Edge k-to-1 Homomorphisms *

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ABSTRACT. A homomorphism from a graph to another graph is an edge preserving vertex mapping. A homomorphism naturally induces an edge mapping of the two graphs. If, for each edge in the image graph, its preimages have k elements, then we have an edge k-to-1 homomorphism. We characterize the connected graphs which admit edge 2-to-1 homomorphism to a path, or to a cycle. A special case of edge k-to-1 homomorphism – k-wrapped quasicovering – is also considered.

1 Introduction

The motivation of this paper can be traced back to two sources: graph homomorphisms and k-to-1 continuous mapping in topological spaces. Graph homomorphism is a widely studied graph theoretical concept. Graph homomorphism is an edge preserving mapping from the vertex set of one graph to the vertex set of another graph. There are quite a few papers considering the computational complexities of homomorphism [2, 3, 6, 12, 21, 23, 29]. Some consider the characterizations of homomorphic preimages of a fixed graph in terms of forbidden homomorphic preimages [13, 14, 15, 20, 26].

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Others concern the interplay of homomorphism and other graph theoretical properties [1, 4, 27, 28]. Still others concern the relationship between graph homomorphisms and languages [11, 22, 25]. Special cases of homomorphisms are considered in [5, 18, 19, 24], in which the preimages of an edge incident to a vertex v are distributed to the preimages of the vertex v in some pattern. It is called a double cover projection [18, 24] if it is a 2-to-1 homomorphism and equally distributed. D. W. Waller [24] found a way to construct all the double covers of a graph by means of a spanning tree, and M. Hofmeistar [18] found a way to count double covers of a graph.

A finite graph can also be viewed as a compact topological space, in which any path, whose inner vertices have degree two, is a compact topological subspace homeomorphic to a real closed interval. Therefore, we can consider a continuous mapping from one graph to another in the sense of topological spaces. A k-to-1 continuous mapping is one for which every inverse image consists of k points. The image graph must have a cycle in order to have a continuous mapping for $k > 1$ [7]. More recently it has been shown that [8] a tree cannot be a k-to-1 finitely discontinuous image of any connected graph if $k > 1$. The discussion of the existence of k-to-1 continuous maps between graphs when k is sufficiently large can be found in [16].

It was also shown in [9] that any tree admits a k-to-1 finitely discontinuous function onto any graph with a cycle for any $k > 2$, but that there cannot be a 2-to-1 finitely discontinuous function from a tree to any graph whether the image has a cycle or not. In [10], those trees that admit a 3-to-1 continuous map onto a cycle were characterized. Those graphs that admit a 3-to-1 continuous map onto a cycle were characterized in [17].

Note that a homomorphism is a special continuous mapping between graphs but it still keeps more graph structures. This leads us to consider the "Heath–Hilton" type questions on homomorphism.

All graphs considered in this paper are finite undirected graphs. Let G be a graph. A *walk* W of G is an alternating sequence of vertices and edges $v_0e_0v_1e_1v_2\cdots e_{k-1}v_k$ such that v_i is incident with e_i and e_{i-1} for $i = 1, 2, \ldots, k$. A walk W is called a *path* if $v_i \neq v_j$ whenever $i \neq j$. A path W is called a *cycle* if $v_0 = v_k$. We use P_n and C_n to represent a path and a cycle of n vertices respectively.

Let G be a connected graph, and H a subgraph of G. We define a relation '$\sim$' on $E(G) - E(H)$ by the condition that $e_1 \sim e_2$ if there exists a walk W such that

1. the first and last edges of W are e_1 and e_2 respectively, and

2. W is internally-disjoint from H (that is, no internal vertex of W is a vertex of H).

It is easy to see that $\sim$ is an equivalence relation on $E(G) - E(H)$. A connected subgraph of $G - E(H)$ induced by an equivalence class under the relation $\sim$ is called a *bridge* of H in G. A bridge which is a path (tree) is called a *path (tree) bridge*. If B is a bridge, a vertex in the set $V(H) \bigcap V(B)$ is called an *attachment* of B to H. We will simply call a bridge or an attachment if there is no confusion.

For convenience, if B is a bridge with only one attachment b (two attachments b_1, b_2), we sometimes write (B, b) $((B, b_1, b_2))$ instead of B.

For two graphs G and H, if there is a mapping h from $V(G)$ to $V(H)$ such that $xy \in E(G)$ implies $h(x)h(y) \in E(H)$, then h is called a *homomorphism* of G to H, denoted by $h : G \to H$. Let $h : G \to H$ be a graph homomorphism. If for each $uv \in E(H)$, $h^{-1}(uv)$ has k elements, we call h the *edge k-to-1 homomorphism*, G the *edge k-to-1 homomorphic preimage*, and H the *edge k-to-1 homomorphic image*.

In the study of graph embeddings, B. Jackson, T.D. Parsons and T. Pisanski [19] introduced a so called wrapped quasicovering which is also a special kind of edge k-to-1 homomorphism. Let G and H be two graphs. An edge k-to-1 homomorphism $h : G \to H$ is called a *k-wrapped quasicovering* of G over H if for each vertex $v \in V(G)$, there is a positive integer $i(v)$ such that, if $v' = h(v)$ then for each edge e of H incident to v', there are $i(v)$ edges of G in $h^{-1}(e)$ incident to v. The number k is called the *multiplicity* of h and $i(v)$ is called the *wrapping index* of h at v.

If $i(v) = 1$ for every vertex v of G, then h is a covering map in the sense of Gross and Tucker [5].

In this paper, we will characterize the graphs which admit edge 2-to-1 homomorphism to a path in Section 2, or to a cycle in Section 3, and we will also characterize the graphs which admit k-wrapped quasicovering over a cycle in Section 4.

2 Connected graphs admitting edge 2-to-1 homomorphisms to a path

We need the following definitions in order to characterize graphs which admit edge 2-to-1 homomorphisms to a path.

Definition 1 *Let T be a tree and $P = u_0u_1 \cdots u_k$ be a path of T. Assume that all the bridges of P in T are path bridges $(T_1, u_{i_1}), \dots, (T_l, u_{i_l})$ where $i_1 < i_2 < \cdots < i_l$.*

If $i_1 = 0$, and T_j is of length $|T_j| = i_{j+1} - i_j$ ($i_{l+1} = k$), for $j = 1, \dots, l$, then T is called a basic I double path with main path P.

If $i_l = k$, and T_j is of length $|T_j| = i_j - i_{j-1}$ ($i_0 = 0$), for $j = 1, \dots, l$, then T is called a basic II double path with main path P.

We call u_0 and u_k the main vertices of T.

Definition 2 *An even cycle* $C_{2m} = u_0u_1 \dots u_mu_{m+1} \dots u_{2m-1}u_0$ *is called a basic III double path with main path* $P = u_0u_1 \dots u_m$ *and main vertices* u_0 *and* u_m.

A *basic double path* is a basic I, II, or III double path.

Definition 3 *Let* H *be a connected graph. If there is a sequence of subgraphs* $H_1, \dots, H_k$ *of* H *such that*

(1) $H_1 \cup \dots \cup H_k = H$,

(2) each H_i *is a basic double path with main path* P_i,

(3) $H_i \cap H_j = \emptyset$ *if* $|i-j| > 1$ *and* $H_i \cap H_{i+1}$ *is a vertex of* H *which is a main end vertex of both* H_i *and* H_{i+1}, *for* $i = 1, \dots k-1$, *then* H, *denoted by* $H = H_1 \circ H_2 \circ \dots \circ H_k$, *is called a double path with main path* $P_1P_2 \cdots P_k$.

Lemma 2.1 *Let* $f : G \to H$ *be an edge* k*-to-*1 *homomorphism. For any vertex* $v \in V(H)$, *if* $f^{-1}(v) = \{x_1, x_2, \dots, x_m\}$, *then* $d(x_1) + \dots + d(x_m) = kd(v)$.

Proof. Let $e_1, \dots, e_l$ be all the edges incident to $x_1, \dots, x_m$. Then $d(x_1) + \dots + d(x_m) = l$ and $f(e_1), \dots, f(e_l)$ are all edges incident to v. Since f is edge k-to-1, $l = kd(v)$. Therefore, $d(x_1) + \dots + d(x_m) = kd(v)$. □

Corollary 2.2 *If* G *admits an edge* 2*-to-*1 *homomorphism to a path, then the maximum degree of* G *is at most* 4, *and the number of vertices in a preimage of any vertex is at most* 4.

Lemma 2.3 *Let* G *be a connected graph admitting an edge* 2*-to-*1 *homomorphism* f *to a path* P_{n+1}. *Then* G *is a double path of size* $2n$.

Proof. Let $P_{n+1} = v_0v_1 \cdots v_n$. Suppose $f(x_i) = v_i$ for $i = 0, n$. Since G is connected, there is a path P' joinnig x_0 and x_n. Let $P' = x_0x_1 \cdots x_{m-1}x_mx_n$. We are going to prove that $m = n-1$. Note that x_1 must be mapped to v_1 since f is a homomorphism. x_2 must be mapped to v_0 or v_2 by the same reason. If $f(x_2) = v_0$, then we have $f(x_3) = v_1$, implying that there are three edges mapped to the edge v_0v_1. Hence $f(x_2) = v_2$. Continuing this way, we see that $f(x_i) = v_i$ for $i = 0, 1, \dots, m$ and $f(x_n) = v_{m+1}$. This proves that $m = n-1$, $P' = x_0x_1 \cdots x_{n-1}x_n$ and $f(x_i) = v_i$. Let T be a bridge of P' in G. If T is not a path, then there is a vertex $x \in T$ such that $d(x) \geq 3$. We have that $f(x) \neq v_0, v_n$ by Lemma 2.1. Let $f(x) = v_j$ for some $0 < j < n$. Then $d(x) + d(x_j) \geq 5$, which is a contradiction. Therefore, in G all vertices of degree 3 and 4 are on the path P'.

Now we prove this lemma by induction on the number of degree 3 and 4 vertices in G.

Let G have no degree 3 and 4 vertices. Then G is a path or a cycle of length $2n$ since G is connected and G admits an edge 2-to-1 homomorphism to P_{n+1}. If only $d(x_0) = 2$ or $d(x_n) = 2$, G is a basic I double path or basic II double path. If $d(x_0) = d(x_n) = 2$, then G is a type III double path or a double path composed of a type I path and a type II path depending G is a cycle or not. Therefore, G is a double path of size $2n$.

Suppose now the lemma is true for graphs having less than m vertices of degree 3 and 4 for $m > 0$. Let G have m vertices of degree 3 and 4 vertices. We have shown that all degree 3 and 4 vertices of G are on the path P'. Let x_i be the last vertex of degree at least 3 on P'. Then $i \neq n$, otherwise G does not admit the edge 2-to-1 homomophism f to P_{n+1} with $f(x_n) = v_n$.

Let $G_1 = f^{-1}(v_0 \cdots v_i)$ and $G_2 = f^{-1}(v_i \cdots v_n)$. Then $G = G_1 \circ G_2$ since $f^{-1}(v_i) = \{x_i\}$. That is G is obtained from G_1 and G_2 by identifying at x_i. Note that $f|G_1$ is an edge 2-to-1 homomorphism from G_1 to the path $v_0 \cdots v_i$. Also note that G_1 has less than m vertices of degree 3 and 4. Hence G_1 is a double path by induction hypothesis. Similarly, G_2 is a double path. Therefore, $G = G_1 \circ G_2$ is a double path of size $2n$. □

Theorem 2.4 *A connected graph G admits an edge 2-to-1 homomorphism to P_{n+1} if and only if G is a double path of size $2n$.*

Proof. The necessity follows from Lemma 2.3.

Sufficiency. Let $P_{n+1} = v_0v_1 \cdots v_n$. Since G is a double path of size $2n$, the main path P of G has length n. Let $P = x_0x_1 \cdots x_n$. Define f on P to be $f(x_i) = v_i$. We extend f to the rest of G as follows.

We can express G as $G_1 \circ G_2 \circ \cdots \circ G_m$, where each G_i is a basic double path with main path P^i which is a subpath of P. Let $P^i = x_px_{p+1} \cdots x_q$. If G_i is a basic III double path with the bridge T, then T has attachments x_p, x_q and length $q-p$. It is easy to see there is a unique way to extend f to T. If G_i is a basic I or basic II double path, let $(T_1, x_{i_1}), \ldots, (T_k, x_{i_k})$ be all bridges with $i_1 < i_2 < \cdots < i_k$. For each T_j, we let f map T_j to $v_{i_j} \cdots v_{i_{j+1}}$ if G_i is a basic I double path, and f map T_j to $v_{i_{j-1}} \cdots v_{i_j}$ if G_i is a basic II double path. The conditions on the lengths of T_j's guarantee such extension on T_j to be 1-to-1 and onto. Therefore, f is an edge 2-to-1 homomorphism from G to P_{n+1}. □

3 Connected graphs admitting edge 2-to-1 homomorphisms to a cycle

In order to characterize connected graphs admitting edge 2-to-1 homomorphisms to a cycle we need the following definition.

Definition 4 *Let H be a double path with main path $P = u_0u_1 \cdots u_n$ and let $Q = v_1 \cdots v_m$, $Q' = v'_1 \cdots v'_m$ be two paths of same length.*

a) A graph G is called a type I double cycle if G is a cycle of even length or G is obtained from H by identifying u_0 with u_n, and the cycle $u_0 \cdots u_{n-1}u_0$ is called the main cycle of G.

b) A graph G is called a type II double cycle if G is obtained from Q, H and Q' by identifying v_m with u_0 and v'_1 with u_n. Denoted by $Q \circ H \circ Q'$.

A *double cycle* of length $2n$ is either a double cycle of type I or a double cycle of type II of size $2n$.

Theorem 3.1 *A connected graph G admits an edge 2-to-1 homomorphism f to an n-cycle C_n if and only if G is a double path of size $2n$ or a double cycle of size $2n$.*

Proof. Let $C_n = v_0v_1 \cdots v_{n-1}$. Let $P_{n+1} = w_0w_1 \cdots w_n$ be a path of n edges.

Sufficiency. Let G be a double path with main path $x_0x_1 \cdots x_n$. By Theorem 2.4, there is an edge 2-to-1 homomophism $f : G \to P_{n+1}$. We can assume that $f(x_i) = w_i$ for $i = 0, \cdots n$. Define F as follows: $F(u) = v_i$ if $f(u) = w_i$ and $u \in V(G) - f^{-1}(\{w_0, w_n\})$, and $F(u) = v_0$ if $u \in f^{-1}(\{w_0, w_n\})$. Then it is easy to see that F is an edge 2-to-1 homomorphism from G to the n-cycle C_n.

Next, let G be a double cycle of type II. Then we can rewrite G as $G = P \circ H \circ P'$, where H is a double path, $P = y_0 \cdots y_k$ and $P' = y'_0 \cdots y'_k$ are two paths of same length. By Theorem 2.4, there is an edge 2-to-1 homomorphism f which maps H to the subgraph $v_0 \cdots v_{n-k-1}$ with $f(y_k) = v_0$ and $f(y'_0) = v_{n-k-1}$. We extend f by letting $f(y'_i) = f(y_{k-i}) = v_{n-k+i-1}$. It is easy to see that f is an edge 2-to-1 homomorphism from G to C_n.

Now suppose that G is a double cycle of type I. If $G = x_0x_1 \cdots x_{2n-1}x_0$ is a cycle of length $2n$, let $f(x_i) = f(x_{i+n}) = v_i$ for $i = 0, 1, 2, ..., n-1$. Then f is an edge 2-to-1 homomorphism from G to C_n.

Assume now G is not a cycle. Then G is obtained from a double path H with main path $P = u_0u_1 \cdots u_n$ by identifying u_0 with u_n. Let u be the vertex in G by identifying u_0 and u_n. We have that there is an edge 2-to-1 homomorphism f from H to C_n. Define $F(v) = f(v)$ for $v \neq u$ and $F(u) = f(u_0)$. It is easy to see that F is an edge 2-to-1 homomorphism from G to C_n.

Necessity. Let a connected graph G admit an edge 2-to-1 homomorphism f to C_n.

Suppose that there is a cycle C in G which is mapped onto C_n. If $|C| > n$, then it is easy to see that $|C| = 2n$ and $G = C$ is a cycle of length $2n$ which is a type I double cycle of size $2n$. If $|C| \leq n$, then we have $|C| = n$. Let $C = x_0 \cdots x_{n-1}$ and suppose that $f(x_i) = v_i$. There is a vertex on C of degree at least 3, say x_0. Let $G' = (G - \{x_0\}) \cup \{x'_0, x''_0\} \cup \{x'_0 x, x''_0 y | x \in f^{-1}(v_1), y \in f^{-1}(v_{n-1})\}$, where x'_0 and x''_0 are two new vertices. We define g such that $g(u) = w_i$ if $f(u) = v_i$ and $u \in V(G') - f^{-1}(v_0)$; $g(x'_0) = w_0, g(x''_0) = w_n$ and $g(u) = w_0$ if $u \in f^{-1}(v_0)$ and u joins $f^{-1}(v_1)$; $g(u) = w_n$ if $u \in f^{-1}(v_0)$ and u joins $f^{-1}(v_{n-1})$. Then g is an edge 2-to-1 homomorphism from G' to P_{n+1}. By Theorem 2.4, G' is a double path. But G is obtained from G by identifying x'_0 and x''_0. Therefore, G is a double cycle of type I.

Suppose now there is no cycle in G which is mapped onto C_n. If G is a double path, we are done. Assume not, then G is not a path. Without loss of generality, let $v_0 v_1 \cdots v_i$ be a longest path in C_n such that $H = f^{-1}(v_0 v_1 \ldots v_i)$ is connected. Then $f|H$ is an edge 2-to-1 homomorphism from H to the path $v_0 v_1 \cdots v_i$. By Theorem 2.4, H is a double path. Now consider $K = f^{-1}(v_i v_{i+1} \cdots v_{n-1} v_0)$. K is not connected, for otherwise K is a double path which implies that there is a cycle in G mapped onto C_n, a contradiction. Since each component of K must join to $f^{-1}(v_0)$ or $f^{-1}(v_i)$ as G is connected, and G admits an edge 2-to-1 homomorphism f to C_n, then K has exactly two components, say K' and K''. If there is no vertex of K' or K'' which maps to v_0 or v_i, then we can obtain a longer path P in C_n such that $f^{-1}(P)$ is connected, which is a contradiction. Therefore, we must have $f(K') = f(K'') = v_i v_{i+1} \cdots v_0$. This implies that both K' and K'' are paths of length $n - i$. It is easy to see that $G = K' \circ H \circ K''$ and hence G is a type II double cycle. □

4 Graphs admitting wrapped quasicovering over cycles

The k-wrapped quasicovering is a special case of edge k-to-1 homomorphism. We will now give a characterization of graphs addmitting k-wrapped quasicovering over cycles for all k. First we need the following definition.

Definition 5 *Let $C_{kn} = v_1 v_2 \cdots v_{kn} v_1$ be a cycle of length kn. Let $V_j = \{v_{j+in} : i = 0, 1, \ldots, k-1\}$ for $j = 1, \ldots, n$. G is called a k-tuple cycle if G is obtained from C_{kn} by identifying some vertices of V_j for each $j = 1, \ldots, n$.*

Theorem 4.1 *Graph G addmits a k-wrapped quasicovering f over an n-cycle C_n if and only if G is a k-tuple cycle.*

Proof. Sufficiency. Let $C_n = u_1 \cdots u_n$. Let G be a k-tuple cycle, i.e., a graph obtained from $C_{kn} = v_1 v_2 \cdots v_{kn}$ by identifying some vertices of V_j for each $j = 1, \ldots, n$ where $V_j = \{v_{j+in} : i = 0, 1, \ldots, k-1\}$ for $j = 1, 2, \ldots, n$. Let V'_j be the vertices of G obtained by identifying some vertices of V_j. Define $f : V(G) \to V(C_n)$ by $f(V'_j) = \{u_j\}$. It is easy to see that f is a k-wrapped quasicovering with index $i(v) = \frac{d(v)}{2}$ for each $v \in V(G)$.

Necessity. First we show that there is an integer k_1 such that G contains a cycle C of length $k_1 n$ and $f|C$ is a k_1-to-1 homomorphism from C to C_n.

Let $U_i = f^{-1}(u_i)$. We claim that for any i and any $v \in U_i$, there is a $w \in U_{i+1}$ such that $vw \in E(G)$, where the subscripts are taken modulo n.

We have that $f(v) = u_i$. For edge $u_i u_{i+1} \in E(C_n)$, there are exactly $i(v)$ edges incident to v which are mapped to $u_i u_{i+1}$ by the definition. Suppose $f(vw) = u_i u_{i+1}$. Then $w \in U_{i+1}$.

Now choose an arbitrary vertex $v_1 \in U_1$. By the above claim, we can choose $v_2 \in U_2$ such that $v_1 v_2 \in E(G)$, then choose $v_3 \in U_3$ such that $v_2 v_3 \in E(G)$, continuing this way, we can have $v_n \in U_n$ such that $v_{n-1} v_n \in E(G)$, then choose $v_{n+1} \in U_1$ such that $v_n v_{n+1} \in E(G)$. If $v_{n+1} = v_1$, then we are done and $k_1 = 1$. If not, then choose $v_{n+2} \in U_2$ such that $v_{n+1} v_{n+2} \in E(G)$, ... At last, we must have a vertex $v_{mn+i} \in U_i$ such that v_{mn+i} is the first vertex we meet which was already chosen, that is $v_{mn+i} = v_{ln+i}$ where $l < m$. Now $C = v_{ln+i} v_{ln+i+1} \cdots v_{mn+i-1} v_{mn+i}$ is a cycle of G which has length $(m-l)n$. Let $k_1 = m - l$. Then $f|C$ is a k_1-to-1 homomorphism from C to C_n. We also note that $f|C$ is a wrapped quasicovering with index $i(v) = \frac{d_C(v)}{2}$.

If $C = G$, then we are done. Suppose $C \neq G$. Let $U = \{v : v \in V(C)$ such that $d_G(v) = 2\}$. Let the graph G' be such that $V(G') = V(G) - U$ and $E(G') = E(G) - E(C)$. Then $f|G'$ is a wrapped quasicovering of multiplicity $k - k_1$ from G' over C_n. By induction, G' is a $(k-k_1)$-tuple cycle, i.e., G' is obtained by identifying some vertices in each set $(f|G')^{-1}(u_i)$. Now it is easy to see that G is obtained from C and G' by identifying some vertices in $V(C) \cap U_i$ with some vertices in $V(G') \cap U_i$. Therefore, G is a k-tuple cycle. □

Corollary 4.2 *G admits a wrapped quasicovering f over C_n of multiplicity k such that f is also k-to-1 on vertex set if and only if $G = C_{kn}$, a cycle of length kn.*

Remark: All the results above can be extended to directed graphs.

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