ADDITIVE PERMUTATIONS WITH REPEATED ELEMENTS

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The purpose of this paper is to extend the well-known concepts of additive permutations and bases of additive permutations to the case when repeated elements are permitted; that means that the basis (an ordered set) can become an ordered multiset. Certain special cases are studied in detail and all bases with repeated elements up to cardinality six are enumerated, together with their additive permutations.

1. Introduction.

The concepts of A-basis (basis of additive permutations) and of an additive permutation were introduced by Kotzig and Laufer [6] and later generalized by Kotzig [5]. An ordered set of relatively prime integers $X = (x_1, x_2, \ldots, x_n)$ is called an A-basis if there exists a permutation $Y = (y_1, y_2, \ldots, y_n)$ of X such that the vector sum $X+Y = (x_1+y_1, x_2+y_2, \ldots, x_n+y_n)$ is again a permutation of X. The purpose of this paper is to generalize the above concepts to the case when X is a multiset. A number of references concerning additive permutations can be found, e.g., in [3, 4].

In every combinatorics course, permutations are one of the first topics discussed, usually followed by permutations with repeated elements. We decided to try this approach for additive permutations, and, as the reader will see, we have obtained some interesting results. For example, it is well known that there is no A-basis of cardinality four, but we will show that there is a unique A-basis of cardinality four with repeated elements. We have also found a family of such A-bases for all cardinalities $n \geq 3$ where we can, for each n, give the exact number of additive permutations (which grows exponentially with n). These results suggest that further investigations of A-bases with repeated elements would be worthwhile. Some new results for additive sequences of permutations with repeated elements will follow in a separate paper.

Let $X = (x_1, x_2, \ldots, x_n)$, where $x_1 \leq x_2 \leq \ldots \leq x_n$, be a multiset of relatively prime integers. Then X is called a *basis of additive permutations with repetitions (repetitive A-basis, RA-basis)* if there exists a permutation $Y = (y_1, y_2, \ldots, y_n)$ of X such that the vector sum $X+Y = (x_1+y_1, x_2+y_2, \ldots, x_n+y_n)$ is again a permutation of X. Then Y will be called an R-additive permutation of X.

As in the case of A-bases, we could prove the following simple statement:

Theorem 1. If $X = (x_1, x_2, \ldots, x_n)$ is an RA-basis, then $\sum_{i=1}^{n} x_i = 0$.

It is clear that, for every n, $(0, 0, \ldots, 0)$ is an RA-basis; this RA-basis will be called *trivial* and will always be excluded from our consideration.

It is well known that there exist no A-bases of cardinality 2 or 4. Obviously, there is no nontrivial RA-basis of cardinality 2, but there exists a unique nontrivial RA-basis of cardinality 4: $X = (-1, 0, 0, 1)$. If we put $Y = (0, 1, 0, -1)$, we have $X+Y = (-1, 1, 0, 0)$.

In [1, Theorem 1] it was proved that any A-basis of cardinality $n \geq 4$ has at most $n-3$ positive elements and at most $n-3$ negative elements. When reading the proof of this theorem, the reader will realize that the above statement remains valid for RA-bases as well. In particular, for $n = 4$, we get that an RA-basis of cardinality 4 contains at most one positive element and at most one negative element; if it is nontrivial, it will contain exactly one positive and exactly one negative element. Theorem 1 and the requirement that the elements of X be relatively prime imply then that $X = (-1, 0, 0, 1)$. It follows from Theorem 4 below that the RA-basis $(-1, 0, 0, 1)$ has exactly four R-additive permutations, enumerated in Sec. 3.

We can extend the result of Theorem 1 in [1] in the following way:

Theorem 2. Let X be an RA-basis of cardinality $n \geq 4$ with at least two positive elements. Then X also has at least two negative elements. (The number of negative elements does not have to be equal to the number of positive elements.)

Proof. For $n = 4$, we have seen that no RA-basis with two positive elements can exist. So we can assume that $n \geq 5$. Let $X = (x_1, x_2, \ldots, x_n)$ be an RA-basis with only one negative element: $x_1 < 0 = x_2 = \ldots = x_r < x_{r+1} \leq x_{r+2} \leq \ldots \leq x_n$, where $2 \leq r \leq n-2$. Let $Y = (y_1, y_2, \ldots, y_n)$ be an R-additive permutation of X. We will first identify the subscripts i for which $x_i+y_i = 0$. Since $\sum_{i=1}^{n} x_i = 0$, we have $x_1 + \sum_{j=r+1}^{n} x_j = 0$, and thus $x_1+x_j < 0$, for $j = r+1, \ldots, n$. This implies that $x_i+y_i = 0$ can only be true if $2 \leq i \leq r$. Then either $y_1 = x_1$ (impossible) or $y_1 = x_i$, with $r+1 \leq i \leq n$ (also impossible). ∎

We should now add that, in general, a higher lower bound on the number of negative elements of X cannot be obtained; this is proved, for every $n \geq 7$, by the following example:

$$X = (-2^{n-5}-1,\ -2^{n-6},\ 0,\ 1,\ 1,\ 2,\ 4,\ \ldots,\ 2^{n-7},\ 2^{n-6},\ 2^{n-6}+1),$$
$$Y = (2^{n-6}+1,\ 2^{n-6},\ -2^{n-5}-1,\ 0,\ 1,\ 2,\ 4,\ \ldots,\ 2^{n-7},\ 1,\ -2^{n-6}),$$
$$X+Y = (-2^{n-6},\ 0,\ -2^{n-5}-1,\ 1,\ 2,\ 4,\ 8,\ \ldots,\ 2^{n-6},\ 2^{n-6}+1,\ 1).$$

One more remark will be useful. If $X = (x_1, x_2, \ldots, x_n)$ is an A-basis and if $Y = (y_1, y_2, \ldots, y_n)$ is an additive permutation of X, and if $y_i = x_j$, for some i, j, $i \neq j$, then $y_j \neq x_i$. This result is not valid for RA-bases and RA-permutations, as the following example shows: $X = (-1, -1, 0, 0, 1, 1)$, $Y = (1, 0, -1, 1, 0, -1)$.

As we will see below, a comparatively rich class of RA-bases with an even richer class of R-additive permutations can be obtained using only certain simple classes of multisets of integers.

2. Symmetric and Elementary RA-bases

An RA-basis $X = (x_1, x_2, \ldots, x_n)$ will be called *symmetric* if $x_{n+1-i} = -x_i$, $i = 1, 2, \ldots, n$. Symmetric A-bases have been studied in [2] but the results obtained there cannot be directly extended to RA-bases; the main result related to them (Theorem 3, below) is not an extension of any of the results obtained in [2]; it states that any ordered multiset $X = (x_1, x_2, \ldots, x_n)$ of integers ($x_1 \leq x_2 \leq \ldots \leq x_n$) which satisfies the symmetry condition $x_{n+1-i} = -x_i$, $i = 1, 2, \ldots, n$, is an RA-basis if it contains a sufficiently large number of zero elements. However, this condition is not necessary: it is violated in Examples 1, 2 and 3 given below.

Theorem 3. Let $p_1 \leq p_2 \leq \ldots \leq p_k$ be relatively prime positive integers. Let $X = (-p_k, -p_{k-1}, \ldots, -p_1, 0, 0, \ldots, 0, p_1, p_2, \ldots, p_k)$ be a multiset in which 0 has multiplicity m. If $m \geq k$, then X is an RA-basis. Moreover, X has at least $\binom{m}{k} 2^k$ R-additive permutations. This lower bound is $\frac{m!}{(m-k)!} 2^k$ if $p_1, \ldots, p_k$ are all different.

Proof. Let $m \geq k$. Let us define an R-additive permutation $Y = (y_1, y_2, \ldots, y_{2k+m})$ as follows. We form the triples $(x_i, x_{i+k}, x_{2k+m+1-i})$ of elements of X; each such triple is really the A-basis $(-1, 0, 1)$ of cardinality three multiplied by $x_i = p_{k+1-i}$, $1 \leq i \leq k$. The A-basis $(-1, 0, 1)$ is known to have exactly two additive permutations, viz., $(0, 1, -1)$ and $(1, -1, 0)$. To the three elements $(x_i, x_{i+k}, x_{2k+m+1-i})$ of X we let correspond in Y the triple $(y_i, y_{i+k}, y_{2k+m+1-i})$, $1 \leq i \leq k$, which is one of the additive permutations of $(-1, 0, 1)$ multiplied by p_{k+1-i}. We put $y_i = 0$, $i = 2k+1, \ldots, k+m$. Then $Y = (y_1, \ldots, y_{2k+m})$ is an R-additive permutation of X. In this way we can generate 2^k R-additive permutations of X. Instead of taking the first k positions corresponding to zero elements of X, we can select any k such positions, and this can be done in $\binom{m}{k}$ ways. The middle elements of the triples $(y_i, y_{i+k}, y_{2k+m+1-i})$ can be placed in these positions in $k!$ ways if the numbers $p_1, \ldots, p_k$ are all different. ■

We observe that the number of R-additive permutations can increase if some of the sums p_i+p_j or some of the differences p_i-p_j are in X.

As an example, let us observe that the RA-basis $X = (-s, -r, 0, 0, r, s)$, where r, s are positive integers, $r < s$, $s \neq 2r$, has $m = k = 2$, and the lower bound for the number of R-additive permutations is $\frac{m!}{(m-k)!} 2^k = 2 \times 4 = 8$; this is also the true number of its R-additive permutations; they will be enumerated later.

In Theorem 3 we have seen that, in the symmetric case, the multiset $X = (x_1, x_2, \ldots, x_n)$ is always an *RA*-basis if it contains a sufficiently large number of zero elements. We will now present some examples showing that the condition involving zero elements is not necessary.

Example 1. The vector $X = (-2, -1, -1, 1, 1, 2)$ is an *RA*-basis with no zero element; $Y = (1, 2, -1, 1, -2, -1)$ is an *R*-additive permutation of X.

Example 2. We will show that, for every positive integer $n \geq 6$, there exists a symmetric *RA*-basis X_n of cardinality n which has at most three zero elements (so that the number of zero elements does not satisfy the condition imposed on the *RA*-basis in Theorem 3). In the construction, we will distinguish two cases according to whether n is even or odd. The cases $n = 6$ and $n = 7$ will be dealt with separately.

a) Let n be even, $n = 2m$. Y_{2m} will denote an *R*-additive permutation of the *RA*-basis X_{2m} defined by

$$X_{2m} = (-2^{m-3}, -2^{m-4}, \ldots, -2^2, -2, -1, -1, 0, 0, 1, 1, 2, 2^2, \ldots, 2^{m-4}, 2^{m-3}),$$

$$Y_{2m} = (2^{m-3}, -2^{m-4}, \ldots, -2^2, -2, -1, 0, -1, 1, 0, 1, 2, 2^2, \ldots, 2^{m-4}, -2^{m-3}),$$

$$X_{2m}+Y_{2m} = (0, -2^{m-3}, \ldots, -2^3, -2^2, -2, -1, -1, 1, 1, 2, 2^2, 2^3, \ldots, 2^{m-3}, 0).$$

b) If n is odd, $n = 2m+1$, we construct X_{2m+1} and Y_{2m+1} as follows. We take X_{2m} and Y_{2m} and insert a zero between the m-th and the $(m+1)$-th terms of both X_{2m} and Y_{2m}.

c) For $m = 3$ we get $X_6 = (-1, -1, 0, 0, 1, 1)$, $X_7 = (-1, -1, 0, 0, 0, 1, 1)$; these *RA*-bases will be discussed, in a more general form, in Theorems 4 and 5 below.

To conclude this part of Section 2, we will present an infinite family of nonsymmetric *RA*-bases.

Example 3. For $n \geq 7$, we put

$$X_n = (-2^{n-6}, -2^{n-6}, 0, 0, 1, 1, 2, 2^2, \ldots, 2^{n-7}, 2^{n-6}),$$

$$Y_n = (0, 2^{n-6}, -2^{n-6}, 1, 1, 0, 2, 2^2, \ldots, 2^{n-7}, -2^{n-6}),$$

$$X_n+Y_n = (-2^{n-6}, 0, -2^{n-6}, 1, 2, 1, 2^2, 2^3, \ldots, 2^{n-6}, 0).$$

Next, we will study the special case of symmetric *RA*-bases which we will call *elementary* bases. They are symmetric bases for which $p_1 = p_2 = \ldots = p_k = 1$.

Theorem 4. The multiset

$$X = (-1, -1, \ldots, -1, 0, 0, \ldots, 0, 1, 1, \ldots, 1),$$

where -1, 0, 1 appear k, m, k times respectively, is an *RA*-basis if, and only if, $1 \leq k \leq m$.

Proof. The sufficiency part follows from Theorem 3; we only have to prove that the condition $m \geq k$ is necessary.

Let us assume that the above multiset X is an *RA*-basis and let $Y = (y_1, y_2,$

..., y_n) be an R-additive permutation of X. Clearly, $y_i = 0$ or 1 for $i = 1, 2, \ldots, k$. Let r and $k-r$ be the number of 0's and of 1's in the multiset $\{y_1, y_2, \ldots, y_k\}$ respectively; $0 \leq r \leq k$. Similarly, $y_i = 0$ or -1 for $i = k+m+1, k+m+2, \ldots, 2k+m$. Let s and $k-s$ be the numbers of 0's and 1's in the multiset $(y_{k+m+1}, y_{k+m+2}, \ldots, y_{2k+m})$ respectively; $0 \leq s \leq k$. Then the multiset $(y_{k+1}, \ldots, y_{k+m})$ contains r elements equal to 1, s elements equal to -1 and $m-r-s$ elements equal to zero. Hence $r+s \leq m$.

This implies that the multiset $X+Y$ contains $r+s$ elements equal to -1. From this we conclude that $r+s = k$. Since $r+s \leq m$, we see that $k \leq m$. ■

Theorem 5. If $m \geq k$, then the RA-basis described in Theorem 4 has exactly

$$\binom{m}{k} \sum_{j=o}^{k} \binom{k}{j}^3$$

additive permutations.

Proof. If $0 \leq j \leq k$, there are $\binom{k}{j}$ ordered multisets $\{y_1, y_2, \ldots, y_k\}$ consisting of j elements equal to 0 and $k-j$ elements equal to 1. Then there are $\binom{m}{k}$ ways in which we can place $m-k$ zeros into the positions $k+1, \ldots, k+m$, and $\binom{k}{j}$ ways to place j elements equal to 1 into the remaining of the positions $k+1, \ldots, k+m$. The positions still left free among $y_{k+1}, \ldots, y_{k+m}$ will be occupied by -1; their number is $k-j$. Finally, we have $\binom{k}{j}$ ways of putting j elements equal to -1 into the remaining k positions; the free $k-j$ positions will be occupied by zero. For a fixed j, we have $\binom{k}{j}\binom{m}{k}\binom{k}{j}\binom{k}{j}$ possibilities. Summing over j yields our statement. ■

We should mention here that, according to Herbert Wilf [8] and Gilbert Labelle [7], there is no simple formula for $\sum_{j=o}^{k} \binom{k}{j}^3$. If $k = 1$, $n \geq 3$ and $m = n-2$, then $X = (-1, 0, \ldots, 0, 1)$ has $2n-4$ R-additive permutations. In general, a lower estimate for $\sum_{j=o}^{k} \binom{k}{j}^3$ can be obtained using the well-known formula $\sum_{j=0}^{k} \binom{k}{j}^2 = \binom{2k}{k}$ and the Stirling formula:

$$\sum_{j=0}^{k} \binom{k}{j}^3 \geq \sum_{j=0}^{k} \binom{k}{j}^2 = \binom{2k}{k} = \frac{(2k)!}{(k!)^2} \approx \frac{2^{2k}}{\sqrt{\pi k}}$$

for k sufficiently large.

3. Enumeration of R-additive Permutations of Small Cardinalities

It is clear that there are no nontrivial RA-bases of cardinalities 1 and 2. It is easy to verify that, for $n = 3$, there is only one RA-basis, $X = (-1, 0, 1)$, which has two additive permutations, (1, -1, 0) and (0, 1, -1).

It is known that there is no additive basis of cardinality four. Now any RA-basis of cardinality $n \geq 4$ contains at most $n-3$ positive elements and at most $n-3$ negative elements. For $n = 4$, this implies that any nontrivial basis has exactly one positive and exactly one negative element. The only such basis is $X = (-1, 0, 0, 1)$. According to Theorem 4, X has $2n-4 = 4$ R-additive permutations. They are $(0, 1, 0, -1)$, $(0, 0, 1, -1)$, $(1, 0, -1, 0)$ and $(1, -1, 0, 0)$.

For $n = 5$, we have a more interesting situation. An RA-basis of cardinality 5 contains at most $n-3 = 2$ positive elements and at most 2 negative elements; that means that it must contain at least one zero. Since we can assume that the number of negative elements does not exceed the number of positive elements (otherwise we would replace the RA-basis X by $-X$), we will consider the following two cases (in which $\pi(X)$, $\nu(X)$ will denote respectively the number of positive and of negative elements of X):

$$1.\ \pi(X) = \nu(X) = 1,\quad 2.\ \pi(X) = \nu(X) = 2.$$

The case $\pi(X) = 2$, $\nu(X) = 1$ is eliminated by Theorem 2.

Case 1 obviously yields only one RA-basis, namely $X = (-1, 0, 0, 0, 1)$ which, according to Theorem 4, has six additive permutations: $(1, -1, 0, 0, 0)$, $(1, 0, -1, 0, 0)$, $(1, 0, 0, -1, 0)$, $(0, 1, 0, 0, -1)$, $(0, 0, 1, 0, -1)$, $(0, 0, 0, 1, -1)$.

Case 2. In this case, $X = (x_1, x_2, x_3, x_4, x_5)$, where $x_1 \leq x_2 < x_3 = 0 < x_4 \leq x_5$. If the sum x_3 in $X+Y$ were obtained as $x_3+x_3 = x_3$, then (x_1, x_2, x_4, x_5) would be an RA-basis of cardinality 4, and thus would be of the form $(-1, 0, 0, 1)$, contradicting our assumption. So there is an i and a j such that $x_i+x_j = x_3 = 0$, i.e. $x_i = -x_j$. By Theorem 1, we may conclude that $x_1 = -x_5$ and $x_2 = -x_4$, i.e. that every RA-basis of Case 2 is of the form $(-a, -b, 0, b, a)$, with a, b positive integers. Moreover, 0 is either $a-a$ or $b-b$.

Suppose first that the sum 0 is obtained as $a-a$. If the sums $-a$ and a were obtained as $0+a = a$ and $-a+0 = -a$, then b and $-b$ would be obtained as sums of b's and $-b$'s, which is not possible. So we must also have $a = 2b$. If the sum 0 is obtained as $b-b$, a similar reasoning leads to the same conclusion. In other words, the only possible RA-basis in Case 2 is $X = (-2, -1, 0, 1, 2)$. By [2, Theorem 2], this $RA-$basis has exactly six additive permutations. They are

$$Y_1 = (2, 0, -2, 1, -1),\ Y_1' = (0, 2, -1, 1, -2),$$
$$Y_2 = (0, 1, 2, -2, -1),\ Y_2' = (1, 2, -2, -1, 0),$$
$$Y_3 = (2, -1, 1, -2, 0),\ Y_3' = (1, -1, 2, 0, -2),$$

where Y' is the inverse permutation of Y.

For $n = 6$, we have found by complete search the following *RA*-bases and their *R*-additive permutations.

1. $X = (-1, 0, 0, 0, 0, 1)$. According to Theorem 5, X has the following eight *R*-additive permutations:

$(0, 1, 0, 0, 0, -1)$ $(0, 0, 0, 1, 0, -1)$ $(1, -1, 0, 0, 0, 0)$ $(1, 0, 0, -1, 0, 0)$
$(0, 0, 1, 0, 0, -1)$ $(0, 0, 0, 0, 1, -1)$ $(1, 0, -1, 0, 0, 0)$ $(1, 0, 0, 0, -1, 0)$

2. $X = (-1. -1, 0, 0, 1, 1)$. According to Theorem 5, X has the following ten *R*-additive permutations:

$(0, 0, 1, 1, -1, -1)$ $(0, 1, -1, 1, -1, 0)$ $(1, 0, 1, -1, 0, -1)$ $(1, 0, -1, 1, 0, -1)$
$(0, 1, 1, -1, -1, 0)$ $(0, 1, 1, -1, 0, -1)$ $(0, 1, -1, 1, 0, -1)$ $(1, 1, -1, -1, 0, 0)$
$(1, 0, 1, -1, -1, 0)$ $(1, 0, -1, 1, -1, 0)$

3. $X = (-4, -2, 0, 1, 2, 3)$. According to [1], X has the following two additive permutations: $(0, 2, 3, 1, -4, -2)$, $(2, 3, -4, 1, -2, 0)$.

4. $X = (-4, -3, 0, 1, 2, 4)$. According to [1], X has the following two additive permutations: $(0, 4, -3, 1, 2, -4)$, $(4, 0, -4, 1, 2, -3)$.

5. $X = (-3, -1, 0, 1, 1, 2)$. X has the following four *R*-additive permutations:

$(0, 2, 1, -1, 1, -3)$ $(0, 2, 1, 1, -1, -3)$ $(2, 1, -3, 1, 0, -1)$ $(2, 1, -3, 0, 1, -1)$

6. $X = (-2, -1, 0, 0, 1, 2)$. X has the following twelve *R*-additive permutations:

$(0, 2, -1, 0, 1, -2)$ $(2, 0, 0, -2, 1, -1)$ $(1, 2, -2, 0, -1, 0)$ $(1, -1, 0, 2, 0, -2)$
$(0, 2, 0, -1, 1, -2)$ $(0, 1, 0, 2, -2, -1)$ $(1, 2, 0, -2, -1, 0)$ $(2, -1, 0, 1, -2, 0)$
$(2, 0, -2, 0, 1, -1)$ $(0, 1, 2, 0, -2, -1)$ $(1, -1, 2, 0, 0, -2)$ $(2, -1, 1, 0, -2, 0)$

7. $X = (-r-s, -s, -r, r, s, r+s)$, where $r \leq s$ are relatively prime positive integers. According to [1], X has the following four *R*-additive permutations (confirmed by complete search):

$(r, -r, r+s, s, -r-s, -s)$ $(s, -r, r+s, -r-s, r, -s)$
$(s, r+s, -s, -r-s, r, -r)$ $(r, r+s, -s, s, -r-s, -r)$

For $r = s = 1$, we obtain the symmetric *RA*-basis $(-2, -1, -1, 1, 1, 2)$ mentioned in Example 1; it only has the four corresponding *R*-additive permutations.

8. $X = (-s, -r, 0, 0, r, s)$, where $r \leq s$ are relatively prime positive integers. This a special case of the *RA*-basis mentioned in Theorem 3. For $r < s$, $s \neq 2r$, X has the following eight *R*-additive permutations:

$(0, r, -r, s, 0, -s)$ $(s, 0, -s, r, -r, 0)$ $(0, 0, r, s, -r, -s)$ $(s, r, -r, -s, 0, 0)$
$(0, r, s, -r, 0, -s)$ $(s, 0, r, -s, -r, 0)$ $(0, 0, s, r, -r, -s)$ $(s, r, -s, -r, 0, 0)$

For $s = 2r$, we obtain the *RA*-basis #6. For $r = s$, we obtain the *RA*-basis #2.

For $n \geq 7$, the problem of enumerating the *RA*-bases and their *R*-additive permutations is still open.

Acknowledgements

The authors gratefully acknowledge the financial support, in the form of Ontario-Québec exchange program grants, awarded to them by their respective provincial governments.

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