Unimodality and the Reflection Principle

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ABSTRACT. We show how lattice paths and the reflection principle can be used to give easy proofs of unimodality results. In particular, we give a "one-line" combinatorial proof of the unimodality of the binomial coefficients. Other examples include products of binomial coefficients, polynomials related to the Legendre polynomials, and a result connected to a conjecture of Simion.

1 Introduction

The Gessel-Viennot lattice path technique [7, 8] has proved useful in proving log concavity results [12]. The purpose of this note is to show that lattice paths, and in particular the Reflection Principle [11], are useful for demonstrating the related property of unimodality. One application is the simplest combinatorial proof we know of that the binomial coefficients are unimodal. Such a proof is implicit in a standard use of the Reflection Principle to show that the ballot numbers (which are differences of binomial coefficients) are nonnegative, see [5, p. 95] or [11, p. 3] However, the connection with unimodality does not seem to have been made before. This technique also applies to the four-step lattice paths studied by DeTemple & Robertson [4], Csáki, Mohanty & Saran [3] , Guy, Krattenthaler & Sagan [9], and Beckenridge et. al. [1]. Lattice paths with diagonal steps, which are related to Legendre polynomials [2], provide another example. Finally, we prove a result similar to a conjecture of Simion [13] and discuss some comments and open questions.

Before beginning with the proofs, let us make some definitions. A sequence of real numbers $(a_k)_{k\geq 0}$ is *unimodal* if there is some index m such that

$$a_0 \leq a_1 \leq \ldots \leq a_m \geq a_{m+1} \geq \ldots \tag{1}$$

Unimodal sequences arise in many areas of mathematics. See Stanley's survey article [14] for details. A related property is log concavity. A sequence is *log concave* if for all indices $i \geq 1$ we have

$$a_{i-1}a_{i+1} \leq a_i^2.$$

The following result is well known and easy to prove.

Proposition 1.1 *Suppose $a_k > 0$ for all k. Then log concavity of $(a_k)_{k\geq 0}$ implies unimodality.*

In Section 3 we will see an example where unimodality implies log concavity.

Let $\mathbf{Z}^2$ denote the two-dimensional integer lattice. A *lattice path*, p, is a sequence $v_0, v_1, \ldots, v_n$ where $v_i \in \mathbf{Z}^2$ for all i. The integer n is called the *length* of the path. For example

$$p = (0,0),\ (0,1),\ (1,1),\ (2,1),\ (3,1),\ (3,2) \tag{2}$$

is a path of length five. All of our paths will start at the origin, i.e., $v_0 = (0,0)$. Thus we will often describe a path in terms of its steps, where the *ith step* is the vector s_i from v_{i-1} to v_i. We will use square brackets to enclose the coordinates of a vector so as to distinguish it from an element of the lattice. If $E = [1,0]$ and $N = [0,1]$ denote the unit steps east and north, respectively, then the path (2) can be written

$$p = N,\ E,\ E,\ E,\ N.$$

Let l be a line in the plane such that reflection in l leaves $\mathbf{Z}^2$ invariant. Consider a lattice path $p = v_0, \ldots, v_n$ that intersects l in at least one lattice point and let v_k be the last such point, i.e., the point with largest index. Then the path associated with p via l by *the reflection principle* is

$$p' = v_0, \ldots, v_k, v'_{k+1}, \ldots v'_n$$

where v'_i is the reflection of v_i in l. By way of illustration, if p is the path in (2) and l is the line $y = x$ then

$$p' = (0,0),\ (0,1),\ (1,1),\ (1,2),\ (1,3),\ (2,3).$$

We now have all the tools we will need to prove our unimodality results.

2 Two-step paths and binomial coefficients

Let $T_v = T_{i,j}$ be the set of all lattice paths starting at the origin, ending at $v = (i,j) \in \mathbf{Z}^2$ and using only steps N and E. As is well known,

$$|T_{i,j}| = \binom{i+j}{i}$$

where $|\cdot|$ denotes cardinality. So we can use such paths to prove the unimodality of the binomial coefficients.

Theorem 2.1 *The sequence*

$$\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$$

is unimodal.

Proof. By symmetry it suffices to find, for $k < n/2$, an injection $T_v \hookrightarrow T_w$ where $v = (k, n-k), w = v + [1,-1]$. Reflection principle using the perpendicular bisector of the line segment $\overline{vw}$ provides such a map.

3 Four-step paths

Now consider lattice paths using steps in any one of the four directions N, E, S, W where $S = [0,-1]$ and $W = [-1,0]$. Let $F_{i,j}(n)$ be the set of all such paths that end at (i,j) and have length n. Note that the length must be specified to make the set finite. Furthermore, we must have $n \equiv i+j \pmod 2$ in order for this set to be nonempty, and we will henceforth assume that this is the case. Also let $F^+_{i,j}(n)$ denote the subset of $F_{i,j}(n)$ consisting of all paths that only use lattice points in the upper half-plane $y \geq 0$. The cardinality of $F_{i,j}(n)$ was first derived in [4]. A combinatorial proof was given in [9] where a formula for $|F^+_{i,j}(n)|$ was also obtained by the reflection principle.

Proposition 3.1 *Let $r = (n-i-j)/2$ and $s = (n+i-j)/2$. Then*

$$\begin{aligned} |F_{i,j}(n)| &= \binom{n}{r}\binom{n}{s}, \\ |F^+_{i,j}(n)| &= \binom{n}{r}\binom{n}{s} - \binom{n}{r-1}\binom{n}{s-1}. \end{aligned}$$

The previous proposition can be used to prove unimodality of a number of sequences involving binomial coefficients. Two of the more interesting follow.

Theorem 3.2 *For any fixed integers n, l, the sequence*

$$\binom{n}{l}\binom{n}{0}, \binom{n}{l-1}\binom{n}{1}, \ldots, \binom{n}{0}\binom{n}{l} \tag{3}$$

is unimodal.

Proof. By Proposition 3.1, if $i = -l + 2k$ and $j = n - l$ then

$$|F_{i,j}(n)| = \binom{n}{l-k}\binom{n}{k}.$$

So by symmetry it suffices to find an an injection $F_{i,j}(n) \hookrightarrow F_{i+2,j}(n)$ when $i < 0$. Reflection principle using the line $x = i + 1$ will do the trick.

We note that this theorem could also be obtained by using the fact that the binomial coefficient sequence is log concave, the result that a pointwise product of log concave sequences is log concave, and Proposition 1.1. As another point of interest, it is amusing to see that this unimodality result *implies* log concavity of the binomial coefficients. Indeed, when $l = 2k$ then comparing terms in the middle of (3) yields

$$\binom{n}{k-1}\binom{n}{k+1} \leq \binom{n}{k}^2.$$

In the next result it is convenient to interpret $\binom{n}{m}$ as zero when $m < 0$.

Theorem 3.3 *For any fixed integers n, l, the sequence*

$$\left\{\binom{n}{l-k}\binom{n}{k} - \binom{n}{l-k-1}\binom{n}{k-1}\right\}_{k\geq 0}$$

is unimodal.

Proof. The proof is the same as the in the previous theorem, replacing $F_{i,j}(n)$ by $F^+_{i,j}(n)$ throughout.

Although this can also be proved using a little bit of elementary calculus, the proof is not as short or as elegant.

4 Path with diagonal steps

We will now deal with paths having steps N, E and D where the last is the diagonal step $D = [1, 1]$. Let q be an indeterminate and consider the polynomial

$$D_{i,j}(q) = \sum_{p} q^{d(p)}$$

where the sum is over all such paths p ending at (i, j) and $d(p)$ is the number of diagonal steps on p. For example, $D_{2,4}(q) = 6q^2 + 20q + 15$. Interest in $D_{i,j}(q)$ stems from the fact that $D_{n,n}(1) = P_n(3)$ where $P_n(x)$ is a Legendre polynomial [2]. These paths are also related to a problem of I. Gessel [6] where he essentially asked for a proof that $|D_{2n,2n+2}(-2)|$ is a Catalan number.

Since we are now dealing with polynomials, we will need an associated notion of unimodality. Let $a(q)$ and $b(q)$ be two polynomials in q. Then we will write $a(q) \leq_q b(q)$ if, for all i, the coefficient of q^i in $a(q)$ is less than or equal to the corresponding coefficient in $b(q)$. We now say that a sequence of polynomials, $\{a_k(q)\}_{k\geq 0}$ is *q-unimodal* if it satisfies (1) with $\leq$ replaced everywhere by $\leq_q$ (and similarly for $\geq$). For more information about q-unimodality, see [12].

Theorem 4.1 *The sequence*

$$D_{0,n}(q), D_{1,n-1}(q), \ldots, D_{n,0}(q) \tag{4}$$

is q-unimodal.

Proof. One can give the same proof as in Theorem 2.1. All that is needed is to note that reflection in the given line leaves the number of D steps on a given path invariant.

Actually this result can be seen as a corollary of the statement of Theorem 2.1, rather than following from its proof, because we have the following explicit formula for our polynomials

$$D_{i,j}(q) = \sum_{d\geq 0} \binom{i+j-d}{i-d,\ j-d,\ d} q^d.$$

To see this, note that a path to (i, j) with d diagonal steps must have $i - d$ horizontal steps and $j - d$ vertical ones. This explains the trinomial coefficient in the sum. Extracting the coefficient of q^d in the terms of the sequence (4), we obtain a sequence of trinomial coefficients which have their third bottom index constant at d. Thus this sequence is unimodal by Theorem 2.1, and they all reach their maximum at the same point, proving that (4) is q-unimodal.

5 Simion's conjecture

A *partition* is a weakly decreasing sequence $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_m)$ of positive integers. It will be convenient to also let $\lambda_i = 0$ for $i > m$.

A path p from $(0, 0)$ to (i, j) using only N and E steps must stay inside the box $0 \leq x \leq i,\ 0 \leq y \leq j$. We now consider paths staying inside such a box with the Ferrers diagram of the partition λ removed from the upperleft corner. To be precise, let $T_{i,j}(\lambda)$ be the set of all N, E-paths p to (i,j) such that any lattice point (x,y) on p satisfies $x \geq \lambda_{j-y}$. For example, if $\lambda = (1)$ then

$$|T_{i,j}(\lambda)| = \binom{i+j}{i} - 1.$$

R. Simion [13, Conjecture 4.3] has made the following conjecture.

Conjecture 5.1 *For any partition λ and integer n, the sequence*

$$|T_{0,n}(\lambda)|,\ |T_{1,n-1}(\lambda)|,\ \ldots,\ |T_{n,0}(\lambda)| \tag{5}$$

is unimodal.

Of course, when $\lambda = \emptyset$ this reduces to Theorem 2.1.

Although we cannot prove this conjecture, we can get a related result. Given $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_m)$ and n, define a sequence of partitions by first removing rows from the Ferrers diagram of λ and then adding columns. Specifically, for $0 \leq i \leq \lfloor n/2 \rfloor$ let

$$\lambda^i = (\lambda_i, \lambda_{i+1}, \ldots, \lambda_m)$$

and

$$\lambda^{\lceil n/2 \rceil + i} = (\lambda_{\lceil n/2 \rceil} + i, \lambda_{\lceil n/2 \rceil + 1} + i, \ldots, \lambda_m + i)$$

where $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ are the round up and round down functions, respectively.

Theorem 5.2 *For any partition λ and integer n, the sequence*

$$|T_{0,n}(\lambda^0)|,\ |T_{1,n-1}(\lambda^1)|,\ \ldots,\ |T_{n,0}(\lambda^n)|$$

is unimodal.

Proof. We will show that the first half of the sequence increases, as the proof that the second half decreases is similar. In fact, the very same map used in the proof of Theorem 2.1 will work. One need only note that if the path p does not contain a point of λ^k then the image of p will not contain a point of λ^{k+1} since we reflect the portion of p beyond the *last* intersection with the line.

6 Comments and open questions

It would be interesting to find applications of this method to other types of lattice paths. One could also try using different lattices and higher dimensional analogs. One problem that needs to be overcome in three or more dimensions is that reflection in a hyperplane which is the perpendicular bisector of the line segment joining opposite vertices of a hypercube does not stabilize the lattice of integer points. Hildebrand and Starkweather [10] have some results in this direction.

Although Simion's conjecture is still open, some special cases have been resolved. In particular, Hildebrand [personal communication] has shown that the conjecture holds when λ consists of a single row or a single column. He has also proved an asymptotic log concavity result when λ is a large rectangle.

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