The Polynomial Model in the Study of Counterexamples to S. Piccard's Theorem

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ABSTRACT. The search for homometric structures, i.e., non-congruent structures sharing the same autocorrelation function, is shown to be of a combinatorial nature and can be studied using purely algebraic techniques. Several results on the existence of certain homometric structures which contradict a theorem by S. Piccard are proved based on a polynomial representation model and the factorization of polynomials over the rationals. Combinatorial arguments show that certain factorizations do not lead to counterexamples to S. Piccard's theorem.

1 Introduction

A *ruler with n marks*, R, is a set of n non-negative integers $0 = r_1 < r_2 < r_3 \ldots < r_n = N$. The *distances measured by R* are all positive integers of the form $r_j - r_i$ with $i < j$, counting multiplicity. This multiset of n(n-1)/2 elements is denoted ΔR. Two n-mark rulers, R and S, are *homometric* iff $\Delta R = \Delta S$. Let $R = \{r_i\}$, $i = 1, 2, \ldots, n$, and let $S = \{s_i\}$, $i = 1, 2, \ldots, n$. R and S are *distinct* rulers unless either $r_i = s_i$ for all i, or $r_i = s_n - s_i$ for all i. R is a *spanning ruler* iff all $n(n-1)/2$ elements of ΔR are distinct.

S. Piccard's 'theorem' [1] asserts that two homometric n-mark spanning rulers cannot be distinct. Bloom [2] noted that $R = (0, 1, 4, 10, 12, 17)$, $S = (0, 1, 8, 11, 13, 17)$ is a counterexample to this 'theorem', and in [3] two different two-parameter families of counterexamples were presented, all involving six-mark rulers. Another family of counterexamples was reported in [4] which covers both previously known families. Moreover, this family was shown to be the *unique* family of counterexamples in the case of six mark rulers. No counterexamples to S. Piccard's 'theorem' have been found for any $n \neq 6$, and it is possible that none exist. We give an analytical proof of the nonexistence of exceptions to S. Piccard's 'theorem' with $n < 6$. Exhaustive search has proven that there are no other counterexamples with rulers having fewer than 13 marks [5].

In this paper, we associate with the ruler $R = (r_i)$ the polynomial $r(x) = \sum_{i=1}^{n} x^{r_i}$, and for any polynomial p(x) of degree N, we define $p^*(x) = x^N p\left(\frac{1}{x}\right)$. Letting $R(x) = r(x)r^*(x)$, the rulers R and S are homometric iff R(x) = S(x), and R is a spanning ruler iff all coefficients of $R(x)$ except the coefficient of x^N (the 'middle coefficient') are restricted to the values 0 and 1. The theory of possible counterexamples to S. Piccard's 'theorem' is developed in terms of the factorizations of the polynomials $R(x)$ and $S(x)$ over the field of rational numbers. Certain factorizations are shown not to lead to counterexamples to S. Piccard's theorem.

2 The Polynomial Model

Let $R = (r_i)$ and $S = (s_i)$ be n-mark homometric rulers of length N. Thus $r_1 = s_1 = 0$ and $r_n = s_n = N$, and with $r(x) = \sum_{i=1}^{n} x^{r_i}$, $s(x) = \sum_{i=1}^{n} x^{s_i}$, we have $R(x) = S(x)$, where $R(x) = r(x)r^*(x)$, $S(x) = s(x)s^*(x)$, with $r^*(x) = x^N r(\frac{1}{x})$ and $s^*(x) = x^N s(\frac{1}{x})$. The trivial solutions to $r(x)r^*(x) = s(x)s^*(x)$ are $r(x) = s(x)$, the case of *identical* rulers, and $r(x) = s^*(x)$, the case of *mirror image* rulers. By the unique factorization theorem for polynomials over the rationals, any *nontrivial* solution, corresponding to *distinct* homometric rulers, will correspond to

$$(r(x), s(x)) = \phi_1(x), \text{ with } 0 < deg\phi_1(x) < degr(x).$$

Let $r(x) = \phi_1(x)\phi_2(x)$, where also $0 < deg\phi_2(x) < degr(x)$, then

$$r(x)r^*(x) = \phi_1(x)\phi_2(x)\phi_1^*(x)\phi_2^*(x) = s(x)s^*(x)$$

which forces $s(x) = \phi_1(x)\phi_2^*(x)$.

Remark 1 *Since $n = r(1) = s(1) = \phi_1(1)\phi_2(1)$, $\phi_1(1), \phi_2(1)$ are integer factors of n (the number of marks), both positive or both negative.*

If we could show that neither $\phi_1(1)$ nor $\phi_2(1)$ can have the values ± 1, we would have a proof that n must be *composite* in all counterexamples to S. Piccard's 'theorem'.

Since all coefficients of $r(x)$ are either 0 or 1, and $r(x) = \phi_1(x)\phi_2(x)$, all coefficients of $\phi_1(x)$ and $\phi_2(x)$ are integers. However, we know from the factorization of $1 + x + x^2 + \ldots + x^{n-1} = \prod_{1<d|n} \Phi_d(x)$ into cyclotomic

polynomials, that without additional assumptions, the coefficients of the factors are not limited to $\{0, +1, -1\}$. Since also $s(x) = \phi_1(x)\phi_2^*(x)$, the *first* and *last* coefficients of both $\phi_1(x)$ and $\phi_2(x)$ are $+1$. The only alternative is that all four of these coefficients are -1, in which case we replace ϕ_1 and ϕ_2 by $-\phi_1$ and $-\phi_2$.

It is easy to generate homometric pairs of rulers which do *not* violate S. Piccard's 'theorem' because they each measure certain distances in more than one way. For example, we get a pair of homometric 9-mark rulers by taking

$$\begin{aligned} r(x) &= \phi_1(x)\phi_2(x) = (1 + x^a + x^b)(1 + x^c + x^d) \\ s(x) &= \phi_1(x)\phi_2^*(x) = (1 + x^a + x^b)(1 + x^{d-c} + x^d) \end{aligned}$$

where we must be careful to pick a, b, c, d so that $r(x) \neq s(x) \neq r^*(x)$ and so that all exponents in $r(x)$ are distinct, and all exponents in $s(x)$ are distinct. In particular, with $a = 1,\ b = 3,\ c = 4,\ d = 9$, we get the homometric rulers

$$\begin{aligned} r(x) &= 1 + x + x^3 + x^4 + x^5 + x^7 + x^9 + x^{10} + x^{12} \\ s(x) &= 1 + x + x^3 + x^5 + x^6 + x^8 + x^9 + x^{10} + x^{12} \end{aligned}$$

It can readily be checked that $R(x) = S(x)$.

With $r(x) = \phi_1(x)\phi_2(x)$ and $s(x) = \phi_1(x)\phi_2^*(x)$, we cannot have $\phi_2(x) = \phi_2^*(x)$ since this would violate $(r(x), s(x)) = \phi_1(x)$. Also, the homometric rulers will not be *distinct* if $\phi_1(x) = \phi_1^*(x)$, since in that case $r^*(x) = \phi_1^*(x)\phi_2^*(x) = \phi_1(x)\phi_2^*(x) = s(x)$. Neither $\phi_1(x)$ nor $\phi_2(x)$ can be a monomial or a binomial, for $\phi_i(x) = 1$ has $\phi_i^*(x) = \phi_i(x)$, and $\phi_i(x) = 1 + x^k$ also has $\phi_i^*(x) = d_i(x)$. Thus we have shown,

Theorem 1 *If $r(x) = \phi_1(x)\phi_2(x)$ and $s(x) = \phi_1(x)\phi_2^*(x)$ correspond to distinct homometric rulers, then both $\phi_1(x)$ and $\phi_2(x)$ must be polynomials of at least* 3 *terms each, with all integer coefficients, with highest and lowest coefficients equal to* $+1$, *and with $\phi_1(x) \neq \phi_1^*(x)$ and $\phi_2(x) \neq \phi_2^*(x)$.*

Next we show

Theorem 2 *If R and S are homometric rulers which violate S. Piccard's 'theorem', with $r(x) = \phi_1(x)\phi_2(x)$ and $s(x) = \phi_1(x)\phi_2^*(x)$, then some coefficient(s) of either $\phi_1(x)$ or $\phi_2(x)$ (or both) must be negative.*

Proof. Suppose the contrary, namely that all non-zero coefficients of both $\phi_1(x)$ and $\phi_2(x)$ are positive. Writing

$$r(x) = \phi_1(x)\phi_2(x) = (1 + x^a + \ldots)(1 + x^b + \ldots) = 1 + x^a + x^b + x^{a+b} + \ldots$$

we observe that no later terms can cancel any of these four, since there will be no negative terms. But both $b = b - 0 = (a+b) - a$ and $a = a - 0 = (a+b) - b$ are distances, i.e., differences of exponents, which are measured in more than one way, violating a hypothesis in the statement of S. Piccard's 'theorem'.

Note that if $r(x)$, the product of $\phi_1(x)$ and $\phi_2(x)$, has many terms, i.e., many marks on the ruler R, it becomes extremely difficult to avoid repeated differences of exponents, i.e., repeated elements in R. It is this phenomenon which *may* be the basis of a proof that there are no counterexamples to S. Piccard's 'theorem' for rulers with more than six marks.

Another necessary condition is the following,

Lemma 1 *Let*

$$\begin{aligned} \phi_1(x) &= 1 \pm x^{a_1} \pm x^{a_2} \ldots + x^{a_m}, \quad \deg\phi_1(x) = a_m \\ \phi_2(x) &= 1 \pm x^{b_1} \pm x^{b_2} \ldots + x^{b_m}, \quad \deg\phi_2(x) = b_m \end{aligned}$$

be the two factors of a ruler $f(x) = \phi_1(x)\phi_2(x)$. Then a necessary condition for $f(x)$ to be a spanning ruler is for at least one of the terms x^{a_m}, x^{b_m} to be cancelled out from the product $\phi_1(x)\phi_2(x)$.

Proof. Assume the contrary, namely that both terms remain in the product. Then $f(x)$ appears as $f(x) = 1 + \ldots + x^{a_m} + \ldots + x^{b_m} + \ldots + x^{a_m+b_m}$, but then, $a_m - 0 = (a_m + b_m) - b_m$, $b_m - 0 = (a_m + b_m) - a_m$, a contradiction.

3 The Known Counterexamples

Two 2-parameter families of counterexamples to S. Piccard's 'theorem' were previously known [3]. In our polynomial model these families can be represented as follows,

<u>FAMILY I</u>

$$\begin{aligned} r(x) &= \phi_1(x)\phi_2(x) = 1 + x^u + x^{u+v} + x^{4u+2v} + x^{6u+2v} + x^{8u+3v} \\ s(x) &= \phi_1(x)\phi_2^*(x) = 1 + x^u + x^{5u+v} + x^{5u+2v} + x^{7u+2v} + x^{8u+3v} \end{aligned}$$

where $\phi_1(x) = 1 + x^u + x^{3u+v}$, $\phi_2(x) = 1 + x^{u+v} - x^{2u+v} + x^{5u+2v}$. The minimum-length counterexample occurs with $u = 1, v = 3$, and length $= 8u + 3v = 17$. Specifically,

$$\begin{aligned} r(x) &= (1 + x + x^6)(1 + x^4 - x^5 + x^{11}) = 1 + x + x^4 + x^{10} + x^{12} + x^{17} \\ s(x) &= (1 + x + x^6)(1 - x^6 + x^7 + x^{11}) = 1 + x + x^8 + x^{11} + x^{13} + x^{17} \end{aligned}$$

FAMILY II

$$\begin{aligned} r(x) &= \phi_1(x)\phi_2(x) = 1 + x^s + x^{s+t} + x^{4s+2t} + x^{6s+4t} + x^{8s+5t} \\ s(x) &= \phi_1(x)\phi_2^*(x) = 1 + x^{s+t} + x^{5s+3t} + x^{5s+4t} + x^{7s+5t} + x^{8s+5t} \end{aligned}$$

where $\phi_1(x) = 1 + x^{s+t} + x^{3s+2t}$, $\phi_2(x) = 1 + x^s - x^{2s+t} + x^{5s+3t}$. The minimum-length counterexample for this case occurs with $s = 1, t = 2$ and length $8s + 5t = 18$. Specifically,

$$\begin{aligned} r(x) &= (1 + x^3 + x^7)(1 + x - x^4 + x^{11}) = 1 + x + x^3 + x^8 + x^{14} + x^{18} \\ s(x) &= (1 + x^3 + x^7)(1 - x^7 + x^{10} + x^{11}) = 1 + x^3 + x^{11} + x^{13} + x^{17} + x^{18} \end{aligned}$$

It has been shown, though, that the above two families are not distinct. Family I can be transformed into Family II under the linear transformation

$$\begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \Leftrightarrow \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix}$$

Yovanof [4] proved that both of the previous two families can be expressed in terms of a single family of counterexamples, which was, moreover, shown to be the *unique* family of counterexamples in the case of six mark rulers. This unique family is given by

FAMILY N

$$\begin{aligned} r(x) &= \phi_1(x)\phi_2(x) = 1 + x^a + x^{b+2a} + x^{2b-a} + x^{2b+a} + x^{3b-a} \\ s(x) &= \phi_1(x)\phi_2^*(x) = 1 + x^a + x^{b-2a} + x^{2b-2a} + x^{2b} + x^{3b-a} \end{aligned}$$

where, $a, b \in \mathcal{Z}^+$, $\phi_1(x) = 1+x^a+x^b$, and $\phi_2(x) = 1+x^{b-2a}-x^{b-a}+x^{2b-a}$. Substituting $a = 1$, $b = 6$ in the above family we get the first counterexample found by Bloom [2] of length 17 with $R = (0, 1, 8, 11, 13, 1, 7)$ and $S = (0, 1, 4, 10, 12, 17)$.

Notice that all these counterexamples involve the product of a trinomial $\phi_1(x)$ with three positive terms, and a quadrinomial $\phi_2(x)$ with three positive and one negative terms. The product then has nine positive terms, three of which cancel with the three negative terms, in such a way that among the surviving six terms there are no repeated differences. This cancellation is so remarkable that it appears likely that when $r(x) = \phi_1(x)\phi_2(x)$ has more than six surviving terms, there will be repeated differences of these terms. If this could be proved, it would show that S. Piccard's 'theorem' has no counterexamples with rulers having more than six marks. The basic polynomial method used here to study homometric rulers, but without reference to its possible application to S. Piccard's 'theorem', is also described in [6].

4 No Counterexamples with Fewer Than 6 Marks

Bloom [7] proved the following theorem.

Theorem 3 *There are no homometric spanning rulers with fewer than six marks on each ruler which violate S. Piccard's 'theorem'.*

It is relatively easy to prove this theorem in the case of rulers with fewer than five marks. For the sake of completeness we present here a simplified proof of the nonexistence of counterexamples with rulers having exactly five marks.

Theorem 4 *If two five-mark rulers measure the same set of distances, and all 10 measured distances are distinct, then the rulers are either identical, or mirror images of each other.*

Proof. Let R and R' be two five-mark rulers which measure the same set of ten distinct differences. Let the distances between consecutive marks, in order, be a, b, c, d on R and a', b', c', d' on R'. Without loss of generality we may assume $a < d$ and $a' < d'$. A ruler is commonly represented in the form of an array, in fact a triangle, which is called the *difference triangle*. The difference triangles for the two rulers are,

```
            R                                R'
    ----------------           ------------------------
    a    b    c    d           a'      b'      c'      d'
     a+b  b+c  c+d               a'+b'   b'+c'   c'+d'
      a+b+c b+c+d                  a'+b'+c' b'+c'+d'
        a+b+c+d                       a'+b'+c'+d'
```

The greatest distance measured by each ruler is its total length L, and since this must be the same for both rulers, $L = a+b+c+d = a'+b'+c'+d'$. Since both rulers measure the same set of ten distances, the sum S of all ten distances must be the same for both rulers, $\sum = 4(a+d) + 6(c+d) = 4(a'+d') + 6(c'+d')$. From these two equations we have $a + d = a' + d'$, and $c + d = c' + d'$.

The second-longest distance measured by R must be either $L-a$ or $L-d$, and since $a < d$, it must be $L - a$. Similarly, the second-longest distance measured by R' must be $L - a'$. Since the two rulers measure the same set of distances, $L - a = L - a'$, so $a = a'$. Since $a + d = a' + d'$, we also get $d = d'$. At this point, the two difference triangles are seen to be

$$\begin{array}{ccccccc}
\multicolumn{7}{c}{\Delta R} \\
\hline
a & & b & & c & & d \\
& a+b & & b+c & & c+d & \\
& & a+b+c & & b+c+d & & \\
& & & a+b+c+d & & &
\end{array}$$

$$\Delta R'$$

a		b'		c'		**d**
	$a+b'$		**b + c**		$c'+d$	
		a + b + c		**b + c + d**		
			a + b + c + d			

The entries in bold characters have been shown to be identical for R and R'. Since the sets of measured distances are the same, we have the set equality $\{b, c, a+b, c+d\} = \{b', c', a'+b', c'+d'\}$. The smallest member of the left set must be either b or c, and of the right set either b' or c'. Since we already know $b+c = b'+c'$, we must have either $b = b', c = c'$ or $b = c', c = b'$. In the former case, R and R' are clearly identical. In the latter case, we have $a'+b' = a+c$, and this measured distance of R' must equal either $a+b$ or $c+d$ of R. However, $a+c = a+b$ gives $c = b$, while $a+c = c+d$ gives $a = d$, each of which contradicts the assumption that the ten measured distances are all distinct. Thus R and R' are identical.

5 One Negative Term in Both Factors

Let $f(x) = \phi_1(x)\phi_2(x)$ be the polynomial representation of a ruler. In this section we concentrate on the case where one of the factors, say $\phi_1(x)$, has all positive terms and the other one has exactly one negative term, i.e.,

$$\begin{aligned} \phi_1(x) &= 1 + x^{a_1} + x^{a_2} + \ldots + x^{a_{p_1-1}} \\ \phi_2(x) &= 1 - x^{b_1} + x^{c_1} + x^{c_2} + \ldots + x^{c_{p_2-1}} \end{aligned}$$

We introduce some terminology which will be used in the proofs of subsequent theorems. Let t_i, $i = 1, 2$, be the number of terms in $\phi_i(x)$, and n_i, p_i, the number of negative and positive terms, respectively, in $\phi_i(x)$. Since $f(x)$ represents a spanning ruler, all negative terms should be canceled out with a properly chosen subset of the positive terms in the product $\phi_1(x)\phi_2(x)$.

Definition 1 *We call* **positive differences** *the differences among the terms in the product $\phi_1(x)\phi_2(x)$, which come from the multiplication of the terms in $\phi_1(x)$ and the positive terms of $\phi_2(x)$. Similarly, we call* **negative differences** *the differences among the terms in the product $\phi_1(x)\phi_2(x)$, which come from the multiplication of the terms of $\phi_1(x)$ and the negative terms of $\phi_2(x)$.*

Definition 2 *We call* **type-i differences** *those ones among the positive differences which incorporate the exponent* i *from $\phi_2(x)$.*

E.g., $(c_1 \rightarrow c_1 + a_1)$ is a type-c_1 difference. Furthermore, let M be the maximum number of canceled differences, T the total number of positive differences, and R the minimum possible number of positive differences left uncanceled.

Obviously, $t_i = n_i + p_i$, $i = 1, 2$, and $T = p_2 \times p_1(p_1 - 1)/2$. In what follows it is convenient to arrange all $t_2 \times p_1(p_1 - 1)/2$ differences of the exponents before any cancellation takes place in t_2 rows and $p_1(p_1 - 1)/2$ columns so that all differences formed by pairs of exponents in $\phi_1(x)$ and a fixed exponent from $\phi_2(x)$, i.e., differences of a fixed type, are in the same row. This arrangement is best illustrated through a specific example. In Figure 1 we depict the differences of the exponents for the product $f(x) = \phi_1(x)\phi_2(x)$, where $\phi_1(x) = 1 + x^d + x^e + x^f$, and $\phi_2(x) = 1 - x^a + x^b + x^c$, i.e., $f(x)$ represents an eight mark ruler. Notice that each positive term appears exactly $(p_1 - 1)$ times and, moreover, it appears in the same row in the set of positive differences. Therefore, any cancellation of a positive term results in the cancellation of $(p_1 - 1)$ positive differences. We can now prove the nonexistence of spanning rulers with a sufficiently large number of marks and a specific factorization of the type that we are dealing with in this section.

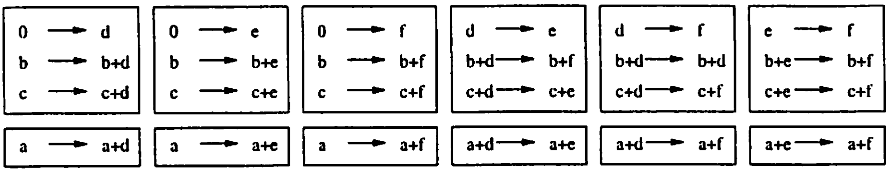

Figure 1.
The total set of differences before any cancellation takes place.

Theorem 5 *If $\phi_1(x)$ has all positive terms (at least 4 terms) and $\phi_2(x)$ has one negative and at least 3 positive terms and if in $f(x) = \phi_1(x)\phi_2(x)$ all negative terms cancel with positive terms, then in $f(x)$ there are repeated differences of exponents.*

Proof.
Case I: $n_1 = 0, \ p_1 = t_1 \geq 4, \quad n_2 = 1, \ p_2 = t_2 - 1 = 3$

In this case we have $p_2 \times t_1(t_1 - 1)/2$ positive differences and $t_1(t_1 - 1)/2$ negative ones. They partition in $t_1(t_1 - 1)/2$ sets of $(p_2 + 1)$ equal differences (case $t_1 = 4$ shown in Figure 1). Our goal is to equate all negative terms in the product $\phi_1(x)\phi_2(x)$ with some properly chosen positive terms so that the resultant polynomial has all formally distinct exponent differences. The only way to achieve this is to cancel out all but at most one of the differences in each and every stack of equal differences in the set of positive ones. We show that this cannot be accomplished.

We have $n_2 \times p_1 = p_1 \geq 4$ negative terms to cancel while we have $p_2 = 3$ different types of differences. Therefore, we have to cancel at least two terms from a single row of differences, w.l.o.g. let's say the row corresponding to type 0-differences. Then

$$M = \underbrace{2(p_1-1) - \binom{2}{2}}_{type-0} + \underbrace{1(p_1-1) - \binom{1}{2}}_{type-b} + \underbrace{1(p_1-1) - \binom{1}{2}}_{type-c}$$

$\Rightarrow\ M \leq p_1(p_1-1)-1$, (recall, $p_1 \geq 4$) but then since $R \geq T - M$,

$\Rightarrow (\ p_2 = 3\ ) \quad R \geq \binom{p_1}{2} + 1.$

By a simple pigeonhole argument, this last inequality implies that there is at least one pair of repeated differences that remains uncancelled in some column in the set of positive differences. Therefore the resultant ruler will have at least one repeated difference.

<u>Case II:</u> $\boxed{n_1 = 0,\ p_1 = t_1 \geq 4, \quad n_2 = 1,\ p_2 = t_2 - 1 > 3}$

In this case: $\quad M \leq p_1(p_1-1)$

$\Rightarrow R \geq p_2 \binom{p_1}{2} - p_1(p_1-1) \Rightarrow$ (since, $p_2 > 3$) $\quad R \geq 2 \times \binom{p_1}{2}$

by the same argument, the resultant ruler is again a non-spanning ruler.

Theorem 6 *A spanning ruler with the factorization:* $f(x) = \phi_1(x)\phi_2(x)$ *where*

$$\phi_1(x) = 1 + x^{a_1} + x^{a_2} + \ldots + x^{p_1-1}$$
$$\phi_2(x) = 1 + x^{c_1} + x^{c_2} + \ldots + x^{c_{p_2}-1} - x^{b_1} - x^{b_2} - \ldots - x^{b_{n_2}}$$

exists only if $p_2 \leq 2n_2 + 1$, *and only if* $p_2 \leq 2n_2$ *in the case that* $n_2p_1 > p_2$.

5.1 An Infinite Family of Counterexamples

We now utilize the polynomial model in order to derive the unique family of counterexamples with six marks. It is worth noticing that the factorization which gives this infinite family is the simplest possible, namely the product of two factors one of which is a trinomial with three all positive terms, so that $\phi_1(1) = 3$, and the other is a quadrinomial with three positive and one negative terms, i.e., $\phi_2(1) = 2$. The product has 12 terms, nine of which are positive and three negative. Our goal is to cancel the three negative terms with three positive ones in such a way that the resulting polynomial has exponents with all distinct differences.

Theorem 7 *If $\phi_1(x)$ has three all positive terms and $\phi_2(x)$ has one negative and three positive terms and if in $f(x) = \phi_1(x)\phi_2(x)$ all negative terms cancel with positive terms, then there exists a* unique *cancellation scheme which gives an infinite parametric family of pairs of spanning homometric rulers. It has the following structure:*

$$
\begin{aligned}
f(x) &= 1 + x^a + x^{b+2a} + x^{2b-a} + x^{2b+a} + x^{3b-a} \\
g(x) &= 1 + x^a + x^{b-2a} + x^{2b-2a} + x^{2b} + x^{3b-a}
\end{aligned}
$$

$a, b \in \mathcal{Z}^+$, and the two factors of $f(x), g(x)$ have the form $\phi_1(x) = 1 + x^a + x^b$, $\phi_2(x) = 1 + x^{b-2a} - x^{b-a} + x^{2b-a}$ $\qquad b \neq 2a$

Proof. Let $f(x) = \phi_1(x)\phi_2(x)$, $g(x) = \phi_1(x)\phi_2^*(x)$ where $\phi_1(x) = 1 + x^a + x^b$, $\phi_2(x) = 1 + x^c - x^d + x^e$, $\phi_2^*(x) = 1 + x^{e-c} - x^{e-d} + x^e$, with $0 < c,\ d < e$, $0 < a < b \in \mathcal{Z}$. Furthermore, let $\phi_1(1)\phi_2(1) = n$, and $\phi_1(x), \phi_2(x)$ be non-symmetric, i.e., $\phi_1(x) \neq \phi_1^*(x)$ and $\phi_2(x) \neq \phi_2^*(x)$. The general expressions for $f(x), g(x)$ are:

$$
\begin{aligned}
f(x) &= x^0 + x^a + x^b + x^c + x^e + x^{a+c} + x^{b+c} + x^{a+e} + x^{b+e} - x^d \\
&\quad - x^{a+d} - x^{b+d} \\
g(x) &= x^0 + x^a + x^b + x^{e-c} + x^e + x^{a+e-c} + x^{b+e-c} + x^{a+e} + x^{b+e} \\
&\quad - x^{e-d} - x^{a+e-d} - x^{b+e-d}
\end{aligned}
$$

The differences formed among the exponents before any cancellations take place are shown in Figure 2. We try to equate the exponents of the negative terms in these two polynomials with exactly three exponents of positive terms in such a way that the remaining terms are all positive and their exponents do not form repeated differences. In this assignment we should satisfy the following constraints:

$$
\begin{gathered}
0 < a < b \qquad b \neq 2a \\
0 < c, d < e \quad c \neq d \\
0 < d < a + d < b + d < b + e
\end{gathered}
$$

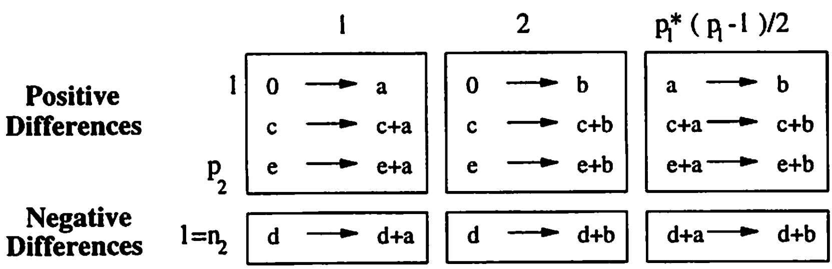

Figure 2.
Differences for a polynomial of weight six resulting in an infinite family of counterexamples.

The possible assignments are:
For $f(x)$

$$d = \left\{ \begin{array}{l} a \\ b \\ b+c \\ a+c \end{array} \right\} \quad a+d = \left\{ \begin{array}{l} c \\ e \\ b+c \\ b \end{array} \right\} \quad b+d = \left\{ \begin{array}{l} c \\ a+c \\ a+e \\ e \end{array} \right\}$$

For $g(x)$

$$e-d = \left\{ \begin{array}{l} a \\ a+e-c \\ b+e-c \\ b \end{array} \right\} \quad a+e-d = \left\{ \begin{array}{l} b \\ e \\ b+e-c \\ e-c \end{array} \right\} \quad b+e-d = \left\{ \begin{array}{l} a \\ e-c \\ e \\ a+e-c \\ a+e \end{array} \right\}$$

A straightforward but tedious backtrack search through the space of possible solutions provides all assignments which do not force a repeated difference. To conserve space, we skip over the details of the proof which can be found in [4]. In the following, we only treat a few special cases indicative of the way this proof methodology works.

Case (1): $\boxed{d = a}$

The assignments that we have to check now are:
For $f(x)$: $a+d = \{c, e, b+c\}$, $b+d = \{c, a+c, a+e, e\}$.
We check all legitimate assignments for $a+d$.

Subcase(1.1)

$$\begin{array}{rcl} d & = & a \\ a+d & = & c \end{array} \quad \Rightarrow \quad \left\{ \begin{array}{rcl} d & = & a \\ c & = & 2a \end{array} \right. \quad \bullet$$

We need to satisfy: $b+d = \{a+e, e\}$, $e-d = \{a, a+e-c, b\}$, $a+e-d = \{b, e\}$, $b+e-d = \{a\}$. Solving the linear equations we get:

$$\left.\begin{array}{rcl} b+d & = & a+e \\ b+e-d & = & a \end{array}\right\} \Rightarrow \quad b = a \quad (\Rightarrow\Leftarrow)$$

$$\left.\begin{array}{rcl} b+d & = & e \\ b+e-d & = & a \end{array}\right\} \Rightarrow \quad 2b = a \quad (\Rightarrow\Leftarrow)$$

Similarly, the remaining cases: $d = a$, $a+d \in \{e, b+c\}$ are shown to lead to a contradiction. Therefore $d \neq a$.

Case (2): $\boxed{d=b}$

The assignments that we now have to check are:
For $f(x)$: $a+d=\{c,e,b+c\}$, $b+d=\{c,a+c,a+e,e\}$.
For $g(x)$

$$e-d=\left\{\begin{array}{l} a \\ a+e-c \\ b+e-c \\ b \end{array}\right\} \quad a+e-d=\left\{\begin{array}{l} b \\ b+e-c \\ e-c \end{array}\right\} \quad b+e-d=\left\{\begin{array}{l} a \\ e \\ a+e-c \end{array}\right\}$$

Once again, tracing through all possible values for $a+d$ and $b+d$ we find two distinct assignments resulting in cancellation schemes that yield two parametric families of pairs of spanning homometric rulers. The first assignment for $f(x)$ is given by: $d=b,\ a+d=c,\ b+d=a+e$. The corresponding assignment for $g(x)$ is,

$$e-d=\left\{\begin{array}{cccc} \neq a & \text{or else} & b=2a & (\Rightarrow\Leftarrow) \\ a+e-c & & & \text{(consistent)} \\ \neq b+e-c & \text{or else} & b=a & (\Rightarrow\Leftarrow) \\ \neq b & \text{or else} & a=0 & (\Rightarrow\Leftarrow) \end{array}\right.$$

Similarly, we get: $a+e-d=b,\ b+e-d=e$. Therefore, since all assignments are legitimately satisfied, we have found a cancellation scheme which gives an infinite parametric family of pairs of spanning homometric rulers. The surviving terms in the two polynomials are

$$\begin{aligned} f(x) &= 1+x^a+x^{a+c}+x^e+x^{b+c}+x^{b+e} \\ g(x) &= 1+x^a+x^{e-c}+x^{b+e-c}+x^{a+e}+x^{b+e} \end{aligned}$$

The two factors of these polynomials expressed in terms of the parameters a,b are: $\phi_1(x)=1+x^a+x^b$, $\phi_2(x)=1+x^{b-2a}-x^{b-a}+x^{2b-a}$. Calculating the products $f(x)=\phi_1(x)\phi_2(x)$, $g(x)=\phi_1(x)\phi_2^*(x)$ we get

FAMILY A

$$\begin{aligned} f(x) &= 1+x^a+x^{b+2a}+x^{2b-a}+x^{2b+a}+x^{3b-a} \\ g(x) &= 1+x^a+x^{b-2a}+x^{2b-2a}+x^{2b}+x^{3b-a} \end{aligned}$$

The second assignment for $f(x)$ that yields another solution is given by: $d=b,\ a+d=e,\ b+d=a+c$. The corresponding assignment for $g(x)$ is,

$$e-d=a \qquad \text{(consistent)}$$

$$a+e-d=\left\{\begin{array}{cccc} b+e-c & & & \text{(consistent)} \\ \neq e-c & \text{or else} & b=0 & (\Rightarrow\Leftarrow) \end{array}\right.$$

$$b+e-d=\begin{cases} \neq a+e-c & \text{or else} \quad b=a \quad (\Rightarrow\Leftarrow) \\ e & \text{(consistent)} \end{cases}$$

Hence, this cancellation scheme gives another infinite parametric family of pairs of spanning homometric rulers. The surviving terms in the two polynomials now are:

$$\begin{aligned} f(x) &= 1+x^a+x^c+x^{b+c}+x^{a+e}+x^{b+e} \\ g(x) &= 1+x^b+x^{e-c}+x^{a+e-c}+x^{a+e}+x^{b+e} \end{aligned}$$

which can equivalently be expressed as

FAMILY B

$$\begin{aligned} f(x) &= 1+x^a+x^{2b-a}+x^{3b-a}+x^{2a+b}+x^{2b+a} \\ g(x) &= 1+x^{2a-b}+x^b+x^{3a-b}+x^{2a+b}+x^{2b+a} \end{aligned}$$

In the same way we complete the backtrack search checking out all possible assignments. The only other case where an assignment results in an infinite family of counterexamples is when: $d=a+c$, $a+d=b$, $b+d=e$. Corresponding to these substitutions the legitimate assignments for $g(x)$ are seen to be:

$$e-d=\begin{cases} \neq a & \text{or else} \quad a=b \quad (\Rightarrow\Leftarrow) \\ b & \text{(consistent)} \end{cases}$$

$$a+e-d=\begin{cases} \neq b & \text{or else} \quad a=0 \quad (\Rightarrow\Leftarrow) \\ e-c & \text{(consistent)} \end{cases}$$

$$b+e-d=\{\ a+e \qquad \text{(consistent)}$$

This assignment pattern gives another infinite parametric family of pairs of spanning homometric rulers:

FAMILY C

$$\begin{aligned} f(x) &= 1+x^a+x^{2b-a}+x^{3b-a}+x^{2a+b}+x^{2b+a} \\ g(x) &= 1+x^{2a-b}+x^b+x^{3a-b}+x^{2a+b}+x^{2b+a} \end{aligned}$$

Obviously this last family coincides with family A. Therefore, the backtracking check has resulted in three sets of cancellation schemes which provide families of homometric rulers. In fact, though, all three families reduce to a single one after relabeling and/or reversing f, g. The difference triangles corresponding to the members of this unique family can be shown to be

FAMILY N

```
0        a        b+2a        2b-a        2b+a        3b-a
--------------------------------------------------------------
     a       b+a        b-3a        2a         b-2a
        b+2a     2b-2a       b-a         b
            2b-a       2b        2b-3a
                2b+a      3b-2a
                    3b-a
```

```
0        a        b-2a        2b-2a        2b        3b-a
--------------------------------------------------------------
     a       b-3a        b          2a         b-a
        b-2a     2b-3a       b+2a        b+a
            2b-2a      2b-a       2b+a
                2b        3b-2a
                    3b-a
```

This unique family of counterexamples can be transformed so that it gives all counterexamples given by the two previously known families I, II. Family N is transformed into family I under the linear transformation

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \Leftrightarrow \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

Family N is transformed into family II under the linear transformation

$$\begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \Leftrightarrow \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix}$$

after setting the elements of sets R, S in increasing order and reversing marks in the first ruler so that it is in standard orientation. We can easily check that under this transformation we get the rulers described by family II.

$$\begin{aligned} R &= \{0, s+t, 5s+3t, 5s+4t, 7s+5t, 8s+5t\} \\ S &= \{0, s, s+t, 4s+2t, 6s+4t, 8s+5t\} \qquad a, b \text{ integers} \end{aligned}$$

5.2 The 2xn case

We investigate now the case where one of the two factors is a trinomial with exactly one negative term and the other factor is a polynomial with all positive terms (at least three of them). In general when the number of terms is fairly large the representation of all positive and negative differences becomes cumbersome and a more convenient one is required. The one that we use in the sequel is best introduced through an example. Specifically we use as the polynomial with all positive terms one with exactly eight terms. In this case $f(x) = \phi_1(x)\phi_2(x)$ where:

$$\begin{aligned} \phi_1(x) &= 1 + x^a + x^b + x^c + x^d + x^d + x^e + x^f + x^g \\ \phi_2(x) &= 1 - x^\alpha + x^w \end{aligned}$$

with $0 < a < b < c < d < e < f < g\,,\ \ 0 < \alpha < w\,,\ \ w \neq 2\alpha$. Then,

$$\begin{aligned} f(x) = \; & x^0 + x^a + x^b + x^c + x^d + x^e + x^f + x^g + \\ & x^w + x^{w+a} + x^{w+b} + x^{w+c} + x^{w+d} + x^{w+e} + x^{w+f} + x^{w+g} \\ & -x^\alpha - x^{\alpha+a} - x^{\alpha+b} - x^{\alpha+c} - x^{\alpha+d} - x^{\alpha+e} - x^{\alpha+f} - x^{\alpha+g} \end{aligned}$$

In Figure 3 we list all exponents of $f(x)$ in the form of a 3×8 array, indicating by circles and boxes the exponents of positive and negative terms, respectively. Notice that, by the way these exponents are listed, the set of exponent pairs which lie in the same row in a given pair of columns all form the same difference.

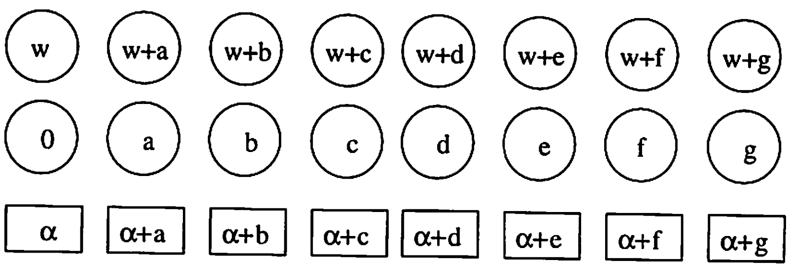

Figure 3. Another way of listing the exponents.

Therefore, any 2×2 subarray of terms from the $2 \times n$ array of positive terms forms a pair of repeated differences. Our objective is to delete a proper subset of the circled entries in such a way that no 2×2 subarray is left with all its four entries non-deleted. The way this is done is by equating n positive terms (circles) with the n negative ones (boxes). Such a cancellation pattern for the specific example of the 2×8 array under consideration is shown in Figure 4. An easy check shows that under this particular cancellation scheme every 2×2 subarray contains at least one deleted positive term.

In the more general case where one of the factors has n all positive terms and the other one has m positive and q negative ones, the aforementioned array partitions into an $m \times n$ subarray with circles and a $q \times n$ subarray of boxes. The number of deleted circles is now $q \times n$.

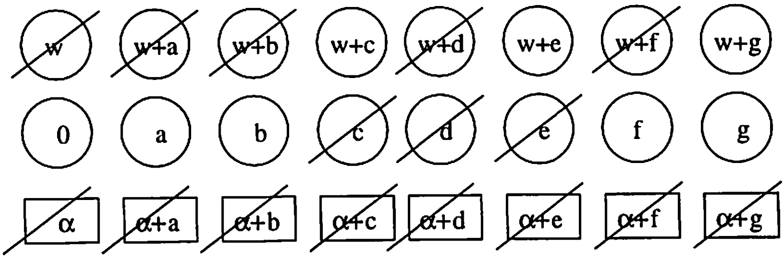

Figure 4. A cancellation scheme which does not automatically force a repeated difference.

The mere fact that the cancellation pattern does not automatically force a repeated difference does not guarantee that a repeated difference will not occur among the remaining exponents. Depending upon the specific relationships among positive and negative terms, a repeated difference is frequently forced among the remaining positive exponents in a way other than the obvious one. In the sequel we shall show that this is always the case when the two factors are such that: $n = 2$, $q = 1$, $m \geq 8$. In order to do that we rearrange the rows of the matrix representing the polynomial exponents so that entries in a single column occur in descending order. A specific assignment of values to negative terms (boxes) is indicated by joining equal terms with a line segment. Figure 5 depicts a particular assignment for the example under consideration.

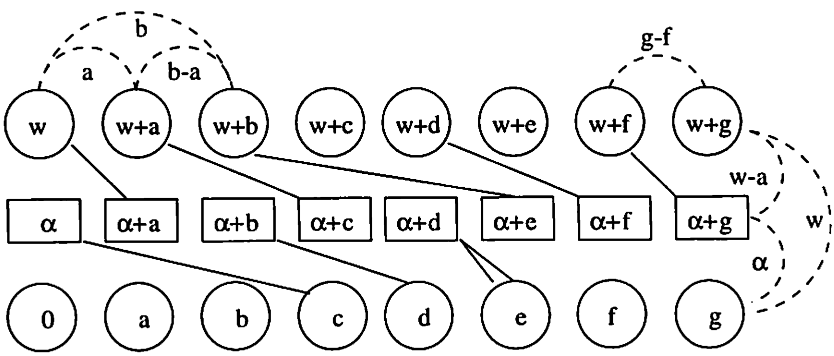

Figure 5. A specific cancellation pattern.

Due to the ordering relation among the terms in this configuration, certain rules apply to the way these line segments occur. It is easy to observe that exactly one such segment emanates from each box. Furthermore, there are only two legitimate directions that such a vector can point to, either to the upper left part of the array or the lower right one. Also, crosses of these line segments are not allowed (or else the order of some two elements would be reversed). An equation of two terms which does not follow the previous two rules leads to a contradiction. Due to this fact, we deduce that the terms $0, w+g$ (the smallest and largest ones, respectively) can never be deleted. Now we are ready to prove the following theorem.

Theorem 8 *There is no spanning ruler $f(x) = \phi_1(x)\phi_2(x)$ when $\phi_1(x)$ has all positive terms (at least eight terms) and $\phi_2(x)$ has exactly one negative and two positive terms.*

Proof. Keeping in mind the relation among exponents and the repeated differences formed by them we can altogether eliminate the labels and deal only with the boxes and circles. We still retain the line segments connecting associated boxes and circles. In order to further facilitate the visualization of these relations, we use *shaded* circles for the deleted ones and *blank* ones representing remaining positive terms. A simple pigeonhole principle shows that we can have at most one column with two shaded circles in which case there exists also a column with two blank ones. A second column with two shaded circles would leave a pair of columns with four non-deleted circles, which results in a repeated difference. One such cancellation scheme is shown in Figure 6. We now demonstrate an algorithm which exhibits the existence of a repeated difference independently of the cancellation scheme, when the number of columns is more than or equal to eight.

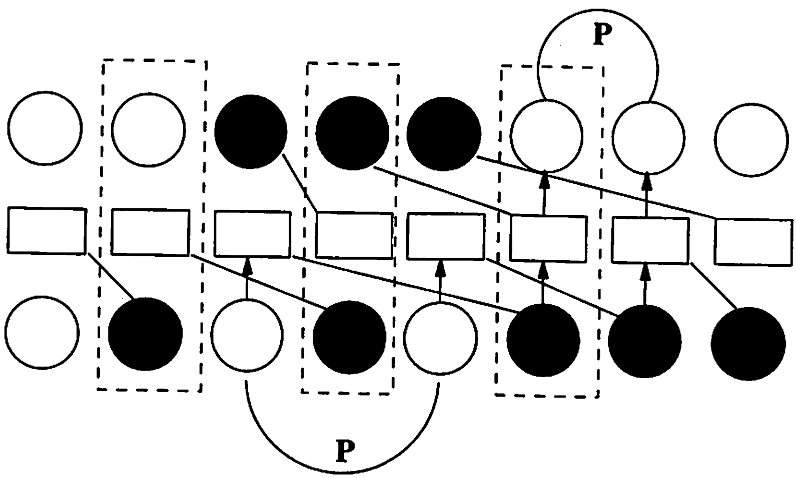

Figure 6. A cancellation scheme

We start by identifying the single column with the two shaded circles (in case there exists any) along with the two columns containing the two boxes associated with the two shaded circles. These three columns are the ones depicted in dotted rectangles in Figure 6. Now we are left with at least five more columns which are guaranteed to contain at least one blank circle each. Therefore, there is one row in this configuration of circles which contains at least three blank circles. For our purposes three such circles are enough. After identifying any three such circles we look at the corresponding boxes. Three vectors are emanating out of these three boxes. At least two of them are pointing towards the same side of the row with the boxes, either up or down. We pick any two of these vectors pointing towards the same direction. These vectors now point towards two shaded circles which are guaranteed to be opposite to two blank ones lying in a single row. Obviously, the difference formed between these two blank circles is equal to the difference formed between the pair of blank circles that we started with. Therefore, we have shown the existence of a single difference formed between two distinct pairs of dots, which proves the theorem.

The algorithm used in the proof of this last theorem when applied to the specific assignment used in Figure 6 gives a pair of repeated differences indicated by P.

5.3 Concluding Remarks: The One Negative Term Case

This section summarizes the results regarding to the existence of homometric spanning rulers with an associated polynomial possessing exactly one negative term among the terms of its two factors $\phi_1(x), \phi_2(x)$. As we have shown, the unique family of counterexamples with six marks falls under this category. A proof of the uniqueness of this family based upon the polynomial approach would require us to disprove the existence of homometric spanning rulers with any other factorization such that $\phi_1(1)\phi_2(1) = 6$. Working along this line we prove the following theorem.

Theorem 9 *There exists no pair of homometric rulers with 6 marks, when the factors of the polynomials have the specific form:* $f(x) = \phi_1(x)\phi_2(x)$, $g(x) = \phi_1(x)\phi_2^*(x)$ *with* $\phi_1(x) = 1+x^c+x^d+x^e+x^f+x^g$, $\phi_2(x) = 1-x^a+x^b$ *where* $0 < c < d < e < f < g \in \mathcal{Z}$, *and* $0 < a < b$, $b \neq 2a$.

The proof of this last theorem is similar to the one for theorem 7. The interested reader should refer to [4] for more details. Theorem 8 in conjuction with the last theorem lends support to the following conjecture:

Conjecture 1 *There is no spanning ruler* $f(x) = \phi_1(x)\phi_2(x)$, *when* $\phi_1(x)$ *has all positive terms (at least four terms) and* $\phi_2(x)$ *has exactly one negative and two positive terms.*

The following table summarizes the results regarding to the existence or not of homometric spanning rulers with associated polynomials assuming the factorization under discussion as deduced from all previously proven theorems.

p_1	p_2	Results
3	2	No Counterexamples (thm. 3)
	3	Infinite Family of Counterexamples (thm. 7)
	≥ 4	No Counterexample (thm. 6)
4	2	No Counterexamples (thm. 3)
	≥ 3	No Counterexample (thm. 6)
5	2	No Counterexamples (thm. 3)
	≥ 3	No Counterexample (thm. 6)
6	2	No Counterexamples (thm. 9)
	≥ 3	No Counterexample (thm. 6)
7	2	?
	≥ 3	No Counterexample (thm. 6)
≥ 8	2	No Counterexample (thm. 8)
	≥ 3	No Counterexample (thm. 6)

From the above results, we see that a proof of conjecture 1 leads also to a proof of the following more general conjecture.

Conjecture 2 *If we restrict the factors $\phi_1(x)$, $\phi_2(x)$ to be of the following form: $\phi_1(x)$ has all positive terms (at least 3 terms), and $\phi_2(x)$ has exactly one negative term (at least 3 terms in total), then the only counterexamples to the theorem of S. Piccard are those given by the infinite parametric family, N.*

6 Factors With More Than One Negative Term

The theorems already proved, where a single negative term exists in both factors, easily generalize to the case where one of the factors has more than one negative terms. In this section we state without proofs a few lemmas and theorems regarding to the latter case. Detailed proofs can be found in [4]. First we exhibit a generalization to the bound derived in theorem 6.

Lemma 2 *A spanning ruler with the factorization: $f(x) = \phi_1(x)\phi_2(x)$ where*

$$\begin{aligned}\phi_1(x) &= 1 + x^{a_1} + x^{a_2} + \ldots + x^{a_{p_1-1}} \\ \phi_2(x) &= 1 + x^{c_1} + x^{c_2} + \ldots + x^{c_{p_2-1}} - x^{b_1} - x^{b_2} - \ldots - x^{b_{n_2}}\end{aligned}$$

exists only if

$$p_2 \binom{p_1}{2} - \left[\sum_{i=1}^{p_2} k_i(p_1 - 1) - \binom{k_i}{2}\right] \geq \binom{p_1}{2}$$

for some k_i's such that $\sum_{i=1}^{p_2} k_i = p_1 n_2$, $p_1 \geq k_i \geq 0$ $\forall i = 1, 2, \ldots, p_2$.

The next theorem covers a specific case of this type of factorization.

Theorem 10 *If $\phi_1(x)$ has 3 positive terms and $\phi_2(x)$ has 2 negative and 4 positive terms, i.e., $\phi_1(x) = 1+x^a+x^b$, $\phi_2(x) = 1-x^c-x^d+x^e+x^f+x^g$, and if in $f(x) = \phi_1(x)\phi_2(x)$ all negative terms cancel with positive terms, then there exists no cancellation scheme yielding a pair of homometric rulers.*

We present now an inequality involving the number of positive terms in the two factors in the case where each one of them possesses exactly one negative term.

Lemma 3 *Assume that $\phi_1(x)$, $\phi_2(x)$ have the form:*

$$\begin{aligned}\phi_1(x) &= 1+x^{a_1}+x^{a_2}+\ldots+x^{a_{p_1-1}}-x^a\\ \phi_2(x) &= 1+x^{b_1}+x^{b_2}+\ldots+x^{b_{p_2-1}}-x^b\end{aligned}$$

then a spanning ruler with the above factorization exists only if:

$$(p_1+p_2)(p_1+p_2-2) \geq (p_1-1)\binom{p_2}{2}+(p_2-1)\binom{p_1}{2} \quad (1)$$

A simple substitution of values into the inequality of the last lemma gives us the following corollary.

Corollary 1 *Assume $n_1 = n_2 = 1$, $p_1 = p_2$. Then a spanning ruler must satisfy:*
$4 \leq p_1 = p_2 \leq 5$.

Proof. (1) $\Rightarrow$ $2(p_1)(2p_1-2) \geq 2(p_1-1)\binom{p_1}{2}$ $\Rightarrow$ $4 \geq p_1 - 1$ $\Rightarrow$ $p_1 = p_2 \leq 5$. But $p_1 = p_2 < 4$ $\Rightarrow$ $\phi_1(1)\phi_2(1) < 6$ in which case we know that no counterexamples exist, hence, $p_1 = p_2 \in \{4, 5\}$.

7 Conclusion

In this paper we have applied the theory of factorization of polynomials over the rationals to investigating the existence of homometric spanning rulers. The existence of such homometric structures contradicts an earlier 'theorem' by S. Piccard. The results obtained demonstrate the power of this approach.

In general, we associate with a pair of spanning rulers a set of two polynomials $r(x)$ and $s(x)$ which can be factored into $r(x) = \phi_1(x)\phi_2(x)$ and $s(x) = \phi_1(x)\phi_2^*(x)$, where $\phi_2(x)$ and $\phi_2^*(x)$ are conjugate polynomials. In the case where both factors $\phi_1(x)$, $\phi_2(x)$ possess only one negative term we have come very close to proving that the only counterexamples to S. Piccard's 'theorem' are those given by a unique infinite parametric family of spanning homometric rulers with six marks.

Furthermore, we have established upper bounds on the number of positive and negative terms that these two factors can have in many other cases. The proof of the nonexistence of counterexamples for certain cases where the number of terms in $\phi_1(x)$, $\phi_2(x)$ is more than 3 and 4 (as a pair), as well as the drastically increasing difficulty with which cancellations of repeated differences can be made as the number of terms increases, appear to support the conjecture that no counterexamples to S. Piccard's 'theorem' exist with rulers having more than six marks. More specifically the following two conjectures of S.W. Golomb seem to be provable.

Conjecture 3 *If the coefficients of* $\phi_1(x), \phi_2(x)$ *are restricted to* $\{-1, 0, +1\}$, *then the only solutions have* $n = 6$ *and are those given by the aforementioned infinite family.*

Conjecture 4 *If, among the coefficients of* $\phi_1(x)$ *and* $\phi_2(x)$ *is a value other than* $\{0, +1, -1\}$, *then in order to get all coefficients of* $f(x)$ *in the set* $\{0, 1\}$ *this will require large* n *(as well as large* N*). But this in turn makes it impossible to keep all differences in* $f(x)$ *distinct.*

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