Non-Isomorphic Smallest Maximally Non-Hamiltonian Graphs

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ABSTRACT. A graph G is maximally non-hamiltonian (MNH) if G is not hamiltonian but becomes hamiltonian after adding an arbitrary new edge. Bondy [2] showed that the smallest size (= number of edges) in a MNH graph of order n is at least $\lceil \frac{3n}{2} \rceil$ for $n \geq 7$. The fact that equality may hold there for infinitely many n was suggested by Bollobás [1]. This was confirmed by Clark, Entringer and Shapiro (see [5, 6]) and by Xiaohui, Wenzhou, Chengxue and Yuanscheng [8] who set the values of the size of smallest MNH graphs for all small remaining orders n. An interesting question of Clark and Entringer [5] is whether for infinitely many n the smallest MNH graph of order n is not unique. A positive answer - the existence of two non-isomorphic smallest MNH graphs for infinitely many n follows from results in [5], [4], [6] and [8]. But, there still exist infinitely many orders n for which only one smallest MNH graph of order n is known.

We prove that for all $n \geq 88$ there are at least $\tau(n) \geq 3$ smallest MNH graphs of order n, where $\lim_{n\to\infty} \tau(n) = \infty$. Thus, there are only finitely many orders n for which the smallest MNH graph is unique.

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1 Introduction

A possible approach to studying hamiltonicity consists in trying to find graphs that provide obstructions for the existence of a hamiltonian cycle. This naturally leads to the main object of study of this paper: We say that a graph is *maximally non-hamiltonian* (MNH) if it is not hamiltonian but becomes hamiltonian after adding an arbitrary new edge. In other words, a non-hamiltonian graph G is MNH if any two non-adjacent vertices of G are ends of a hamiltonian path in G. At a first glance, such requirement seems to force the graph G to be "rich" (in terms of size, i.e., the number of edges). Denoting by $f(n)$ the smallest size of a MNH graph of order n, the lower bound $f(n) \geq \lceil \frac{3n}{2} \rceil$ for $n \geq 7$ (due to Bondy [2]) might therefore appear to be far from exact. Surprisingly, as we shall see, almost always it is as exact as it can be, and this is true for all orders $n \geq 19$ (it is true for orders 6, 10 - 13, 17, 19 as well).

Some examples of cubic (= 3-valent) MNH graphs were known at the time when Bondy's paper [2] was published. The two most prominent ones have been the Petersen graph and the Coxeter graph, giving an equality in Bondy's bound for $n = 10$ and $n = 28$. Some years later, Bollobás [1] suggested that equality may hold there for infinitely many values of n. A breakthrough in determining the exact value of $f(n)$ was achieved by Clark and Entringer [5] and Clark, Entringer and Shapiro [6]. They proved that the Isaacs' flower snarks and their appropriate modifications provide examples of MNH graphs of order n and size $\lceil \frac{3n}{2} \rceil$ for all even $n \geq 36$ and all odd $n \geq 55$, showing that $f(n) = \lceil \frac{3n}{2} \rceil$ for almost all n. The remaining cases of orders n were set by Xiaohui, Wenzhou, Chengxue and Yuanscheng [8].

Clark and Entringer [5] asked whether for infinitely many n the smallest MNH graph of order n is not unique. Combining the results from [5, 6, 8] it can be seen that for infinitely many n there are two non-isomorphic smallest MNH graphs of order n, but there still exist infinitely many orders n for which only one smallest MNH graph is known.

Let $p(n, k)$ denote the number of partitions of n into k parts, that is, the number of distinct ways to write n as a sum of k positive integers, disregarding order. It is well-known that $p(n, k) = \sum_{i=1}^{k} p(n-k, i)$. Similarly, let $p'(n, k)$ denote the number of partitions of n into k parts where each summand is at least two; obviously, $p'(n, k) = p(n, k) - \sum_{i=1}^{k} p'(n-i, k-i)$. Let us put $Q(0) = 2$, $Q(1) = Q(2) = 3$, $Q(3) = Q(4) = 4$, $Q(5) = Q(6) = 5$ and $Q(7) = 6$. Let $n = 8a + b \geq 88$, where $0 \leq b < 8$ and let $k = 2a + 1$. In this paper we prove that for all $n \geq 88$ there exist at least $\tau(n) = \sum_{i=0}^{\lfloor \frac{k-2-3Q(b)}{14} \rfloor} p'(k-2-Q(b)-6i, Q(b)+4i) \geq 3$ non-isomorphic smallest MNH graphs of order n, where $lim_{n \to \infty} \tau(n) = \infty$. The main ingredients of our constructions are Isaacs' flower snarks, again.

2 Modifications of Isaacs' Snarks

We first describe the construction of Isaacs' snarks J_k, $k \geq 3$, k odd (see [7]).

Definition 1 *For $k \geq 3$, k odd, the graph J_k has vertex set $\{v_0, v_1, \ldots, v_{4k-1}\}$ and edge set $E_0 \cup E_1 \cup E_2 \cup E_3$, where $E_0 = \cup_{j=0}^{k-1}\{v_{4j}v_{4j+1}, v_{4j}v_{4j+2}, v_{4j}v_{4j+3}\}$, $E_1 = \{v_{4j+1}v_{4j+7} \mid 0 \leq j \leq k-1\}$, $E_2 = \{v_{4j+2}v_{4j+6} \mid 0 \leq j \leq k-1\}$, $E_3 = \{v_{4j+3}v_{4j+5} \mid 0 \leq j \leq k-1\}$.*

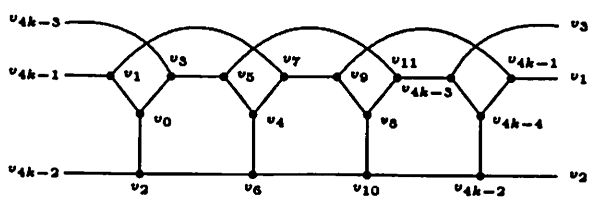

Figure 1. The graph J_k with its labelled vertices

Now we define two operations of "expanding a vertex to a triangle" and "replacing an edge by a bowtie", which we then apply to Isaacs' snarks.

Definition 2 *Let v be a vertex of degree 3 in a graph G and let v_1, v_2, v_3 be its neighbours. Let $T = K_3$ with vertices labeled u_1, u_2, u_3.*
By $G(v)$ we denote the graph obtained from G by replacing v by T, i.e.,

$$G(v) = [(G - v) \cup T] + u_1v_1 + u_2v_2 + u_3v_3.$$

The graph $G(v)$ is said to be obtained from G by "expanding v to a triangle".

If each of the vertices $v_1, v_2, \ldots, v_k$ is expanded to a triangle in G we denote the resulting graph by $G(v_1, v_2, \ldots, v_k)$. By $N(u)$ we denote the set of neighbours of u.

Definition 3 *The graph $\bowtie = K_5 - u_1u_2 - u_1u_3 - u_4u_3 - u_4u_2$, where $u_i, i = 1, 2, 3, 4, 5$ are the vertices of K_5, will be called a bowtie. Let G be a graph and let v_i, v_j be vertices of degree 3 in G such that v_iv_j is an edge of G. By the symbol $G(v_iv_j)$ we denote the graph obtained from G in the following way,*

$$G(v_iv_j) = [(G - \{v_i, v_j\}) \cup \bowtie] + u_1v_k + u_4v_l + u_2v_m + u_3v_n,$$

where $v_j, v_k, v_l \in N(v_i)$ and $v_i, v_m, v_n \in N(v_j)$. Shortly, we say that $G(v_iv_j)$ is obtained from G by "replacing the edge v_iv_j by a bowtie".

To be able to review known results we define further construction "extending an edge to a triangle".

Definition 4 *Let uv be an edge in a graph G such that the degree of u and v is 3 and let w be a vertex not in G. By $G[uv]$ we denote the graph obtained from G in the following way*

$$G[uv] = [G \cup w] + vw + uw.$$

The graph $G[uv]$ is said to be obtained from G by "extending uv to a triangle".

The previous modifications applied to Isaacs' snarks yield smallest MNH graphs. In the following Table we briefly recall these constructions. The Table is to be read with $k = 2p + 1$ and $p \geq 7$. The cases $p \leq 6$ are solved as well, but using different constructions. Moreover, for $p \geq 7$ these results were achieved in [8] and [5, 6].

order	Clark et al. [5, 6]	Xiaohui et al. [8]	uniqueness
8p	$J_{k-2}(v_2, v_{14})$	$J_{k-2}(v_0, v_4)$	non-isomorphic
8p+1	$J_{k-2}(v_{14}, v_{26})[v_0v_2]$	$J_{k-2}(v_0, v_4)[v_{16}v_{18}]$	non-isomorphic
8p+2	$J_{k-2}(v_2, v_{14}, v_{26})$	$J_{k-2}(v_0, v_4, v_8)$	non-isomorphic
8p+3	$J_{k-2}(v_{14}, v_{26}, v_{38})[v_0v_2]$	$J_{k-2}(v_0, v_4, v_8)[v_{16}v_{18}]$	non-isomorphic
8p+4	J_k	J_k	isomorphic
8p+5	$J_k[v_0v_2]$	$J_k[v_{16}v_{18}]$	isomorphic
8p+6	$J_k(v_2)$	$J_k(v_0)$	non-isomorphic
8p+7	$J_k(v_{14})[v_0v_2]$	$J_k(v_0)[v_{16}v_{18}]$	non-isomorphic

Table 1. Modifications of Isaacs' snarks

Our aim is to apply first two operations on Isaacs' snarks to obtain the required number of smallest MNH graphs. We shall need a number of auxiliary results. The first one was proved in [5].

Lemma 1 *[5] If u and v are non-adjacent vertices of $J_k, k \geq 5$, and u_1, u_2 and u_3 are the neighbours of u then each edge $uu_i, i = 1, 2, 3$ lies in a hamiltonian $u - v$ path of J_k.* □

The proof of the following Lemma is trivial, thus we omit it.

Lemma 2 *Let G be a graph and let u and v be vertices of degree 3 in G.*
(1) If $G(v)$ ($G(uv)$) is hamiltonian, then G is hamiltonian as well.
(2) If there is a hamiltonian $x - y$ path in G, where $x, y \neq u$ ($x, y \neq u, v$), then there is a hamiltonian $x - y$ path in $G(v)$ ($G(uv)$) as well. □

Let $d(u, v)$ denote the distance between the vertices u and v.

Lemma 3 *The graphs J_k, $k \geq 5$, k odd, contain hamiltonian $v_{4i+2} - v_{4j+2}$ paths $P(v_{4(i+1)+2}, v_{4(j+1)+2})$ and $P(v_{4i}, v_{4j})$ where $d(v_{4i+2}, v_{4j+2}) \geq 3$, $P(v_{4(i+1)+2}, v_{4(j+1)+2}) = (v_{4i+2}v_{4(i+1)+2} \dots v_{4(j+1)+2}v_{4j+2})$ and $P(v_{4i}, v_{4j}) = (v_{4i+2}v_{4i} \dots v_{4j}v_{4j+2})$.*

Proof. Because of the symmetry of J_k it is sufficient to describe the following hamiltonian $P(v_{4(i+1)+2}, v_{4(j+1)+2})$ and $P(v_{4i}, v_{4j})$ paths with $i = 0$. A list of such paths follows. Indices throughout are to be taken modulo $4k$.

The construction of the path $P_1 = P(v_6, v_{4(j+1)+2})$ if $|\{v_6, v_{10}, \ldots, v_{4(j-1)+2}\}| = 2p+1$, $p \geq 1$, $|\{v_{4(j+1)+2}, v_{4(j+2)+2}, \ldots, v_{4(k-1)+2}\}| = 2m \geq 4$ and $4|2m$: Let $S(i) = v_{4i+1}\ v_{4(i-1)+3}\ v_{4(i-1)}$.
Let $R(i) = v_{4i+1}\ v_{4(i-1)+3}\ v_{4(i-2)+1}\ v_{4(i-2)}\ v_{4(i-2)+2}\ v_{4(i-1)+2}\ v_{4(i-1)}\ v_{4(i-1)+1}\ v_{4(i-2)+3}\ v_{4(i-3)+1}\ v_{4(i-4)+3}\ v_{4(i-4)}\ v_{4(i-4)+2}\ v_{4(i-3)+2}\ v_{4(i-3)}\ v_{4(i-3)+3}$.
Then, $P_1 = (v_2\ v_6 \ldots v_{4(j-1)+2}\ v_{4(j-1)}\ v_{4(j-1)+3}\ v_{4j+1}\ v_{4j}\ v_{4j+3}\ S(j-1)\ S(j-2) \ldots S(1)\ R(0)\ R(k-4) \ldots R(j+9)\ v_{4(j+5)+1}\ v_{4(j+4)+3}\ v_{4(j+3)+1}\ v_{4(j+3)}\ v_{4(j+3)+2}\ v_{4(j+4)+2}\ v_{4(j+4)}\ v_{4(j+4)+1}\ v_{4(j+3)+3}\ v_{4(j+2)+1}\ v_{4(j+1)+3}\ v_{4(j+1)}\ v_{4(j+1)+1}\ v_{4(j+2)+3}\ v_{4(j+2)}\ v_{4(j+2)+2}\ v_{4(j+1)+2}\ v_{4j+2})$.

The construction of the path $P_2 = P(v_6, v_{4(j+1)+2})$ if $|\{v_6, v_{10}, \ldots, v_{4(j-1)+2}\}| = 2p+1$, $p \geq 1$, $|\{v_{4(j+1)+2}, v_{4(j+2)+2}, \ldots, v_{4(k-1)+2}\}| = 2m \geq 2$ and $4 \nmid 2m$: Let $S(i) = v_{4i+1}\ v_{4(i-1)+3}\ v_{4(i-1)}$.
Let $R(i) = v_{4i+1}\ v_{4(i-1)+3}\ v_{4(i-2)+1}\ v_{4(i-2)}\ v_{4(i-2)+2}\ v_{4(i-1)+2}\ v_{4(i-1)}\ v_{4(i-1)+1}\ v_{4(i-2)+3}\ v_{4(i-3)+1}\ v_{4(i-4)+3}\ v_{4(i-4)}\ v_{4(i-4)+2}\ v_{4(i-3)+2}\ v_{4(i-3)}\ v_{4(i-3)+3}$.
Then, $P_2 = (v_2\ v_6 \ldots v_{4(j-1)+2}\ v_{4(j-1)}\ v_{4(j-1)+3}\ v_{4j+1}\ v_{4j}\ v_{4j+3}\ S(j-1)\ S(j-2) \ldots S(1)\ R(0)\ R(k-4) \ldots R(j+7)\ v_{4(j+3)+1}\ v_{4(j+2)+3}\ v_{4(j+1)+1}\ v_{4(j+1)}\ v_{4(j+1)+3}\ v_{4(j+2)+1}\ v_{4(j+2)}\ v_{4(j+2)+2}\ v_{4(j+1)+2}\ v_{4j+2})$.

The construction of the path $P_3 = P(v_6, v_{4(j+1)+2})$ if $|\{v_6, v_{10}, \ldots, v_{4(j-1)+2}\}| = 2p$, $p \geq 2$ and $4 \mid 2p$: Let $S(i) = v_{4i+1}\ v_{4i}\ v_{4i+3}$.
Let $R(i) = v_{4i+1}\ v_{4(i+1)+3}\ v_{4(i+1)}\ v_{4(i+1)+2}\ v_{4i+2}\ v_{4i}\ v_{4i+3}\ v_{4(i+1)+1}\ v_{4(i+2)+3}\ v_{4(i+3)+1}\ v_{4(i+3)}\ v_{4(i+3)+2}\ v_{4(i+2)+2}\ v_{4(i+2)}\ v_{4(i+2)+1}\ v_{4(i+3)+3}$.
Then, $P_3 = (v_2\ v_6\ v_{10}\ v_8\ v_{11}\ v_5\ v_4\ v_7\ v_9\ v_{15}\ v_{17}\ v_{16}\ v_{18}\ v_{14}\ v_{12}\ v_{13}\ v_{19}\ R(22)\ R(38) \ldots R(j-4)\ S(j)\ S(j+1) \ldots S(k-2)\ v_{4(k-1)+1}\ v_3\ v_0\ v_1\ v_{4(k-1)+3}\ v_{4(k-1)}\ v_{4(k-1)+2}\ v_{4(k-2)+2} \cdots v_{4j+2})$.

The construction of the path $P_4 = P(v_6, v_{4(j+1)+2})$ if $|\{v_6, v_{10}, \ldots, v_{4(j-1)+2}\}| = 2p$, $p \geq 1$ and $4 \nmid 2p$: Let $S(i) = v_{4i+1}\ v_{4(i+1)+3}\ v_{4(i+1)}$.
Let $R(i) = v_{4i+1}\ v_{4(i+1)+3}\ v_{4(i+2)+1}\ v_{4(i+2)}\ v_{4(i+2)+2}\ v_{4(i+1)+2}\ v_{4(i+1)}\ v_{4(i+1)+1}\ v_{4(i+2)+3}\ v_{4(i+3)+1}\ v_{4(i+4)+3}\ v_{4(i+4)}\ v_{4(i+4)+2}\ v_{4(i+3)+2}\ v_{4(i+3)}\ v_{4(i+3)+3}$.
Then, $P_4 = (v_2\ v_6\ v_{10}\ v_8\ v_{11}\ v_5\ v_4\ v_7\ R(10)\ R(26) \ldots R(j-5)\ S(j-1)\ S(j) \ldots S(k-3)\ v_{4(k-2)+1}\ v_{4(k-1)+3}\ v_1\ v_0\ v_3\ v_{4(k-1)+1}\ v_{4(k-1)}\ v_{4(k-1)+2}\ v_{4(k-2)+2} \cdots v_{4j+2})$.

The construction of the path $P_5 = P(v_0, v_{4j})$ if $|\{v_6, v_{10}, \ldots, v_{4(j-1)+2}\}| = 2p$, $p \geq 2$ and $4 \mid 2p$: Let $S(i) = v_{4i+3}\ v_{4(i-1)+1}\ v_{4(i-1)}$.

Let $R(i) = v_{4i+1}\ v_{4(i-1)+3}\ v_{4(i-2)+1}\ v_{4(i-2)}\ v_{4(i-2)+2}\ v_{4(i-1)+2}\ v_{4(i-1)}\ v_{4(i-1)+1}\ v_{4(i-2)+3}\ v_{4(i-3)+1}\ v_{4(i-4)+3}\ v_{4(i-4)}\ v_{4(i-4)+2}\ v_{4(i-3)+2}\ v_{4(i-3)}\ v_{4(i-3)+3}$.
Then, $P_5 = (v_2\ v_0\ v_3\ v_{4(k-1)+1}\ v_{4(k-1)}\ v_{4(k-1)+2}\ v_{4(k-2)+2} \cdots v_{4(j+1)+2}\ v_{4(j+1)}\ v_{4(j+1)+3}\ R(j)\ R(j-4) \ldots R(38)\ v_{21}\ v_{19}\ v_{13}\ v_{12}\ v_{15}\ v_{17}\ v_{16}\ v_{18}\ v_{14}\ v_{10}\ v_6\ v_4\ v_5\ v_{11}\ v_8\ v_9\ v_7\ v_1\ S(k-1)\ S(k-2) \ldots S(j+3)\ v_{4(j+2)+3}\ v_{4(j+1)+1}\ v_{4j+3}\ v_{4j}\ v_{4j+2})$.

The construction of the path $P_6 = P(v_0, v_{4j})$ if $|\{v_6, v_{10}, \ldots, v_{4(j-1)+2}\}| = 2p$, $p \geq 1$ and $4 \not| 2p$: Let $S(i) = v_{4i+3}\ v_{4(i-1)+1}\ v_{4(i-1)}$.
Let $R(i) = v_{4i+1}\ v_{4(i-1)+3}\ v_{4(i-2)+1}\ v_{4(i-2)}\ v_{4(i-2)+2}\ v_{4(i-1)+2}\ v_{4(i-1)}\ v_{4(i-1)+1}\ v_{4(i-2)+3}\ v_{4(i-3)+1}\ v_{4(i-4)+3}\ v_{4(i-4)}\ v_{4(i-4)+2}\ v_{4(i-3)+2}\ v_{4(i-3)}\ v_{4(i-3)+3}$.
Then, $P_6 = (v_2\ v_0\ v_3\ v_{4(k-1)+1}\ v_{4(k-1)}\ v_{4(k-1)+2}\ v_{4(k-2)+2} \cdots v_{4(j+1)+2}\ v_{4(j+1)}\ v_{4(j+1)+3}\ R(j)\ R(j-4) \ldots R(30)\ v_{13}\ v_{11}\ v_5\ v_4\ v_6\ v_{10}\ v_8\ v_9\ v_7\ v_1\ S(k-1)\ S(k-2) \ldots S(j-3)\ v_{4(j-2)+3}\ v_{4(j-1)+1}\ v_{4j+3}\ v_{4j}\ v_{4j+2})$. □

Lemma 4 *The graphs J_k, $k \geq 5$, k odd, contain hamiltonian $v_{4i} - v_{4j+2}$ paths $P(v_{4i+2}, v_{4(j+1)+2})$ and $P(v_{4i+2}, v_{4(j-1)+2})$ where $d(v_{4i}, v_{4j+2}) \geq 4$, $P(v_{4i+2}, v_{4(j+1)+2}) = (v_{4i}v_{4i+2} \ldots v_{4(j+1)+2}v_{4j+2})$ and $P(v_{4i+2}, v_{4(j-1)+2}) = (v_{4i}v_{4i+2} \ldots v_{4(j-1)+2}v_{4j+2})$.*

Proof. Because of the symmetry of J_k it is sufficient to describe the following hamiltonian $P(v_{4i+2}, v_{4(j+1)+2})$ paths with $i = 0$. A list of such paths follows. Indices throughout are to be taken modulo $4k$.

The construction of the path $P_1 = P(v_2, v_{4(j+1)+2})$ if $|\{v_6, v_{10}, \ldots, v_{4(j-1)+2}\}| = 2p+1$, $p \geq 1$ and $4 \mid 2p+2$: Let $S(i) = v_{4i+3}\ v_{4(i+1)+1}\ v_{4(i+1)}$.
Let $R(i) = v_{4i+2}\ v_{4(i+1)+2}\ v_{4(i+1)}\ v_{4(i+1)+3}\ v_{4i+1}\ v_{4i}\ v_{4i+3}\ v_{4(i+1)+1}\ v_{4(i+2)+3}\ v_{4(i+3)+1}\ v_{4(i+3)}\ v_{4(i+3)+3}\ v_{4(i+2)+1}\ v_{4(i+2)}\ v_{4(i+2)+2}\ v_{4(i+3)+2}$.
Then, $P_1 = (v_0\ v_2\ v_6\ R(10)\ R(14) \ldots R(j-6)\ v_{4(j-2)+2}\ v_{4(j-1)+2}\ v_{4(j-1)}\ v_{4(j-1)+1}\ v_{4(j-2)+3}\ v_{4(j-2)}\ v_{4(j-2)+1}\ S(j-1)\ S(j) \ldots S(k-3)\ v_{4(k-2)+3}\ v_{4(k-1)+1}\ v_3\ v_5\ v_4\ v_7\ v_1\ v_{4(k-1)+3}\ v_{4(k-1)}\ v_{4(k-1)+2}\ v_{4(k-2)+2} \cdots v_{4j+2})$.

The construction of the path $P_2 = P(v_2, v_{4(j+1)+2})$ if $|\{v_6, v_{10}, \ldots, v_{4(j-1)+2}\}| = 2p+1$, $p \geq 2$ and $4 \not| 2p+2$: Let $S(i) = v_{4i+3}\ v_{4(i+1)+1}\ v_{4(i+1)}$.
Let $R(i) = v_{4i+2}\ v_{4(i+1)+2}\ v_{4(i+1)}\ v_{4(i+1)+3}\ v_{4i+1}\ v_{4i}\ v_{4i+3}\ v_{4(i+1)+1}\ v_{4(i+2)+3}\ v_{4(i+3)+1}\ v_{4(i+3)}\ v_{4(i+3)+3}\ v_{4(i+2)+1}\ v_{4(i+2)}\ v_{4(i+2)+2}\ v_{4(i+3)+2}$.
Then, $P_2 = (v_0\ v_2\ v_6\ R(10)\ R(14) \ldots R(j-8)\ v_{4(j-4)+2}\ v_{4(j-3)+2}\ v_{4(j-3)}\ v_{4(j-3)+3}\ v_{4(j-4)+1}\ v_{4(j-4)}\ v_{4(j-4)+3}\ v_{4(j-3)+1}\ v_{4(j-2)+3}\ v_{4(j-1)+1}\ v_{4(j-1)}\ v_{4(j-1)+2}\ v_{4(j-2)+2}\ v_{4(j-2)}\ v_{4(j-2)+1}\ S(j-1)\ S(j) \ldots S(k-3)\ v_{4(k-2)+3}\ v_{4(k-1)+1}\ v_3\ v_5\ v_4\ v_7\ v_1\ v_{4(k-1)+3}\ v_{4(k-1)}\ v_{4(k-1)+2}\ v_{4(k-2)+2} \cdots v_{4j+2})$.

The construction of the path $P_3 = P(v_2, v_{4(j+1)+2})$ if $|\{v_6, v_{10}, \ldots, v_{4(j-1)+2}\}| = 2p \geq 2$, $|\{v_{4(j+1)+2}, v_{4(j+2)+2}, \ldots, v_{4(k-1)+2}\}| = 2m+1 \geq 3$ and $4|2m+2$: Let $S(i) = v_{4i+3}\ v_{4(i-1)+1}\ v_{4(i-1)}$.

Let $R(i) = v_{4i+1}\ v_{4(i-1)+3}\ v_{4(i-2)+1}\ v_{4(i-2)}\ v_{4(i-2)+2}\ v_{4(i-1)+2}\ v_{4(i-1)}\ v_{4(i-1)+1}\ v_{4(i-2)+3}\ v_{4(i-3)+1}\ v_{4(i-4)+3}\ v_{4(i-4)}\ v_{4(i-4)+2}\ v_{4(i-3)+2}\ v_{4(i-3)}\ v_{4(i-3)+3}$.
Then, $P_3 = (v_0\ v_2\ v_{4(k-1)+2}\ v_{4(k-1)}\ v_{4(k-1)+3}\ v_1\ v_7\ v_4\ v_6\ v_{10} \dots v_{4(j-1)+2}\ v_{4(j-2)}\ v_{4(j-2)+1}\ v_{4j+3}\ v_{4j}\ v_{4j+1}\ S(j-1)\ S(j-2) \dots S(14)\ v_{11}\ v_5\ v_3\ R(k-1)\ R(k-5) \dots R(j+7)\ v_{4(j+3)+1}\ v_{4(j+2)+3}\ v_{4(j+1)+1}\ v_{4(j+1)}\ v_{4(j+1)+3}\ v_{4(j+2)+1}\ v_{4(j+2)}\ v_{4(j+2)+2}\ v_{4(j+1)+2}\ v_{4j+2})$.

The construction of the path $P_4 = P(v_2, v_{4(j+1)+2})$ if $|\{v_6, v_{10}, \dots, v_{4(j-1)+2}\}| = 2p \geq 2$, $|\{v_{4(j+1)+2}, v_{4(j+2)+2}, \dots, v_{4(k-1)+2}\}| = 2m+1 \geq 5$ and $4 \not| 2m+2$: Let $S(i) = v_{4i+3}\ v_{4(i-1)+1}\ v_{4(i-1)}$.
Let $R(i) = v_{4i+1}\ v_{4(i-1)+3}\ v_{4(i-2)+1}\ v_{4(i-2)}\ v_{4(i-2)+2}\ v_{4(i-1)+2}\ v_{4(i-1)}\ v_{4(i-1)+1}\ v_{4(i-2)+3}\ v_{4(i-3)+1}\ v_{4(i-4)+3}\ v_{4(i-4)}\ v_{4(i-4)+2}\ v_{4(i-3)+2}\ v_{4(i-3)}\ v_{4(i-3)+3}$.
Then, $P_4 = (v_0\ v_2\ v_{4(k-1)+2}\ v_{4(k-1)}\ v_{4(k-1)+3}\ v_1\ v_7\ v_4\ v_6\ v_{10} \dots v_{4(j-1)+2}\ v_{4(j-2)}\ v_{4(j-2)+1}\ v_{4j+3}\ v_{4j}\ v_{4j+1}\ S(j-1)\ S(j-2) \dots S(14)\ v_{11}\ v_5\ v_3\ v_{4(k-1)+1}\ v_{4(k-2)+3}\ v_{4(k-3)+1}\ v_{4(k-3)}\ v_{4(k-3)+2}\ v_{4(k-2)+2}\ v_{4(k-2)}\ R(k-2)\ R(k-6) \dots R(j+8)\ v_{4(j+4)+1}\ v_{4(j+3)+3}\ v_{4(j+2)+1}\ v_{4(j+1)+3}\ v_{4(j+1)}\ v_{4(j+1)+1}\ v_{4(j+2)+3}\ v_{4(j+2)}\ v_{4(j+2)+2}\ v_{4(j+1)+2}\ v_{4j+2})$. $\square$

3 Main Results

Proposition 1 *The graph $J_k(v_{4i_1+2}, v_{4i_2+2}, \dots, v_{4i_m+2})$ is MNH for $k \geq 5$, k odd, where $m \geq 1$, $0 \leq i_l \leq k-1$ for $l = 1, \dots, m$ and $d(v_{4i_l+2}, v_{4i_p+2}) \geq 3$ for $l \neq p$.*

Proof. In [5] it was shown that the graph $J_k(v_2)$ is MNH for $k \geq 5$. From the symmetry of J_k we have that $J_k(v_{4i+2})$ is MNH for odd $k \geq 5$ and $i = 0, 1, \dots, k-1$.

Now we show that $J_k(v_{4i+2}, v_{4j+2})$ is MNH where $k \geq 5$ is odd and $d(v_{4i+2}, v_{4j+2}) \geq 3$. Let the vertex v_{4l+2} be expanded to the triangle $T_l = (u_l^1 u_l^2 u_l^3)$ for $l = i, j$. It follows from Lemma 2 that this graph is not hamiltonian. Because of the symmetry of J_k, $J_k(v_{4i+2})$, $J_k(v_{4j+2})$ it suffices to show that there are hamiltonian paths joining each of the following pairs of vertices of $J_k(v_{4i+2}, v_{4j+2})$ (see Figure 2).

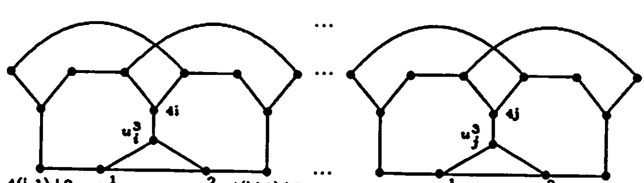

Figure 2.

(i) u_i^1 and v_{4i}, u_i^1 and $v_{4(i+1)+2}$, u_i^1 and w where $w \notin \{v_{4i}, v_{4(i+1)+2}, v_{4(i-1)+2}, u_i^2, u_i^3, u_j^1, u_j^2, u_j^3\}$, (use Lemma 1),

(ii) u_i^1 and u_j^p, $1 \leq p \leq 3$, (use Lemma 3),

(iii) u_i^3 and $v_{4(i+1)+2}$, u_i^3 and w where $w \notin \{v_{4i}, v_{4(i+1)+2}, v_{4(i-1)+2}, u_i^1, u_i^2, u_j^1, u_j^2, u_j^3\}$, (use Lemma 1),

(iv) u_i^3 and u_j^1, u_i^3 and u_j^3, (use Lemma 3),

(v) u_i^2 and u_j^1, (use Lemma 3),

(vi) u and w where u and w are non-adjacent vertices both different from u_i^l and u_j^l for $l = 1, 2, 3$, (use Lemma 1).

Finding these paths using Lemmas 1-3 is trivial but time-consuming exercise which we leave to the reader.

Assume that each $J_k(v_{4i_1+2}, v_{4i_2+2}, \ldots, v_{4i_n+2})$ is MNH where $n \leq m - 1 < \lfloor \frac{k}{2} \rfloor$. Let $G = J_k(v_{4i_1+2}, v_{4i_2+2}, \ldots, v_{4i_m+2})$ where $k \geq 5$ and $d(v_{4i_p+2}, v_{4i_q+2}) \geq 3$ for all $p \neq q$. The graph G is non-hamiltonian by Lemma 2. Let x, y be two non-adjacent vertices of G. Since $m \geq 3$, there is at least one l such that $1 \leq l \leq m$ and $x, y \neq u_l^1, u_l^2, u_l^3$ (these are the vertices of the triangle T_l by which the vertex v_{4i_l+2} is expanded). Let $G' = J_k(v_{4i_1+2}, v_{4i_2+2}, \ldots, v_{4i_{l-1}+2}, v_{4i_{l+1}+2}, \ldots, v_{4i_m+2})$. From the induction hypothesis there is a hamiltonian $x - y$ path in G', and Lemma 2 guaranties a hamiltonian $x - y$ path in G.

The proof is complete. □

Proposition 2 *The graph* $J_k(v_{4i_1+2}, v_{4i_2+2}, \ldots, v_{4i_m+2})(v_{4i_{m+1}+2}v_{4i_{m+1}})$ *is MNH for* $k \geq 5$, *k odd, where* $m \geq 0$, $0 \leq i_l \leq k-1$ *for* $l = 1, \ldots, m+1$ *and* $d(v_{4i_l+2}, v_{4i_p+2}) \geq 3$ *for* $l \neq p$.

Proof. First we show that the graph $J_k(v_{4i+2}v_{4i})$ is MNH for $k \geq 5$ and $0 \leq i \leq k-1$. From the symmetry of J_k we can assume that $i = 0$ and that the edge v_2v_0 is replaced by the bowtie on the vertex set $\{u_1, \ldots, u_5\}$ (see Figure 3). From Lemma 2, this graph is non-hamiltonian. Because of the symmetry of J_k it suffices to show that there are hamiltonian paths joining each of the following pairs of vertices of $J_k(v_2v_0)$ (see Figure 3):

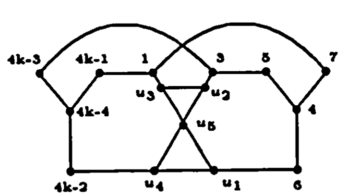

Figure 3.

(i) u_4 and v_6, u_4 and u_3, u_4 and u_2, u_4 and w where $w \notin \{v_6, v_{4k-2}, u_1, u_2, u_3, u_5\}$, (use Lemma 1),

(ii) u_5 and v_1, u_5 and v_6, u_5 and w where $w \notin \{v_1, v_3, v_6, v_{4k-2}, u_1, u_2, u_3, u_4\}$, (use Lemma 1),

(iii) u_3 and v_3, u_3 and w where $w \notin \{v_1, v_3, u_1, u_2, u_4, u_5\}$, (use Lemma 1),

(iv) u and w where u and w are non-adjacent vertices both different from u_l for $l = 1, \ldots, 5$ (use Lemma 1).

Now we show that the graph $J_k(v_{4i+2})(v_{4j+2}v_{4j})$, where $d(v_{4i+2}, v_{4j+2}) \geq 3$ is MNH for $k \geq 5$. By Lemma 2 this graph is non-hamiltonian. From the symmetry of J_k we can assume that $j = 0$. Further assume that the edge v_2v_0 is replaced by the bowtie on the vertex set $\{u_1, \ldots, u_5\}$ and that the vertex v_{4i+2} is expanded to the triangle $T = (u_6u_7u_8)$ (see Figure 4). Because of the symmetry of J_k, $J_k(v_{4i+2})$ and $J_k(v_{4j+2}v_{4j})$ it suffices to show that there are hamiltonian paths joining each of the following pairs of vertices of $J_k(v_{4i+2})(v_2v_0)$.

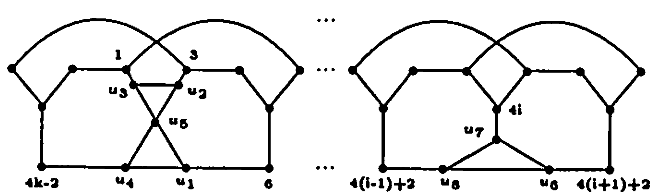

Figure 4.

(i) u_4 and v_6, u_4 and u_3, u_4 and u_2, u_4 and w where $w \notin \{v_6, v_{4k-2}, u_1, u_2, u_3, u_5, u_6, u_7, u_8\}$, (use Lemma 1),

(ii) u_4 and u_6, u_4 and u_7, u_4 and u_8, (use Lemma 3),

(iii) u_5 and v_1, u_5 and w where $w \notin \{v_1, v_3, u_1, u_2, u_3, u_4, u_6, u_7, u_8\}$, (use Lemma 1),

(iv) u_5 and u_6, u_5 and u_7, u_5 and u_8, (use Lemma 3),

(v) u_3 and v_3, u_3 and w where $w \notin \{v_1, v_3, u_1, u_2, u_4, u_5, u_6, u_7, u_8\}$, (use Lemma 1),

(vi) u_3 and u_6, u_3 and u_7, u_3 and u_8, (use Lemma 4),

(vii) u_1 and u_6, u_1 and u_7, u_1 and u_8, (use Lemma 3),

(viii) u_2 and u_6, u_2 and u_7, u_2 and u_8, (use Lemma 4),

(ix) u_8 and v_{4i}, u_8 and $v_{4(i+1)+2}$, u_8 and w where $w \notin \{v_{4(i-1)+2}, v_{4i}, v_{4(i+1)+2}, u_i, i = 1, \ldots, 7\}$, (use Lemma 1),

(x) u_7 and $v_{4(i+1)+2}$, u_7 and w where $w \notin \{v_{4(i-1)+2}, v_{4i}, v_{4(i+1)+2}, u_8, u_i, i = 1, \ldots, 6\}$, (use Lemma 1),

(xi) u and w where u and w are non-adjacent vertices both different from u_i, $i = 1, \ldots, 8$ (use Lemma 1).

Again, finding these paths using Lemmas 1-4 is trivial but time-consuming exercise which we leave to the reader. The rest of the proof can be handled similarly to the corresponding part of the previous proof. □

We are ready to state and prove the main result.

Theorem 1 *For all $n = 8a + b \geq 88$, $0 \leq b \leq 7$, there exist at least*

$$\tau(n) = \sum_{j=0}^{\lfloor \frac{k-2-3Q(b)}{14} \rfloor} p'(k - 2 - Q(b) - 6j, Q(b) + 4j) \geq 3$$

non-isomorphic smallest MNH graphs, where $k = 2a + 1$.

Proof. For $j = 0, \ldots, \lfloor \frac{k-2-3Q(b)}{14} \rfloor$, it holds that $k - 2 - 2j \geq 5$, and then by Propositions 1 and 2 there are MNH graphs

for $b = 2m$ where $m = 0, 1, 2, 3$

$J_{k-2-2j}(v_{4i_1+2}, v_{4i_2+2}, \ldots, v_{4i_{4j+m+2}+2})$,

for $b = 2m + 1$ where $m = 0, 1$

$J_{k-2-2j}(v_{4i_1+2}, v_{4i_2+2}, \ldots, v_{4i_{4j+m+1}+2})(v_{4i_{4j+m+2}+2} v_{4i_{4j+m+2}})$,

for $b = 2m + 1$ where $m = 2, 3$

$J_{k-2-2j}(v_{4i_1+2}, v_{4i_2+2}, \ldots, v_{4i_{4j+m+2}+2})(v_{4i_{4j+m+3}+2} v_{4i_{4j+m+3}})$.

The order of these graphs is n and the size is $\lceil \frac{3n}{2} \rceil$, thus all these graphs are smallest MNH. It is a matter of routine to observe that for $j = 0, \ldots, \lfloor \frac{k-2-3Q(b)}{14} \rfloor$, for each partition of $k - 2 - 6j - Q(b)$ into $4j + Q(b)$ parts where each summand is at least 2, there is at least one location for the $4j + Q(b)$ vertices (which are replaced by a triangle in J_{k-2-2j}) - the distances between these vertices along the cycle $(v_2 v_6 \ldots v_{4(k-2-2j-1)+2} v_2)$ will be equal to the values of the $4j + Q(b)$ summands of the partition of $k - 2 - 6j - Q(b)$. Thus, there exist at least $\tau(n) = \sum_{j=0}^{\lfloor \frac{k-2-3Q(b)}{14} \rfloor} p'(k - 2 - Q(b) - 6j,$

$Q(b) + 4j)$ non-isomorphic smallest MNH graphs of order n. Since $k \geq 23$, and since $p'(15, 6) = 3$, $\tau(n) \geq 3$. Obviously, $\lim_{n\to\infty} \tau(n) = \infty$. □

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