On (a, d)-Antimagic Prisms

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ABSTRACT. We deal with (a, d)-antimagic labelings of the prisms. A connected graph $G = (V, E)$ is said to be (a, d)-antimagic if there exist positive integers a, d and a bijection $f\colon E \rightarrow \{1, 2, \ldots, |E|\}$ such that the induced mapping $g_f\colon V \rightarrow \mathbf{N}$, defined by $g_f(v) = \sum\{f(u, v)\colon (u, v) \in E(G)\}$, is injective and $g_f(V) = \{a, a + d, \ldots, a + (|V| - 1)d\}$.

We characterize (a, d)-antimagic prisms with even cycles and we conjecture that prisms with odd cycles of length n, $n \geq 7$, are $\left(\frac{n+7}{2}, 4\right)$-antimagic.

1 Introduction and Definitions

The graphs considered here will be finite, undirected and simple. The symbols $V(G)$ and $E(G)$ will denote the vertex set and the edge set of G. The weight $w(v)$ of a vertex $v \in V(G)$ under a edge labeling f is the sum of values $f(e)$ assigned to all edges incident to a given vertex v.

A connected graph $G = (V, E)$ is said to be (a, d)-antimagic if there exist positive integers a, $d \in N$ and bijection $f\colon E(G) \rightarrow \{1, 2, \ldots, |E(G)|\}$ such that the induced mapping $g_f\colon V(G) \rightarrow W$ is also a bijection, where $W = \{w(v) \mid v \in V(G)\} = \{a, a + d, \ldots, a + (|V| - 1)d\}$ is the set of weights of vertices.

If $G = (V, E)$ is (a, d)-antimagic and $f\colon E(G) \rightarrow \{1, 2, \ldots, |E(G)|\}$ is a corresponding bijective mapping of G then f is said to be an (a, d)-antimagic labeling of G.

In [8], Hartsfield and Ringel introduce the concept of an antimagic graph. The concept of an (a, d)-antimagic labeling is defined in [3] and (a, d)-antimagic labelings of the special graphs called parachutes are described in [4].

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The prism D_n, $n \geq 3$, is a trivalent graph which can be defined as the Cartesian product $P_2 \times C_n$ of a path on two vertices with a cycle on n vertices. The prism can also be defined as the Cayley graph of the dihedral group of order $2n$ (using a rotation and a reflection as generators, see [5]).

Let $I = \{1, 2, \ldots, n\}$ and $J = \{1, 2\}$ be index sets. We will denote the vertex set of D_n by $V = \{x_{1,1}, x_{1,2}, \ldots, x_{1,n}, x_{2,1}, x_{2,2}, \ldots, x_{2,n}\}$ and edge set by $E = \{(x_{j,i}x_{j,i+1}) \mid j \in J \text{ and } i \in I\} \cup \{(x_{1,i}x_{2,i}) \mid i \in I\}$.

We make the convention that $x_{j,n+1} = x_{j,1}$ and $x_{j,0} = x_{j,n}$ (for $j \in J$) to simplify later notations.

Various types of labelings of prisms have been intensively studied by several authors. In [2] it is proved that $P_2 \times C_n$ is graceful when $n \equiv 0 \pmod 4$. A complete proof of all cases was given by Frucht and Gallian [5] in 1988. Graceful labeling of prisms and prisms related graphs were given by Gallian [6] in 1988.

Gallian, Prout and Winters [7] proved that $P_2 \times C_n$ is harmonious when $n \neq 4$. Magic and consecutive labelings of prisms are described in [1,9].

In this paper we characterize all (a, d)-antimagic graphs of prisms D_n when n is even and show that if n is odd the prisms D_n are $(\frac{5n+5}{2}, 2)$-antimagic.

2 Necessary conditions

The following theorem gives the necessary conditions for an (a, d)-antimagic labeling of D_n.

Theorem 1. *Let D_n be (a, d)-antimagic graph.*
If n is even, then either $d = 1$ and $a = \frac{7n+4}{2}$ or $d = 3$ and $a = \frac{3n+6}{2}$.
If n is odd, then either $d = 2$ and $a = \frac{5n+5}{2}$ or $d = 4$ and $a = \frac{n+7}{2}$.

Proof: Assume that D_n is (a, d)-antimagic and $W = \{w(v) \mid v \in V(D_n)\} = \{a, a + d, a + 2d, \ldots, a + (2n - 1)\, d\}$ is the set of weights of vertices. Clearly, the sum of weights in the set W is

$$\sum_{v \in V(D_n)} w(v) = n\,[2a + d(2n - 1)]. \qquad (1)$$

Since the edges of G are labeled by the set of integers $\{1, 2, \ldots, 3n\}$ and since each of these labels is used twice in the computation of the weights of vertices, the sum of all the edge labels used to calculate the weights of vertices is equal to

$$2\,(1 + 2 + \cdots + |V(D_n)|) = (3n + 1)3n. \qquad (2)$$

Thus the following the Diophantine equation holds

$$3(3n + 1) = 2a + d(2n - 1). \qquad (3)$$

From (3) we have

$$d = \frac{3(3n+1) - 2a}{2n-1}. \tag{4}$$

By putting $a \geq 6$ ($a = 6$ is the minimal value of weight which can be assigned to a vertex of degree three) we get the upper bound on the value d, i.e. $0 < d < \frac{9}{2}$.

From (4) it follows that

(i) if n is even, then d is odd, and we obtain exactly two solutions (a, d) of the Diophantine equation (3):

$$(a,d) = \left(\frac{7n+4}{2}, 1\right) \text{ and } (a,d) = \left(\frac{3n+6}{2}, 3\right).$$

(ii) if n is odd, then d is even, and this means that equation (3) has exactly two solutions:

$$(a,d) = \left(\frac{5n+5}{2}, 2\right) \text{ and } (a,d) = \left(\frac{n+7}{2}, 4\right).$$

This completes the proof.

3 Prisms with odd cycles

Theorem 2. *If n is odd, $n \geq 3$, then the prism D_n has a $(\frac{5n+5}{2}, 2)$-antimagic labeling.*

Proof: We assume that $n \geq 3$ is odd. The desired $(\frac{5n+5}{2}, 2)$-antimagic labeling of D_n can be described by the following formulae: $f_1(x_{j,i}x_{j,i+1}) = \left[\frac{i+1}{2} + n(j-1)\right]\delta(i) + \left[\frac{n+i+1}{2} + n(j-1)\right]\delta(i+1)$ for $i \in I$ and $j \in J$, $f_1(x_{1,i}x_{2,i}) = 2n + i$ for $i \in I$, where

$$\delta(x) = \begin{cases} 0 & \text{if } x \equiv 0 \pmod 2 \\ 1 & \text{if } x \equiv 1 \pmod 2 \end{cases}. \tag{5}$$

First, we shall show that the labeling f_1 uses each integer $1, 2, \ldots, |E(D_n)|$.

If i is odd, $i \in I$ and $j \in J$, then $f_1(x_{j,i}x_{j,i+1})$ is equal successively to $\left[1, 2, 3, \ldots, \frac{n+1}{2}\right]$ and $\left[n+1, n+2, \ldots, \frac{3n+1}{2}\right]$.

if i is even, $i \in I$ and $j \in J$, then $f_1(x_{j,i}x_{j,i+1})$ successively assume values of $\left[\frac{n+3}{2}, \frac{n+5}{2}, \ldots, n\right]$ and $\left[\frac{3n+3}{2}, \frac{3n+5}{2}, \ldots, 2n\right]$.

If $i \in I$, then $f_1(x_{1,i}x_{2,i})$ successively attain values $[2n+1, 2n+2, \ldots, 3n]$.

Let us denote the weights (under a edge labeling f) of vertices $x_{1,i}$ and $x_{2,i}$ of D_n by

$$w_f(x_{1,i}) = f(x_{1,i}x_{1,i+1}) + f(x_{1,i-1}x_{1,i}) + f(x_{1,i}x_{2,i}) \tag{6}$$

$$w_f(x_{2,i}) = f(x_{2,i}x_{2,i+1}) + f(x_{2,i-1}x_{2,i}) + f(x_{1,i}x_{2,i}), \text{ for } i \in I. \tag{7}$$

The weights of vertices under the labeling f_1 constitute the sets

$$W_1 = \{w_{f_1}(x_{1,i}) \mid i \in I\} = \left\{\frac{5n+5}{2}, \frac{5n+5}{2}+2, \dots, \frac{5n+5}{2}+2n-2\right\}$$

and

$$W_2 = \{w_{f_1}(x_{2,i}) \mid i \in I\} = \left\{\frac{5n+5}{2}+2n, \frac{5n+5}{2}+2n+2, \dots, \frac{5n+5}{2}+4n-2\right\}.$$

We can see that each vertex of D_n receives exactly one label of weight from $W_1 \cup W_2$ and each number from $W_1 \cup W_2$ is used exactly once as a label of weight and further that the set $W = W_1 \cup W_2 = \{a, a+d, \dots, a+(|V|-1)\,d\}$, where $a = \frac{5n+5}{2}$ and $d = 2$.

This proves that f_1 is $(\frac{5n+5}{2}, 2)$-antimagic labeling.

Lemma 1. *Prism D_5 is not $(6,4)$-antimagic.*

Proof: Assume that D_5 is $(6,4)$-antimagic. Then the set of weights of vertices is $\{6, 10, 14, 18, 22, 26, 30, 34, 38, 42\}$. The smallest value of weight of vertex ($w(x_1) = a = 6$) can be obtained only under the triple of values of adjacent edges $(1,2,3)$. The following value of weight of vertex ($w(x_2) = a + d = 10$) can be obtained under four triples $(1,2,7)$, $(1,3,6)$, $(2,3,5)$ and $(1,4,5)$, but that leaves only the triple $(1,4,5)$, because the vertices x_1 and x_2 (must be adjacent) have an unique common edge (its value is $f(x_1x_2) = 1$). The value of weight of vertex x_3 ($w(x_3) = a + 2d = 14$) can be obtained under ten triples $(1,2,11)$, $(1,3,10)$, $(1,4,9)$, $(1,5,8)$, $(1,6,7)$, $(2,3,9)$, $(2,4,8)$, $(2,5,7)$, $(3,4,7)$ and $(3,5,6)$. The first five triples we can exclude because they contain the value 1. Each of the remaining 5 triples contains a pair of values from the set $\{2,3,4,5\}$, therefore it is impossible to arrange these triples on the edges of vertex x_3 and this contradicts the fact that D_5 is $(6,4)$-antimagic.

We know $(\frac{n+7}{2}, 4)$-antimagic labelings for D_7, D_9, and D_{11} (Figures 1, 2 and 3). This prompts us to propose the following:

Conjecture 1. *If n is odd, $n \geq 7$, then the prism D_n is $(\frac{n+7}{2}, 4)$-antimagic.*

4 Prisms with even cycles

Theorem 3. *If n is even, $n \geq 4$, then the prism D_n has a $\left(\frac{7n+4}{2}, 1\right)$-antimagic labeling.*

Proof: We assume that $n \geq 4$ is even. In this case the following formulae, where again the function $\delta(x)$ defined in (5) is used, yield the desired $\left(\frac{7n+4}{2}, 1\right)$-antimagic labeling.

$$f_2(x_{j,i}x_{j,i+1}) = \left[\frac{i+1}{2} + 2n(j-1)\right]\delta(i) + \left[3n + \frac{2-i}{2} + n(1-j)\right]\delta(i+1)$$
$$\text{for } i \in I \text{ and } j \in J,$$

and

$$f_2(x_{1,i}x_{2,i}) = \left[n + \frac{i-1}{2}\right]\delta(i) + \left[n + \frac{2-i}{2}\right]\delta(i+1),$$

but this time only for $i \in I - \{1\}$, while now

$$f_2(x_{1,1}x_{2,1}) = \frac{3n}{2}.$$

It is tedious, but not difficult, to check that the values of f_2 are $1, 2, \ldots, |E(D_n)|$ and further that the sets

$$W_3 = \{w_{f_2}(x_{1,i}) \mid i \in I\} = \left\{\frac{7n}{2} + 1 + i \mid i \in I\right\}$$

and

$$W_4 = \{w_{f_2}(x_{2,i}) \mid i \in I\} = \left\{\frac{9n}{2} + 1 + i \mid i \in I\right\}$$

are the sets of weights of vertices of D_n, where $w_f(x_{1,i})$ and $w_f(x_{2,i})$ have been defined above in (6) and (7) respectively. Moreover, it can be seen that the induced mapping $g\colon V(D_n) \to W_3 \cup W_4$ is bijective.

Theorem 5. *Prisms D_n with even cycles admit a $\left(\frac{3n+6}{2}, 3\right)$-antimagic labeling.*

Proof: The proof breaks up into four cases depending upon the congruence class of n (mod 8). To simplify the notation we let $F(i,j,n) = \left(\left[\frac{3n}{2}(j-1) + 2i\right]\delta(i) + \left[\frac{3n}{2}(j-1) + 2i - 3\right]\delta(i+1)\right)\lambda\left(i, \frac{n}{2}\right) + \left(\left[\frac{n}{2}(3j+1)\right.\right.$ $\left.+1 - 2i\right]\delta(i) + \left[\frac{n}{2}(3j+1) + 4 - 2i\right]\delta(i+1)\big)\lambda\left(\frac{n}{2}+1, i\right)$, where

$$\lambda(x,y) = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{if } x > y \end{cases} \tag{8}$$

and the function $\delta(x)$ is defined in (5).

Case 1. $n \equiv 2 \pmod 8$.

For $n \geq 10$ we construct the edge labeling f_3 in the following way:

$$f_3(x_{j,i}x_{j,i+1}) = F(i,j,n) \text{ for } i \in I - \left\{\frac{n}{2}+1\right\} \text{ and } j \in J;$$

$$\text{for } i = \frac{n}{2}+1 \text{ and } j \in J \text{ let}$$

$$f_3(x_{j,i}x_{j,i+1}) = \frac{n}{2}(3j-1)-1.$$

$$f_3(x_{1,i}x_{2,i}) = \begin{cases} \frac{3n}{2}+2-i & \text{if } 2 \leq i \leq \frac{n}{2}+1, \\ 3n & \text{if } i = 1, \\ 2n+2+i & \text{if } i \equiv 3 \pmod 4 \text{ and } \frac{n}{2}+2 \leq i \leq n-3, \\ 2n-1+i & \text{if } i \equiv 0 \pmod 4 \text{ and } \frac{n}{2}+3 \leq i \leq n-2, \\ 2n-1+i & \text{if } i \equiv 1 \pmod 4 \text{ and } \frac{n}{2}+4 \leq i \leq n-1, \\ 2n-4+i & \text{if } i \equiv 2 \pmod 4 \text{ and } \frac{n}{2}+5 \leq i \leq n. \end{cases}$$

Case 2. $n \equiv 6 \pmod 8$.

For $n = 6$ see Figure 4. For $n \geq 14$ use the following labeling f_4: $f_4(x_{j,i}x_{j,i+1}) = F(i,j,n)$ for $i \in I - \left\{\frac{n}{2}+1, n-4, n-3\right\}$ and $j \in J$. For $i \in \left\{\frac{n}{2}+1, n-4, n-3\right\}$ and $j \in J$ let

$$f_4(x_{j,\frac{n}{2}+1}x_{j,\frac{n}{2}+2}) = \frac{n}{2}(3j-1)-1,$$

$$f_4\left(x_{j,n-4}x_{j,n-3}\right) = 7 + \frac{3n}{2}(j-1)$$

and

$$f_4\left(x_{j,n-3}x_{j,n-2}\right) = 12 + \frac{3n}{2}(j-1).$$

$$f_4(x_{1,i}x_{2,i}) = \begin{cases} 3n & \text{if } i = 1, \\ \frac{3n}{2}+2-i & \text{if } 2 \leq i \leq \frac{n}{2}+1, \\ 2n+2+i & \text{if } i \equiv 1 \pmod 4 \text{ and } \frac{n}{2}+2 \leq i < n-2, \\ 2n-1+i & \text{if } i \equiv 2 \pmod 4 \text{ and } \frac{n}{2}+3 \leq i < n-4, \\ 2n-1+i & \text{if } i \equiv 3 \pmod 4 \text{ and } \frac{n}{2}+4 \leq i < n-3, \\ 2n-4+i & \text{if } i \equiv 0 \pmod 4 \text{ and } \frac{n}{2}+5 \leq i < n-2, \\ \frac{5n-8+i}{2} & \text{if } n-4 \leq i \leq n \text{ is even}, \\ \frac{7n-5-i}{2} & \text{if } n-3 \leq i \leq n-1 \text{ is odd}. \end{cases}$$

Case 3. $n \equiv 4 \pmod 8$.

For $n = 4$ see Figure 5. For $n \geq 12$ we define the edge labeling f_5 as follows:

$$f_5(x_{j,i}x_{j,i+1}) = F(i,j,n) \text{ for } i \in I \text{ and } j \in J.$$

$$f_5(x_{1,i}x_{2,i}) = \begin{cases} \frac{5n}{2}+1 & \text{if } i = \frac{n}{2}+1, \\ n+1 & \text{if } i = \frac{n}{2}+2, \\ 3n & \text{if } i = 1, \\ \frac{3n}{2}+2-i & \text{if } 2 \le i \le \frac{n}{2}, \\ 2n+2+i & \text{if } i \equiv 1 \pmod 4 \text{ and } \frac{n}{2}+3 \le i < n, \\ 2n-1+i & \text{if } i \equiv 2 \pmod 4 \text{ and } \frac{n}{2}+4 \le i < n, \\ 2n-1+i & \text{if } i \equiv 3 \pmod 4 \text{ and } \frac{n}{2}+5 \le i < n, \\ 2n-4+i & \text{if } i \equiv 0 \pmod 4 \text{ and } \frac{n}{2}+6 \le i \le n. \end{cases}$$

Case 4. $n \equiv 0 \pmod 8$.

For $n = 8$ see Figure 6. For $n \ge 16$ and $i \in I - \{n-4, n-3\}$, $j \in J$ use the following labeling f_6: $f_6(x_{j,i}x_{j,i+1}) = F(i,j,n)$; for $n-4 \le i \le n-3$ and $j \in J$ let

$$f_6\left(x_{j,n-4}x_{j,n-3}\right) = 7 + \frac{3n}{2}(j-1),$$

$$f_6\left(x_{j,n-3}x_{j,n-2}\right) = 12 + \frac{3n}{2}(j-1),$$

$$f_6(x_{1,i}x_{2,i}) = \begin{cases} 3n & \text{if } i = 1, \\ \frac{3n}{2}+2-i & \text{if } 2 \le i \le \frac{n}{2}, \\ \frac{5n+1}{2} & \text{if } i = \frac{n}{2}+1, \\ n+1 & \text{if } i = \frac{n}{2}+2, \\ 2n+2+i & \text{if } i \equiv 3 \pmod 4 \text{ and } \frac{n}{2}+3 \le i \le n-5, \\ 2n-1+i & \text{if } i \equiv 0 \pmod 4 \text{ and } \frac{n}{2}+4 \le i < n-4, \\ 2n-1+i & \text{if } i \equiv 1 \pmod 4 \text{ and } \frac{n}{2}+5 \le i < n-3, \\ 2n-4+i & \text{if } i \equiv 2 \pmod 4 \text{ and } \frac{n}{2}+6 \le i < n-2, \\ \frac{5n-8+i}{2} & \text{if } n-4 \le i \le n \text{ is even}, \\ \frac{7n-5-i}{2} & \text{if } n-3 \le i \le n-1 \text{ is odd}. \end{cases}$$

It is simple to verify that the labelings f_3, f_4, f_5 and f_6 are the bijections from the edge set $E(D_n)$ onto the set $\{1, 2, \ldots, 3n\}$. The weights of vertices of D_n under the labeling f_k, $3 \le k \le 6$, constitute the sets

$$W_5 = \{w_{f_k}(x_{1,i}) \mid i \in I, 3 \le k \le 6\} = \left\{\frac{3n}{2} + 3i \mid i \in I\right\}$$

and

$$W_6 = \{w_{f_k}(x_{2,i}) \mid i \in I, 3 \le k \le 6\} = \left\{\frac{9n}{2} + 3i \mid i \in I\right\},$$

where $w_f(x_{1,i})$ and $w_f(x_{2,i})$ are defined in (6) and (7) respectively.

We see that the set $W = W_5 \cup W_6 = \{a, a+3, a+6, \dots, a+(2n-1)3\}$, where $a = \frac{3n+6}{2}$, is the set of weights of all vertices of D_n and it can be seen that the induced mapping $g\colon V(D_n) \to W = W_5 \cup W_6$ is bijective.

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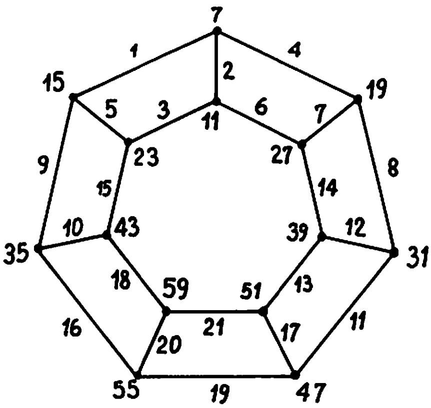


Figure 1

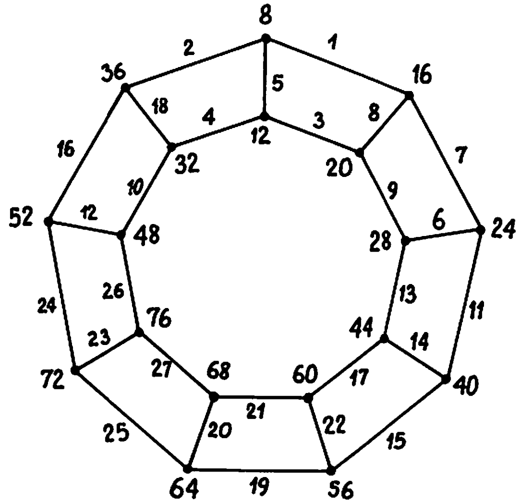
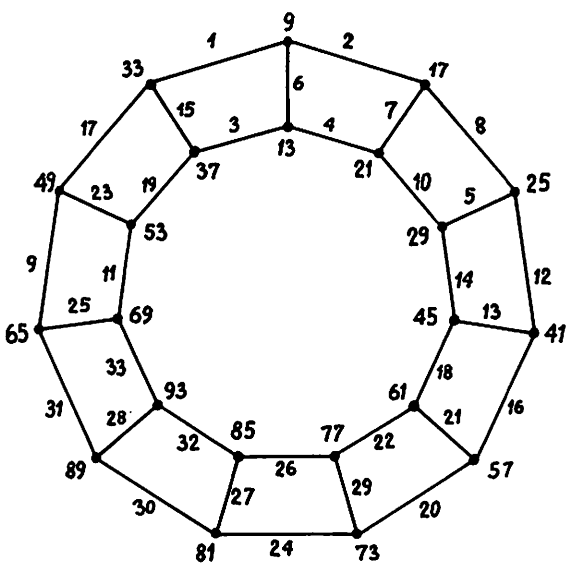


Figure 2



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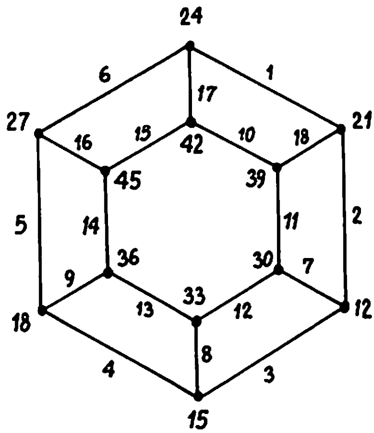

Figure 4

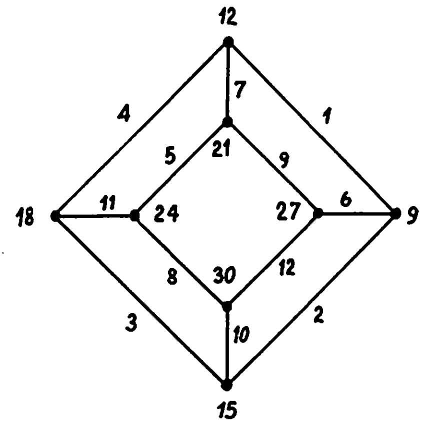

Figure 5

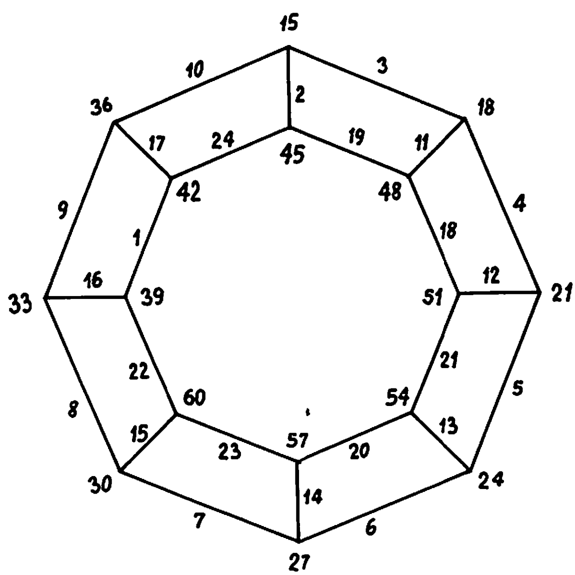

Figure 6