provided every matrix obtained from M by replacing one of its nonzero entries by 1 or -1 is not sign-nonsingular. The matrices in (2) are maximal sign-nonsingular matrices. If M and N are maximal sign-nonsingular matrices, then so is $M \star N$ [LMV]. Thus, by Theorem 2.6 and Corollary 2.7, a fully indecomposable, sign-nonsingular matrix which allows orthogonality is maximal.

Let S be the sign-pattern of a fully indecomposable orthogonal matrix. We say that S is a *minimal orthogonal sign pattern* provided no fully indecomposable matrix obtained from S by replacing some (at least one) of its nonzero entries by 0's is the sign-pattern of an orthogonal matrix. We conclude this note with a result that relates maximal sign-nonsingular matrices and minimal orthogonal sign-patterns.

Corollary 2.8 *Let A be a fully indecomposable sign-nonsingular matrix which allows orthogonality. Then A is a minimal orthogonal sign pattern.*

It is easy to verify that if M and N are orthogonal matrices, then $M \star N$ is an orthogonal matrix. The next lemma shows that the analagous result holds for symplectic matrices.

Lemma 2.5 *If M and N are symplectic matrices, then $M \star N$ is symplectic.*

Proof. Let M', N', x and y be defined as in (6), and let

$$M^{-1} = \left[\begin{array}{cc} E & v \end{array} \right] \text{ and } N^{-1} = \left[\begin{array}{c} u^T \\ F \end{array} \right].$$

Since $MM^{-1} = I$,

$$M'E = I, M'v = 0, y^T E = O \text{ and } y^T v = 1.$$

Since $NN^{-1} = I$, $xu^T + N'F = I$. Using these facts it is easy to verify that

$$(M \star N) \left[\begin{array}{cc} E & vu^T \\ O & F \end{array} \right] = I.$$

It follows that M is symplectic. □

We now characterize the fully indecomposable, sign-nonsingular matrices which allow symplecticity.

Theorem 2.6 *Let A be an n by n fully indecomposable, $(0, 1, -1)$-matrix with $n \geq 2$. Then A is sign-nonsingular and allows symplecticity if and only if there exist permutation matrices P and Q, signature matrices D and E, and a sequence M_1, M_2, ..., M_k $(k \geq 1)$ of matrices such that*

(i) *M_1 is one of the matrices in (2);*

(ii) *For $i = 1, \ldots, k-1$, M_i either can be obtained from M_{i-1} be transposing and permuting rows and columns, or after possibly permuting rows and columns of M_{i-1}, $M_i = N_i \star M_{i-1}$ or $M_i = M_{i-1} \star N_i$ where N_i is one of the matrices in (2);*

(iii) *$DPAQE = M_n$.*

Proof. First suppose that A is sign-nonsingular and allows symplecticity. If A is doubly indecomposable, then by Lemma 2.3, there exist permutation matrices P and Q and signature matrices D and E such that $DPAQE$ is one of the two matrices in (2). Suppose A is not doubly indecomposable. Then there exist integers p and q such that $p+q = n-1$ and a p by q zero submatrix O of A. We may assume that p and q are chosen so that $\min\{p, q\}$ is minimal. By replacing A by A^T if necessary, we may assume that $p \leq q$. By permuting the rows and

Proof. By the previous discussion A is necessarily a maximal sign-nonsingular matrix. Any fully indecomposable, sign-pattern S obtained from A by replacing some of its nonzero entries by 0 is a non-maximal sign-nonsingular matrix, and hence S cannot support orthogonality. Therefore, A is a minimal orthogonal sign pattern. □

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