Neighbourhood Unions and Edge-Pancyclicity

Wen Song Lin and Zeng Min Song
Department of Mathematics
Southeast University
Nanjing, 210096
P.R. China

ABSTRACT. Let G be a 2-connected simple graph with order n ($n \geq 5$) and minimum degree δ. This paper proves that if $|N(u) \cup N(v)| \geq n - \delta + 2$ for any two nonadjacent vertices $u, v \in V(G)$, then G is edge-pancyclic, with a few exceptions. Under the same condition, we prove that if $u, v \in V(G)$ and $\{u, v\}$ is not a cut set and $N(u) \cap N(v) \neq \phi$ when $uv \in E(G)$, then there èxist $u - v$ paths of length from $d(u, v)$ to $n - 1$.

We consider only simple undirected graphs. Let $n = |V(G)| \geq 5$, G is said to be pancyclic if for every k ($3 \leq k \leq n$), G contains a cycle of length k. Let $K \subset N = \{3, 4, \ldots, n\}$, G is said to be K^--*edge-pancyclic* (K^--*vertex-pancyclic*) if for any edge e (vertex x) and any integer $k \in N \setminus K$, G contains a cycle of length k containing $e(x)$. When $K = \phi$, G is called *edge-pancyclic* (*vertex-pancyclic*). If $K = \{k\}$, G is called k^--*edge-pancyclic*. G is called *panconnected* if for every two vertices u, v there exist paths from u to v of length from $d_G\ (u, v)$ to $|V(G)| - 1$. For simplicity, we call a cycle of length t a t-cycle. Suppose $C = v_1 v_2 \ldots v_t v_1$ is a t-cycle, we use $v_i \overrightarrow{C} v_j$ to denote the path $v_i v_{i+1} \ldots v_{j-1} v_j$ on C, $v_i \overleftarrow{C} v_j$ to denote the path $v_i v_{i-1} v_{i-2} \ldots v_{j+1} v_j$ on C and A_i to denote $N(v_i) \setminus V(C)$ ($i = 1, 2, \ldots, t$). Other terminology and notation follow [1] and [2].

Pancyclic graphs were first considered by Bondy in [3]. Since then, many sufficient conditions for a graph to be hamiltonian have been proved to be sufficient conditions for a graph to be pancylic. In recent years, people began to consider vertex-pancyclic graphs. For example, in [4] and [5] the authors gave sufficient conditions for vertex-pancyclic graphs which involve degree sum or neighborhood intersections.

In [7], Faudree, Gould, Jacobson and Lesniak conjectured that if G has order n, connectivity t, minimum degree δ and for any two nonadjacent vertices u, v of G there holds $|N(u) \cup N(v)| \geq n - t$ with $\delta \geq t + 1$, then G is vertex-pancyclic. In [6], Song reproposed this conjecture in the form that if each pair of nonadjacent vertices u and v in a 2-connected graph of order n and minimal degree δ satisfies $|N(u) \cup N(v)| \geq n - \delta + 1$, then G is vertex-pancyclic. Obviously, Song's conjecture can imply the conjecture by Faudree et al. In [8], the authors solved Song's conjecture. Our purpose in this paper is to prove that if G satisfies the neighborhoods union condition with $|N(u) \cup N(v)| \geq n - \delta + 2$, then G is edge-pancyclic unless G belongs to some special graphs.

Before giving the theorem we first describe three families of special graphs: M_1, M_2 and M_3.

$G \in M_1$, if and only if $V(G) = \{v_1, v_2\} \cup S_1 \cup S_2 \cup T$, $v_1v_2 \in E(G)$, $N(v_1) = S_1 \cup \{v_2\}$, $N(v_2) = S_2 \cup \{v_1\}$ and $G[S_1 \cup S_2 \cup T]$ is any graph with $N(w) \cap S_i \neq \phi$ $(i = 1, 2)$ $(w \in T)$.

$G \in M_2$ if and only if $V(G) = \{v_1, v_2\} \cup S_1 \cup S_2 \cup T$, $v_1v_2 \in E(G)$, $G[S_1 \cup \{v_1\}]$ and $G[S_2 \cup \{v_2\}]$ are complete graphs. And $u_1u_2 \notin E(G)$ for any $u_1 \in S_1$, $u_2 \in S_2$. For any $w \in T$, $N(w) \cap S_i \neq \phi$ $(i = 1, 2)$, $G[T]$ is any graph.

$G \in M_3$ if and only if $V(G) = \{v_1, v_2\} \cup A \cup T$, $v_1v_2 \in E(G)$, $G[A]$ and $G[T]$ are complete graphs. $G[A \cup \{v_1, v_2\}]$ and $G[T \cup \{v_1, v_2\}]$ are 2-connected graphs. For any $u \in A$ and $v \in T$, $uv \notin E(G)$.

Obviously $M_2 \subseteq M_1$.

Theorem 1. *Let G be a 2-connected simple graph of order n $(n \geq 5)$ and minimum degree δ. If $|N(u) \cup N(v)| \geq n - \delta + 2$ for any two nonadjacent vertices u, v, then G is edge-pancyclic, unless $G \in M_1$ or M_3.*

Proof: If $\delta(G) \leq 3$. Suppose $u, v \in V(G)$ then $uv \in E(G)$. Otherwise, if $uv \notin E(G)$, then $|N(u) \cup N(v)| \leq n - 2 < n - \delta + 2$, a contradiction. Thus G is a complete graph. Obviously Theorem 1 holds. Hence we can assume $\delta \geq 4$.

Suppose $e = v_1v_2$ is an arbitrary edge in G. We will show that e lies on a 5-cycle. If e lies on a 3-cycle $v_1v_2v_3v_1$. Let $u_1 \in A_1$. Suppose $A_1 \cap A_3 = A_2 \cap A_3 = \phi = N(u_1) \cap A_2$. Then $|N(u_1) \cup N(v_3)| \leq n - \delta$. Hence $N(u_1) \cap A_2 \neq \phi$ or $A_1 \cap A_3 \neq \phi$ or $A_2 \cap A_3 \neq \phi$. In each case e lies on a 4-cycle. Suppose that e lies on $v_1v_2v_3v_4v_1$. Let $u_2 \in A_2$. If $N(u_2) \cap A_4 = \phi$ and $A_3 \cap A_4 = A_3 \cap A_2 = \phi$ then $|N(u_2) \cup N(v_3)| \leq n - 2 - |A_4| \leq n - \delta + 1$, a contradiction. Thus $N(u_2) \cap A_4 \neq \phi$ or $A_3 \cap A_4 \neq \phi$ or $A_3 \cap A_2 \neq \phi$, and e lies on a 5-cycle.

If there exists no 3-cycle containing e. Let $S_1 = N(v_1) \setminus \{v_2\}$, $S_2 = N(v_2) \setminus \{v_1\}$, $T = V(G) \setminus (S_1 \cup S_2 \cup \{v_1, v_2\})$. For any $w \in T$, if $N(w) \cap S_1 = \phi$, then $|N(w) \cup N(v_2)| \leq n - \delta$, contradiction. Thus $N(w) \cap S_1 \neq \phi$, similarly $N(w) \cap S_2 \neq \phi$. Thus $G \in M_1$. If there exist $x \in S_1$, $y \in S_2$ and that

$xy \in E(G)$, then e lies on a 4-cycle. From the discussion above, e lies on a 5-cycle. If for every $x \in S_1$ and $y \in S_2$, $xy \notin E(G)$, then for every two vertices $x_1, x_2 \in S_1$, from $|N(x_1) \cup N(v_2)| \leq n - \delta$, we have $x_1x_2 \in E(G)$, i.e. $G[S_1]$ is a complete graph. Similarly, $G[S_2]$ is a complete graph. For an arbitrary $w \in T$, from $|N(w) \cup N(v_i)| \leq n - \delta + 1$ $(i = 1, 2)$, we have $N(w) \cap S_i \neq \phi$ $(i = 1, 2)$. Obviously $G \in M_2$ and e must lie on a 5-cycle.

Below we will prove that if e lies on an l-cycle $(5 \leq l \leq n)$ then e lies on an $(l+1)$-cycle.

Suppose $C = v_1v_2 \ldots v_lv_1$ is an l-cycle containing e. Obviously there exists $u \in V(G) \setminus V(C)$ such that $d_C(u) > 0$. Let $N_C(u) = \{v_{i_1}, v_{i_2}, \ldots, v_{i_m}\}$ $(i_1 < i_2 < \cdots < i_m)$ and $B = N(u) \setminus V(C)$, $C_j = \{v_{i_j+1}, v_{i_j+2}, \ldots, v_{i_{j+1}-1}\}$. Assume e is on the path $v_{i_1} \overrightarrow{C} v_{i_2}$. Obviously $|C_j| \geq 1$ $(j = 2, 3, \ldots, m)$.

We divide the remainder of the proof into three cases.

Case 1: $m \geq \delta - 1$

If $\{v_{i_1-1}, v_{i_3-1}, v_{i_4-1}, \ldots, v_{i_m-1}\} \cup \{u\}$ is an independent set, then $|N(v_{i_1-1}) \cup N(v_{i_3-1})| \leq n - \delta + 1$, a contradiction. Hence there exists $s, t \neq 2$ such that $v_{i_s-1}v_{i_t-1} \in E(G)$, obviously e lies on an $(l+1)$-cycle.

Case 2: $3 \leq m \leq \delta - 2$

For any $y \in B$, $v_{i_k-2}y \notin E(G)$ $(k \neq 2)$, otherwise $v_{i_k-2}yuv_{i_k} \overrightarrow{C} v_{i_k-2}$ is an $(l+1)$-cycle containing e. Clearly $\{v_{i_1-2}, v_{i_3-2}, \ldots, v_{i_m-2}\}$ is not an independent set (otherwise $|N(v_{i_1-2}) \cup N(v_{i_3-2})| \leq n - \delta + 1$). Without loss of generality, we assume $v_{i_1-2}v_{i_3-2} \in E(G)$. Because $\{v_{i_1-1}, v_{i_3-1}, \ldots, v_{i_m-1}\}$ is an independent set, there must exist $y \in B$ such that yv_{i_1-1} or $yv_{i_3-1} \in E(G)$ (otherwise $|N(u_{i_1-1}) \cup N(v_{i_3-1})| \leq n - \delta + 1$). If $yv_{i_1-1} \in E(G)$, then $v_{i_1-2}v_{i_3-2} \overleftarrow{C} v_{i_1-1}yuv_{i_3} \overrightarrow{C} v_{i_1-2}$ is an $(l+1)$-cycle containing e; If $yv_{i_3-1} \in E(G)$, then $v_{i_1-2}v_{i_3-2} \overleftarrow{C} v_{i_1}uyv_{i_3-1} \overrightarrow{C} v_{i_1-2}$ is an $(l+1)$-cycle containing e.

Case 3: $m \leq 2$

Let $R = V(G) \setminus V(C)$, obviously $|R| \geq \delta - 1 \geq 3$. Without loss of generality, we assume $|N_C(u) \leq 2$ for every $u \in R$.

Suppose that, $|N_C(u)| = 2$ for some $u \in R$. Suppose $N_C(u) = \{v_{i_1}, v_{i_2}\}$, without loss of generality, we assume $e \in v_{i_1} \overrightarrow{C} v_{i_2}$. If $|C_1| \geq 3$, by symmetry we can assume $e \neq v_{i_2}v_{i_2-1}$ or $v_{i_2-1}v_{i_2-2}$. The proof follows as in case 2, by considering $\{v_{i_1-2}, v_{i_2-2}\}$ and $\{v_{i_1-1}, v_{i_2-1}\}$. If $|C_1| = 2$ and $e = v_{i_1}v_{i_1+1}$ (or $v_{i_2}v_{i_2-1}$), similarly we can prove e lies on an $(l+1)$-cycle. Suppose $e = v_iv_{i+1}$. By the discussion above, we have $N_C(u) \in Q = \{\{v_i, v_{i+1}\}, \{v_{i-1}, v_{i+2}\}, \{v_i, v_{i+2}\}, \{v_{i-1}, v_{i+1}\}\}$. The argument is split into three cases.

Case 3.1: $l \geq \delta + 1$

Let $u_1, u_2 \in R$. If $u_1u_2 \notin E(G)$, then $|N(u_1) \cup N(u_2)| \leq n - 2 - (l-4) \leq n - \delta + 1$, a contradiction. Hence $u_1u_2 \in E(G)$, that is, $G[R]$ is a complete graph.

Since G is 2-connected, there must exist at least four different vertices $v_i, v_j \in V(C)$ and $y_i, y_j \in R$ such that $v_iy_i, v_jy_j \in E(G)$. We can assume $i < j$ and $e \in v_i \vec{C} v_j$. First suppose $i+1 \neq j$ and $i \neq j+1$ (modulo l), and $e \neq v_{i+1}v_i$ without loss of generality.

If there exists $y_{j+2} \in R$ such that $v_{j+2}y_{j+2} \in E(G)$ then $v_jy_jy_{j+2}v_{j+2} \vec{C} v_j$ is an $(l+1)$-cycle containing e (Clearly by $N_C(y_j) \in Q$ we have $y_{j+2} \neq y_j$). Hence we can assume $N_R(v_{j+2}) = \phi$. Similarly we can assume $N_R(v_{j+3}) = \phi$ (notice that $G[R]$ is a complete graph and $|R| \geq 3$). If $N_R(v_{j+1}) \neq \phi$. It is easy to prove that $N_R(v_{j+4}) = \phi$. Thus $v_{j+2}v_{j+4} \in E(G)$ (otherwise, $|N(v_{j+2}) \cup N(v_{j+4})| \leq n - \delta + 1$). If $N_R(v_{i+1}) \neq \phi$, suppose $y_{i+1} \in R$ such that $v_{i+1}y_{j+1} \in E(G)$. Clearly $y_i \neq y_{i+1}$ and $v_iy_iy_{i+1}v_{i+1} \vec{C} v_{j+2}v_{j+4} \vec{C} v_i$ is an $(l+1)$-cycle containing e. Hence we can assume $N_R(v_{i+1}) = \phi$. Thus clearly $v_{i+1}v_{j+2} \in E(G)$ and $v_{i+1}v_{j+2} \vec{C} v_iy_iy_jv_j \vec{C} v_{i+1}$ is an $(l+1)$-cycle containing e. If $N_R(v_{j+1}) = \phi$, it is clear that $v_{j+1}v_{j+3} \in E(G)$, thus $N_R(v_{i+1}) = \phi$. Similarly $v_{i+1}v_{j+2} \in E(G)$ and so e lies on an $(l+1)$-cycle.

If $i+1 = j$ or $i = j+1$ and except for v_i, v_j there is no vertex on C that is adjacent to vertices in R, then a similar argument shows that $G[V(C) \setminus \{v_i, v_j\}]$ is a complete graph. Thus $G \in M_3$.

Case 3.2: $5 \leq l \leq \delta$

Since $l \leq \delta$, we have $|N_R(v_i)| \geq 1$ for any $v_i \in V(C)$.

If $l = \delta$, suppose $e = v_iv_{i+1}$ and there exist $y_i, y_{i-2} \in R$ such that $v_{i-2}y_{i-2} \in E(G)$. Obviously $y_i \neq y_{i-2}$ (otherwise $N_C(y_i) \notin Q$). If $y_iy_{i-2} \in E(G)$ then e lies on an $(l+1)$-cycle. If $y_iy_{i-2} \notin E(G)$, since $|N_C(y_{i-2})| \leq 1$ we have $|N(y_i) \cup N(y_{i-2})| \leq n - (l-3) - 2 = n - \delta + 1$, a contradiction. Hence we can assume that $5 \leq l < \delta$.

Since $l < \delta$ we have $|A_i| \geq \delta - l + 1 \geq 2$. Suppose $e = v_1v_2$. If $l \geq 7$, $A_2 \cap A_4 = A_2 \cap A_6 = A_4 \cap A_6 = \phi$ (otherwise there exists $u \in R$ such that $N_C(u) \notin Q$). Suppose $y_2 \in A_2$ and $y_4 \in A_4$. Then if $y_2y_4 \in E(G)$ there exists an $(l+1)$-cycle containing e. Hence we can assume that $y_2y_4 \notin E(G)$. Since $|N(y_2) \cup N(y_4)| \geq n - \delta + 2$, there are at most $\delta - l + 1$ vertices in $R \setminus \{y_2, y_4\}$ that are nonadjacent to both vertices y_2 and y_4. (Notice that $|N_C(y_4)| \leq 1$). Because $|A_2| \geq \delta - l + 1$ there exists $y_2' \in A_2$ such that y_2y_2' or $y_4y_2' \in E(G)$. If $y_4y_2' \in E(G)$ we have an $(l+1)$-cycle containing e. Hence we can assume $y_2y_2' \in E(G)$. This together with $|A_6| \geq \delta - l + 1$

shows that there exist $y_6, y_6' \in A_6$ such that y_6 and y_6' are adjacent to y_2 or y_4. If y_6y_4 or $y_6'y_4 \in E(G)$ we obtain an $(l+1)$-cycle containing e. Hence we may assume $y_6y_2, y_6'y_2 \in E(G)$. Let $C' = v_1v_2y_2'y_2y_6v_6 \overrightarrow{C} v_1$. Then C' is an l-cycle containing e. Notice that $N_{C'}(y_6') \supseteq \{v_6, v_2\}$. If $|N_{C'}(y_6')| \geq 3$, a similar argument to that of Case 2 shows that e lies on an $(l+1)$-cycle. If $|N_{C'}(y_6')| = 2$, the fact that $N_{C'}(y_6') \notin Q$ contradicts the discussion at the beginning of Case 3.

If $l = 5$. We can assume $A_1 \cap A_2 \neq \phi$ (The discussion of the case $A_1 \cap A_2 = \phi$ is similar to that of $l \geq 7$). Suppose $y_2 \in A_1 \cap A_2$ and $y_4 \in A_4$. We have $y_2y_4 \notin E(G)$ and there are at most $\delta - l + 1$ vertices in $R \backslash \{y_2, y_4\}$ that are nonadjacent to y_2 and y_4. Hence there exist $y_{2'} \in A_2$ and $y_1 \in A_1$ such that $y_2y_2' \in E(G)$, $y_1y_2 \in E(G)$ (y_1, y_2' may be identical). It is obvious that $A_3 \cap A_2 = \phi$. Thus there exist $y_3, y_3' \in A_3$ such that both y_3 and y_3' are adjacent to y_2 or y_4. If y_3y_2 or $y_3'y_2 \in (G)$ then $v_1v_2v_3y_3(y_3')y_2y_1v_1$ is a 6-cycle containing e. Hence we assume $y_3y_4, y_3'y_4 \in E(G)$. Similarly there exists $y_5, y_5' \in A_5$ such that $y_5y_4, y_5'y_4 \in E(G)$. Thus $v_1v_4 \notin E(G)$ (otherwise $v_1v_2v_3y_3y_4v_4v_1$ is a 6-cycle containing e). Similarly $v_2v_4 \notin E(G)$. Hence $|A_4| \geq \delta - 2$. Clearly $N(v_3) \cap A_4 = \phi$ and $N(y_2) \cap A_4 = \phi$. Since $A_3 \cap A_2 = \phi$ we have $v_3y_2 \notin E(G)$, but $|N(v_3) \cup N(y_2)| \leq n - 2 - |A_4| \leq n - \delta$, a contradiction.

If $l = 6$. As $l < \delta$ we have $\delta \geq 7$. Clearly $v_3y_2 \notin E(G)$ and $N(v_3) \cap A_4 = N(v_3) \cap A_5 = N(y_2) \cap A_4 = \phi$. If $N(y_2) \cap A_5 = \phi$, then $|N(v_3) \cup N(y_2)| \leq n - 2 - |A_4| - |A_5| \leq n - \delta - 2(\delta - 5) < n - \delta + 1$, a contradiction. Hence $N(y_2) \cap A_5 \neq \phi$. Suppose $y_2y_5 \in E(G)$. By the discussion above we have there exists $y_2' \in A_2$ such that $y_2y_2' \in E(G)$. We obtain the 7-cycle $v_1v_2y_2'y_2y_5v_5v_6v_1$ containing e.

This completes the proof of Theorem 1.

Corollary 1. *Let G be a graph satisfying the conditions of Theorem 1, then G is $\{3,4\}$-edge-pancyclic, unless $G \in M_3$.*

Corollary 2. *Let G satisfy the conditions of Theorem 1, then G is 3^--edge-pancyclic, unless $G \in M_2$ or M_3.*

Theorem 2. *Let G be a 2-connected simple graph with order n and minimum degree δ, if $|N(u) \cup N(v)| \geq n - \delta + 2$ for every two nonadjacent vertices u and v, then for every two vertices $u, v \in V(G)$, there exists paths between u and v of length from $d(u,v)$ to $n-1$, unless $\{u,v\}$ is a cut set of G or $uv \in E(G)$ and $N(u) \cap N(v) = \phi$.*

Proof: For any two vertices $u, v \in V(G)$. If $uv \in E(G)$, by the assumptions of Theorem 2, we have $N(u) \cap N(v) \neq \phi$ and $\{u,v\}$ is not the cut set of G. By the proof of Theorem 1, there exist cycles of length from 3 to n containing uv, thus there exist paths of length from 1 to $n-1$.

If $uv \notin E(G)$. Let $G' = G+uv$. Obviously G' is subject to the conditions of Theorem 1. $\{u, v\}$ is not a cut set of G'. If $N_{G'}(u) \cap N_{G'}(v) \neq \phi$ then there exist cycles of length from 3 to n containing uv thus there exist paths of length from 2 to $n-1$ between u and v. If $N_{G'}(u) \cap N_{G'}(v) = \phi$, then $d(u, v) = 3$ or 4. Thus G' is 3^--edge pancyclic or $\{3, 4\}^-$-edge pancyclic. In both cases the result holds.

This completes the proof of Theorem 2.

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