Partial signed domination in graphs

Johannes H. Hattingh and Elna Ungerer
Department of Mathematics
Rand Afrikaans University
Auckland Park, South Africa

Michael A. Henning
Department of Mathematics
University of Natal
Pietermaritzburg, South Africa

Abstract

Let $G = (V, E)$ be a graph. For any real valued function $f : V \to \mathbf{R}$ and $S \subseteq V$, let $f(S) = \sum_{u \in S} f(u)$. Let c, d be positive integers such that $\gcd(c, d) = 1$ and $0 < \frac{c}{d} \leq 1$. A $\frac{c}{d}$-dominating function f is a function $f : V \to \{-1, 1\}$ such that $f[v] \geq 1$ for at least $\frac{c}{d}$ of the vertices V. The $\frac{c}{d}$-domination number of G, denoted by $\gamma_{\frac{c}{d}}(G)$, is defined as $\min\{f(V) \mid f$ is a $\frac{c}{d}$-dominating function on $G\}$. We determine a sharp lower bound on $\gamma_{\frac{c}{d}}(G)$ for regular graphs G, determine the value of $\gamma_{\frac{c}{d}}$ for an arbitrary cycle C_n and show that the decision problem **PARTIAL SIGNED DOMINATING FUNCTION** is NP-complete.

1 Introduction

Let $G = (V, E)$ be a graph and let v be a vertex in V. The *open neighborhood* of v is defined as the set of vertices adjacent to v, i.e., $N(v) = \{u | uv \in E\}$. The *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$. For a set S of vertices, we define the open neighborhood $N(S)$ as $\cup_{v \in S} N(v)$, and the closed neighborhood $N[S]$ as $N(S) \cup S$. A set S of vertices is a *dominating set* if $N[S] = V$. The *domination number* of a graph G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set in G.
For any real valued function $f : V \to \mathbf{R}$ and $S \subseteq V$, let $f(S) = \sum_{u \in S} f(u)$. The *weight* of f is defined as $f(V)$. We will also denote $f(N[v])$ by $f[v]$, where $v \in V$. If $v \in V$ and $f[v] \geq 1$, then we say that the vertex v is *covered* under f. We denote the set of all vertices of V that are covered

under f by C_f.
A *minus dominating function* is defined in [4] as a function $f : V \to \{-1,0,1\}$ such that $f[v] \geq 1$ for every $v \in V$. The *minus domination number* of a graph G is $\gamma^-(G) = \min\{f(V) \mid f$ is a minus dominating function on $G\}$.
A *signed dominating function* is defined in [5] as a function $f : V \to \{-1,1\}$ such that $f[v] \geq 1$ for every $v \in V$. The *signed domination number* of a graph G is $\gamma_s(G) = \min\{f(V) \mid f$ is a signed dominating function on $G\}$.
A *majority dominating function* is defined in [1] as a function $f : V \to \{-1,1\}$ such that $f[v] \geq 1$ for at least half the vertices $v \in V$. The *majority domination number* of a graph G is $\gamma_{maj}(G) = \min\{f(V) \mid f$ is a majority dominating function on $G\}$.
Let $k \in \mathbf{Z}^+$ such that $1 \leq k \leq |V|$. A *k-subdominating function* (kSF) *to* $\{-1,1\}$ for G is defined in [3] as a function $f : V \to \{-1,1\}$ such that $f[v] \geq 1$ for at least k vertices of G. The *k-subdomination number to* $\{-1,1\}$ of a graph G, denoted by $\gamma_{ks}^{-11}(G)$, is equal to $\min\{f(V) \mid f$ is a kSF to $\{-1,1\}$ of $G\}$. In the special cases where $k = |V|$ and $k = \lceil\frac{|V|}{2}\rceil$, $\gamma_{ks}^{-11}(G)$ is respectively the signed domination number and the majority domination number.
Let $k \in \mathbf{Z}^+$ such that $1 \leq k \leq |V|$. A *k-subdominating function* (kSF) *to* $\{-1,0,1\}$ for G is defined in [2] as a function $f : V \to \{-1,0,1\}$ such that $f[v] \geq 1$ for at least k vertices of G. The *k-subdomination number to* $\{-1,0,1\}$ of a graph G, denoted by $\gamma_{ks}^{-101}(G)$, is equal to $\min\{f(V) \mid f$ is a kSF to $\{-1,0,1\}$ of $G\}$. In the special case where $k = |V|$, $\gamma_{ks}^{-11}(G)$ is the minus domination number. Since every kSF to $\{-1,1\}$ is also a kSF to $\{-1,0,1\}$, we have that $\gamma_{ks}^{-101}(G) \leq \gamma_{ks}^{-11}(G)$ for an arbitrary graph G.
Let c, d be positive integers such that $\gcd(c,d) = 1$ and $0 < \frac{c}{d} \leq 1$. A *$\frac{c}{d}$-dominating function* f is a function $f : V \to \{-1,1\}$ such that $f[v] \geq 1$ for at least $\frac{c}{d}$ of the vertices V. The *$\frac{c}{d}$-domination number* of G, denoted by $\gamma_{\frac{c}{d}}(G)$, is defined as $\min\{f(V) \mid f$ is a $\frac{c}{d}$-dominating function on $G\}$. In the special cases where $\frac{c}{d} = 1$ and $\frac{c}{d} = \frac{1}{2}$, $\gamma_{\frac{c}{d}}(G)$ is respectively the signed domination number and the majority domination number.
In this paper, we determine a lower bound on $\gamma_{\frac{c}{d}}(G)$ for regular graphs G, determine the value of $\gamma_{\frac{c}{d}}$ for an arbitrary cycle C_n and show that the decision problem **PARTIAL SIGNED DOMINATING FUNCTION** is NP-complete.

2 A lower bound on $\gamma_{\frac{c}{d}}(G)$ for regular graphs G

Theorem 1 *Let c, d be positive integers such that* $\gcd(c,d) = 1$ *and* $0 <$

$q = \frac{c}{d} \leq 1$. *For every r-regular* $(r \geq 2)$ *graph* $G = (V, E)$ *of order p,*

$$\gamma_q(G) \geq \begin{cases} p(q\frac{r+3}{r+1} - 1) & \text{for } r \text{ odd} \\ p(q\frac{r+2}{r+1} - 1) & \text{for } r \text{ even,} \end{cases}$$

and these bounds are best possible.

Proof. Let $f : V \to \{-1, 1\}$ be any q-dominating function on G for which $f(V) = \gamma_q(G)$. Let P and M (standing for "positive" and "minus") be the sets of vertices in G that are assigned the values $+1$ and -1, respectively, under f. Then $|P|+|M| = p$. Further, let P^+ and P^- be the sets of vertices in P whose closed neighborhood sum under f is positive and nonpositive, respectively. Define M^+ and M^- analogously. Then $P = P^+ \cup P^-$ and $M = M^+ \cup M^-$. Further, let $|M^+| = a$, $|P^+| = b$ and $|P^-| = c$. Then, since f is a q-dominating function, $a+b \geq qp$. We consider two possibilities.
Case 1. $a < qp\frac{\lfloor \frac{r}{2} \rfloor}{r+1}$.
Then, since $|P| = b + c \geq b \geq qp - a$, it follows that

$$|P| > qp - qp\frac{\lfloor \frac{r}{2} \rfloor}{r+1} = qp\left(1 - \frac{\lfloor \frac{r}{2} \rfloor}{r+1}\right).$$

Hence,

$$\begin{aligned} \gamma_q(G) &= |P| - |M| \\ &= 2|P| - p \\ &> 2qp\left(1 - \frac{\lfloor \frac{r}{2} \rfloor}{r+1}\right) - p \end{aligned}$$

which yields the desired result.
Case 2. $a \geq qp\left(\frac{\lfloor \frac{r}{2} \rfloor}{r+1}\right)$.
Let ℓ be the number of edges joining a vertex of M^+ and a vertex of P. Then, since each vertex of M^+ must be adjacent to at least $\lceil \frac{r}{2} \rceil + 1$ vertices of P, we have that $\ell \geq \left(\lceil \frac{r}{2} \rceil + 1\right) a$. On the other hand, although a vertex of P^- may be adjacent to as many as r vertices of M, each vertex of P^+ is adjacent to at most $\lfloor \frac{r}{2} \rfloor$ vertices of M. It follows that $\ell \leq \lfloor \frac{r}{2} \rfloor b + rc$. Consequently,

$$\left(\left\lceil \frac{r}{2} \right\rceil + 1\right) a \leq \left\lfloor \frac{r}{2} \right\rfloor b + rc.$$

Hence it follows that,

$$\begin{aligned}
|P| &= b + c \\
&\geq b + \left((\lceil \tfrac{r}{2} \rceil + 1)a - \lfloor \tfrac{r}{2} \rfloor b\right)/r \\
&= \left(1 - \tfrac{1}{r}\lfloor \tfrac{r}{2} \rfloor\right) b + \left(\lceil \tfrac{r}{2} \rceil + 1\right) \tfrac{a}{r} \\
&\geq \left(1 - \tfrac{1}{r}\lfloor \tfrac{r}{2} \rfloor\right) (qp - a) + \left(\lceil \tfrac{r}{2} \rceil + 1\right) \tfrac{a}{r} \\
&= \left(1 - \tfrac{1}{r}\lfloor \tfrac{r}{2} \rfloor\right) qp + \tfrac{a}{r}\left(\lceil \tfrac{r}{2} \rceil + \lfloor \tfrac{r}{2} \rfloor + 1 - r\right) \\
&= \left(1 - \tfrac{1}{r}\lfloor \tfrac{r}{2} \rfloor\right) qp + \tfrac{a}{r}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\gamma_q(G) &= 2|P| - p \\
&\geq 2qp\left(1 - \tfrac{1}{r}\lfloor \tfrac{r}{2} \rfloor\right) + \tfrac{2a}{r} - p \\
&\geq 2qp\left(1 - \tfrac{1}{r}\lfloor \tfrac{r}{2} \rfloor\right) + \tfrac{2qp\lfloor \frac{r}{2} \rfloor}{r(r+1)} - p \\
&= 2qp\left(1 - \tfrac{(r+1)\lfloor \frac{r}{2} \rfloor - \lfloor \frac{r}{2} \rfloor}{r(r+1)}\right) - p \\
&= 2qp\left(1 - \tfrac{\lfloor \frac{r}{2} \rfloor}{r+1}\right) - p.
\end{aligned}$$

That these lower bounds are best possible, may be seen as follows. Let G be the (disjoint) union of the graphs $H_i \cong K_{r+1}$, $i = 1, \ldots, c$, and the graph $F \cong (d-c)K_{r+1}$. For $i = 1, \ldots, c$, let $M_i \subseteq V(H_i)$ with $|M_i| = \lfloor \frac{r}{2} \rfloor$ and let $P_i = V(H_i) - M_i$. Note that $|P_i| = \lceil \frac{r}{2} \rceil + 1$. Define $f : V(G) \to \{-1, 1\}$ by $f(v) = 1$ for $v \in \cup_{i=1}^{c} P_i$ and $f(v) = -1$ for $v \in (\cup_{i=1}^{c} M_i) \cup V(F)$. Then f is a q-dominating function on G in which every vertex of $\cup_{i=1}^{c} H_i$ has positive neighborhood sum under f. Hence

$$\begin{aligned}
\gamma_q(G) &\leq f(V(G)) \\
&= c(\lceil \tfrac{r}{2} \rceil + 1 - \lfloor \tfrac{r}{2} \rfloor) - (d - c)(r + 1) \\
&= c(\lceil \tfrac{r}{2} \rceil + 1 - \lfloor \tfrac{r}{2} \rfloor + r + 1) - d(r + 1) \\
&= \begin{cases} c(r+3) - d(r+1) & \text{for } r \text{ odd} \\ c(r+2) - d(r+1) & \text{for } r \text{ even.} \end{cases}
\end{aligned}$$

The lower bound of our theorem implies that

$$\gamma_q(G) \geq \begin{cases} \frac{c}{d}(d(r+1))^{\frac{r+3}{r+1}} - d(r+1) & \text{for } r \text{ odd} \\ \frac{c}{d}(d(r+1))^{\frac{r+2}{r+1}} - d(r+1) & \text{for } r \text{ even.} \end{cases}$$

$$= \begin{cases} c(r+3) - d(r+1) & \text{for } r \text{ odd} \\ c(r+2) - d(r+1) & \text{for } r \text{ even.} \end{cases}$$

This completes the proof of our theorem. ■

In [5] and [10] the following lower bounds on $\gamma_s(G)$ for r-regular graphs G of order p for r even and odd, respectively, are established.

Theorem 2 *For every r-regular ($r \geq 2$) graph G of order p,*

$$\gamma_s(G) \geq \begin{cases} \frac{2p}{r+1} & \text{for } r \text{ odd} \\ \frac{p}{r+1} & \text{for } r \text{ even.} \end{cases}$$

Zelinka [11] established the following lower bound on $\gamma_{maj}(G)$ for a cubic graph G.

Theorem 3 *For every cubic graph G of order p, $\gamma_{maj}(G) \geq -\frac{p}{4}$ and this bound is best possible.*

Henning [8] generalised the result of Theorem 3 to r-regular graphs.

Theorem 4 *For every r-regular ($r \geq 2$) graph $G = (V, E)$ of order p,*

$$\gamma_{maj}(G) \geq \begin{cases} \left(\frac{1-r}{2(r+1)}\right)p & \text{for } r \text{ odd} \\ \left(\frac{-r}{2(r+1)}\right)p & \text{for } r \text{ even,} \end{cases}$$

and these bounds are best possible.

Note that, if $q = 1$ in the statement of Theorem 1, then we obtain the result of Theorem 2 and if $q = \frac{1}{2}$, then we obtain the result of Theorem 4.

3 The value of $\gamma_{ks}^{-11}(C_n)$

Cockayne and Mynhardt (see [3]) determined γ_{ks}^{-11} for an arbitrary path P_n.

Theorem 5 *If $n \geq 2$ is an integer and $1 \leq k \leq n$, then $\gamma_{ks}^{-11}(P_n) = 2\lfloor \frac{2k+4}{3} \rfloor - n$.*

We now calculate $\gamma_{ks}^{-11}(C_n)$. We begin by noting that $\gamma_s(C_n) = n - 2\lceil \frac{n}{3} \rceil$ (cf. [5]) and that

$$2\left\lfloor \frac{2k+4}{3} \right\rfloor = \begin{cases} \lceil \frac{k}{3} \rceil + k + 2 & \text{if } k \equiv 0 \pmod 3 \text{ or } k \equiv 1 \pmod 3 \\ \lceil \frac{k}{3} \rceil + k + 1 & \text{if } k \equiv 2 \pmod 3. \end{cases}$$

Theorem 6 *If $n \geq 3$ is an integer and $1 \leq k \leq n-1$, then*

$$\gamma_{ks}^{-11}(C_n) = \begin{cases} \frac{n-2}{3} & \text{if } k = n-1 \text{ and } k \equiv 1 \pmod 3 \\ 2\lfloor \frac{2k+4}{3} \rfloor - n & \text{otherwise.} \end{cases}$$

Proof. We first prove the upper bounds for $\gamma_{ks}^{-11}(C_n)$. Let $P_n : v_1, v_2, \ldots, v_n$ be a path on n vertices, and let C_n be the cycle obtained from P_n by joining the vertices v_1 and v_n.

Case 1 $k \equiv 0 \pmod 3$.

The function $f : V \to \{-1, 1\}$ defined by $(f(v_1), \ldots, f(v_n)) = (\underbrace{1, 1, -1, 1, 1, -1, \ldots, 1, 1, -1}_{k}, 1, -1, \ldots, -1)$ is a kSF for P_n of weight $\lceil \frac{k}{3} \rceil + k - n + 2 = 2\lfloor \frac{2k+4}{3} \rfloor - n$ which is also a kSF for C_n. Hence $\gamma_{ks}^{-11}(C_n) \leq 2\lfloor \frac{2k+4}{3} \rfloor - n$.

Case 2 $k \equiv 1 \pmod 3$.

If $k = n-1$, then $f : V \to \{-1, 1\}$ defined by $(f(v_1), \ldots, f(v_n)) = (\underbrace{1, 1, -1, 1, 1, -1, \ldots, 1, 1, -1, 1}_{k}, -1)$ is a $(k-1)SF$ for P_n of weight $\frac{n-2}{3}$ but a kSF for C_n. Hence $\gamma_{ks}^{-11}(C_n) \leq \frac{n-2}{3}$.

If $k \leq n-2$, then $f : V \to \{-1, 1\}$ defined by $(f(v_1), \ldots, f(v_n)) = (\underbrace{1, 1, -1, 1, 1, -1, \ldots, 1, 1, -1, 1}_{k}, 1, -1, \ldots, -1)$ is a kSF for P_n of weight $\lceil \frac{k}{3} \rceil + k - n + 2 = 2\lfloor \frac{2k+4}{3} \rfloor - n$ which is also a kSF for C_n. Hence $\gamma_{ks}^{-11}(C_n) \leq 2\lfloor \frac{2k+4}{3} \rfloor - n$.

Case 3 $k \equiv 2 \pmod 3$.

The function $f : V \to \{-1, 1\}$ defined by $(f(v_1), \ldots, f(v_n)) = (\underbrace{1, 1, -1, 1, 1, -1, \ldots, 1, 1, -1, 1, 1}_{k}, -1, \ldots, -1)$ is a kSF for P_n of weight $\lceil \frac{k}{3} \rceil + k - n + 1 = 2\lfloor \frac{2k+4}{3} \rfloor - n$ which is also a kSF for C_n. Hence $\gamma_{ks}^{-11}(C_n) \leq 2\lfloor \frac{2k+4}{3} \rfloor - n$.

We now prove the lower bounds for $\gamma_{ks}^{-11}(C_n)$. Since (see [7])

$$\gamma_{ks}^{-101}(C_n) = \begin{cases} \lceil \frac{n-2}{3} \rceil & \text{if } k = n-1 \text{ and } k \equiv 0 \text{ or } 1 \pmod 3 \\ 2\lfloor \frac{2k+4}{3} \rfloor - n & \text{otherwise,} \end{cases}$$

and $\gamma_{ks}^{-11}(C_n) \geq \gamma_{ks}^{-101}(C_n)$, the lower bound for $\gamma_{ks}^{-11}(C_n)$ will follow if we can prove the following result:

Proposition 1 *If $k = n-1$ and $k \equiv 0 \pmod 3$, then*

$$\gamma_{ks}^{-11}(C_n) \geq \frac{n+2}{3}.$$

Proof. Let $C_n : v_0, v_1, \ldots, v_{n-1}, v_0$ be the cycle on n vertices and let $V = V(C_n)$. Let f be a minimum kSF to $\{-1, 1\}$ for C_n.
If f covers all of C_n's vertices, then f is a signed dominating function of C_n, so that $\gamma_{ks}^{-11}(C_n) = f(V) \geq \gamma_s(C_n) = n - 2\lfloor \frac{n}{3} \rfloor = \frac{n+2}{3}$.
In what follows, we assume that there is exactly one uncovered vertex under f, say v_{n-1}. Note that $f(v_{n-1}) = 1$, for if this is not the case, then $f(v_0) = -1$ or $f(v_{n-2}) = -1$. But then we have two uncovered vertices, which contradicts the fact that $k = n-1$. Note further that $f(v_0) = f(v_{n-2}) = -1$.
For all $v_i \neq v_{n-2}$, we have that if $f(v_i) = -1$, then $f(v_{i+1}) = f(v_{i+2}) = 1$, where addition is taken modulo n. If $f(v_{i+3}) = 1$, then we construct the cycle C_n' from the cycle C_n by removing the vertex v_{i+1}, joining the vertices v_i and v_{i+2} and inserting v_{i+1} between v_{n-2} and v_{n-1}. Note that v_i, v_{i+1} and v_{i+2} are still covered by f, while f now covers the previously uncovered v_{n-1}. By relabeling the vertices of C_n', we obtain a minimum kSF which is also a signed dominating function. This case was handled previously.
This implies that f consists of a sequence of -1's and 1's such that each -1 is adjacent to two 1's and each 1, except the value for v_{n-1}, is adjacent to a -1 and a 1. In this case $n \equiv 2 \pmod 3$, which contradicts our assumption that $n \equiv 1 \pmod 3$.
This contradiction shows that, in this case, a minimum kSF to $\{-1, 1\}$ for C_n is actually a signed dominating function of C_n. ■
This result generalises the following result of Broere, Hattingh, Henning and McRae (see [1]).

Theorem 7 *If $n \geq 3$ is an integer, then*

$$\gamma_{maj}(C_n) = \gamma_{maj}(P_n).$$

Let c, d be positive integers such that $\gcd(c, d) = 1$ and $0 < q = \frac{c}{d} \leq 1$. By letting $k = \lceil q|V(C_n)|\rceil$, we have

$$\gamma_q(C_n) = \begin{cases} n - 2\lfloor \frac{n}{3} \rfloor & \text{if } \lceil q|V(C_n)|\rceil = n \\ \frac{n-2}{3} & \text{if } \lceil q|V(C_n)|\rceil = n-1 \text{ and} \\ & \lceil q|V(C_n)|\rceil \equiv 1 (\text{mod} 3) \\ 2\lfloor \frac{2\lceil q|V(C_n)|\rceil + 4}{3} \rfloor - n & \text{otherwise.} \end{cases}$$

4 Complexity results

In this section we show that the problem
PARTIAL SIGNED DOMINATING FUNCTION (PSDF)
INSTANCE: A graph G, positive integers c, d such that $\gcd(c, d) = 1$ and $0 < \frac{c}{d} \leq 1$ and an integer k.
QUESTION: Is there a $\frac{c}{d}$-dominating function of weight k or less for G?
is NP-complete by describing a polynomial transformation from the following problem (see [6]):
DOMINATING SET
INSTANCE: A planar 4-regular graph $G = (V, E)$ and a positive integer $k \leq \frac{|V|}{2}$.
QUESTION: Is there a dominating set of cardinality k or less for G?
If $\frac{c}{d} = 1$, then **PSDF** is the NP-complete problem **SIGNED DOMINATION** (see [9]). Hence, we also assume that $0 < \frac{c}{d} < 1$. For convenience, we set $q = \frac{c}{d}$.
We will need the following lemma.

Lemma 1 *If c, d, p are positive integers such that $0 < q = \frac{c}{d} < 1$, then there exist positive integers ℓ and r such that $8 \leq \ell \leq d^2(\lceil \frac{p}{2} \rceil + 4)$, $r < d^2(\lceil \frac{p}{2} \rceil + 4)$ and $q = \frac{p+r}{2p+r+\ell}$.*

Proof. Since $c < d$, we have $c \geq 1$, $d \geq 2$ and $d - c \geq 1$. Let $t = \lceil \frac{p}{2} \rceil + 4$. Then $dt(d-c) \geq 2t$ and $cdt \geq 2t$. However, $2t \geq p + 8$, whence $dt(d-c) \geq p+8$ and $cdt > p$. Let t be the smallest positive integer such that $dt(d-c) \geq p+8$ and $cdt > p$. It follows that $t \leq \lceil \frac{p}{2} \rceil + 4$. Let $r = cdt - p$ and $\ell = ddt - cdt - p$. Note that r and ℓ are both positive integers such that $r, \ell < ddt \leq d^2(\lceil \frac{p}{2} \rceil + 4)$. Furthermore, $\ell \geq 8$ and $q = \frac{p+r}{2p+r+\ell}$. ∎

Theorem 8 *The decision problem* **PSDF** *is NP-complete.*

Proof. Obviously, **PSDF** is in NP.
Let G be a 4-regular planar graph, $p = p(G)$ and k be an integer such that $k \leq \frac{p}{2}$. By Lemma 1, there exists positive integers r, ℓ such that $\ell \geq 8$ and $q = \frac{p+r}{2p+r+\ell}$. Let H be the graph constructed from G as follows:

Take a complete graph F on $p+\ell$ vertices, a fixed subset $U \subseteq V(F)$ with $|U| = 4$ and an empty graph L on r vertices, and let H be obtained from the disjoint union of F, G, and L by joining each vertex of U to every vertex in $V(G) \cup V(L)$. Since $p(H) = 2p + r + \ell < 2(p + d^2(\lceil \frac{p}{2} \rceil + 4))$, the graph H can be constructed from G in polynomial time.
We start by showing that if S is a dominating set of G of cardinality at most k, then there is a q-dominating function f of H of weight at most $2k - 2p - r - \ell + 8$. Define $f : V(H) \to \{-1, 1\}$ by $f(v) = 1$ if $v \in S \cup U$, while $f(v) = -1$ otherwise. If $v \in S$, then $f(v) = 1$ and since G is 4-regular and $f(U) = 4$, it follows that $f[v] \geq 1$. If $v \in V(G) - S$, then v is adjacent to some vertex u in S for which $f(u) = 1$. Again it follows that $f[v] \geq 1$. It is clear that $f[w] = 3$ for each vertex $w \in V(L)$, so that $f[v] \geq 1$ for at least $p+r = q(2p+r+\ell) = qp(H)$ vertices. This shows that f is a q-dominating function of H of weight $2|S| - 2p - r - \ell + 8 \leq 2k - 2p - r - \ell + 8$.
For the converse, assume that $\gamma_q(H) \leq 2k - 2p - r - \ell + 8$. Among all the minimum q-dominating functions of H, let f be one that assigns the value $+1$ to as many vertices of U as possible. Let P and M be the sets of vertices in H that are assigned the values $+1$ and -1, respectively, under f. Then $|P| + |M| = 2p + r + \ell$, and $|P| - |M| = \gamma_q(H)$. Before proceeding further we prove three claims.
Claim 1 $|P| \leq k + 4$.
Proof. Suppose $|P| \geq k + 5$. Then $|M| \leq 2p + r + \ell - k - 5$, so that $\gamma_q(H) = |P| - |M| \geq 2k - 2p - r - \ell + 10$, which contradicts the fact that $\gamma_q(H) \leq 2k - 2p - r - \ell + 8$. □
Claim 2 $f[v] \leq 0$ for all $v \in V(F)$.
Proof. Suppose there exists a $v \in V(F)$ such that $f[v] \geq 1$. If $v \in U$, then, since v dominates H, it follows that $0 < 1 \leq f[v] = f(V(H)) = \gamma_q(H) \leq 2k - 2p - r - \ell + 8$, whence $p + \frac{r}{2} < k$, which is a contradiction. Hence $v \in V(F) - U$. Since $N[v] = V(F)$, it follows that more than half of the vertices of the F have the value 1 assigned to them under f. This implies that $|P| > \frac{p+\ell}{2} = \frac{p}{2} + \frac{\ell}{2} \geq \frac{p}{2} + 4$. By Claim 1 and the fact that $k \leq \frac{p}{2}$, it follows that $|P| \leq \frac{p}{2} + 4$, which is a contradiction. □
By Claim 2, it follows that $f[v] \geq 1$ for all $v \in V(G) \cup V(L)$.
Claim 3 $f(U) = 4$.
Proof. Suppose that $f(u) = -1$ for some $u \in U$. If $f(v) = -1$ for all $v \in V(G)$, then $f[v] \leq -3$ for all $v \in V(G)$, which is a contradiction. It follows that there exists a $v \in V(G)$ such that $f(v) = 1$. Define $g : V(H) \to \{-1, 1\}$ by $g(w) = f(w)$ if $w \in V(H) - \{u, v\}$, $g(v) = -1$ and $g(u) = 1$. Note that if $x \notin N[v]$, then $g[x] = f[x] + 2$, while if $x \in N[v]$, then $g[x] = f[x]$. It follows that $g[v] \geq 1$ for at least $\frac{c}{d}$ of the vertices of H while the weights of g and f are equal. Hence g is a q-dominating function of H of weight $\gamma_q(H)$ that assigns the value $+1$ to more vertices of U than does f, contradicting our choice of f. □

Let $S = P \cap V(G)$. Since $f[v] \geq 1$ for all $v \in V(G)$, it follows that either $f(v) = 1$ or there is a $u \in N_G[v]$ with $f(u) = 1$. Hence, each vertex in G is either in S or adjacent to some vertex of S, which shows that S is a dominating set of G. Since $f(U) = 4$, Claim 1 implies that $|S| \leq k$, which completes the proof. ■

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