A LOWER BOUND FOR THE NUMBER OF POLYGONIZATIONS OF N POINTS IN THE PLANE

A. GARCÍA and J. TEJEL
Dpto. Métodos Estadísticos
Facultad de Matemáticas
Univ. Zaragoza. España

ABSTRACT

Let $\Phi(N)$ be the maximum number of simple polygons that can be drawn using as vertices a set V of N points in the plane. By counting the number of simple polygons of a particular configuration of V, an improved lower bound for $\Phi(N)$ is obtained. It is proved that $\Phi(N)^{\frac{1}{N}}$ is asymptotically greater than 3.6.

Key words: Simple polygon, convex hull, generating function, algebraic functions, formula of Cauchy-Hadamard.

1. INTRODUCTION

Let V be a set of N points in the plane. The number of simple polygons that can be drawn using as vertices the points of V depends on the relative position of these points. We are interested in knowing $\Phi(N)$, the maximum of the above numbers among the different configurations of V.

This problem was first studied by Newborn and Moser [7]; they proved that $\Phi(N)$ is asymptotically greater than 2.15^N. Later, Ajtai et al. [1] proved that $\limsup_{N\to\infty} \Phi(N)^{\frac{1}{N}}$ is finite, but very little is known about the value of this limit, c. The best bounds for c are:

$$3.268461786 < c \leq 1384000$$

where the former figure is due to Hayward [5], and the latter to Smith [9]. The latter author conjectures 6 as a probable value of c.

Hayward [5] obtains the lower bound 3.268461786 analyzing the particular case in which the points are in three spiralled arcs emanating from the origin, and counting a subset of the polygons of that configuration.

In this paper we will count the polygons that can be drawn when the configuration of the set V is as shown in figure 1: the points lie in two

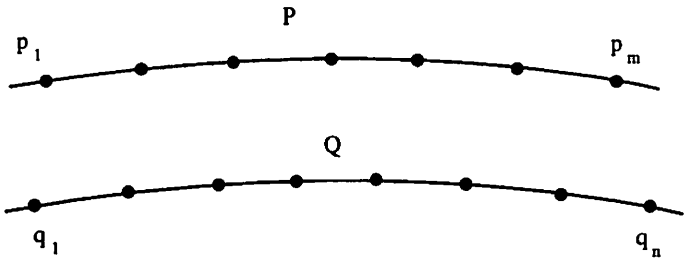

Figure 1

concave lines P and Q, such that, if $p_1, p_2, \ldots, p_m$ are the points on P and $q_1, q_2, \ldots, q_n$ are these on Q, the segments $p_i q_j$ do not cross Q. In addition, Q must be outside the convex hull formed by $\{p_1, p_2, \ldots, p_m\}$. We will prove that the number of polygons that can be drawn for this configuration is asymptotically greater than 3.6^N, where $N = m + n$.

To prove this, we will use one result (theorem 1) found in García and Tejel [4]. In that paper, the following situation is studied: P and Q are two convex polygons with Q inside P; any simple polygon with vertices on $P \cup Q$, traversed in clockwise direction, visits the points of Q in a particular order. In that paper, these possible orders are characterized.

Definition 1. We will say that a permutation of $(1, 2, \ldots, n)$ has "the subdivision property" if it is of the form $(1, \sigma)$, where σ is a permutation of the indices $2, 3, \ldots, n$ formed by two permutations σ_1 and σ_2 of consecutive indices, and the permutations σ_1 and σ_2 can be subdivided in the same way as σ, as long as they have at least two indices.

For example, the permutation $(1, 3, 4, 2, 7, 5, 6, 9, 8)$ has this property because:

$$\begin{gathered}(1, (3, 4, 2, 7, 5, 6, 9, 8)) \to (1, ((3, 4, 2), (7, 5, 6, 9, 8))) \to \\ (1, (((3, 4), (2)), ((7, 5, 6), (9, 8)))) \to \\ (1, ((((3), (4)), (2)), (((7), (5, 6)), ((9), (8))))) \to \\ (1, ((((3), (4)), (2)), (((7), ((5), (6))), ((9), (8)))))\end{gathered}$$

but $(1, 3, 5, 2, 4)$ does not have this property.

Definition 2. Given a permutation of n indices, we will say that six indices $i_1 < i_2 < i_3 < i_4 < i_5 < i_6$ form a six point "star" if they appear in the order $(\ldots, i_1, \ldots, i_2, \ldots, i_5, \ldots, i_6, \ldots, i_3, \ldots, i_4, \ldots)$ or in the order $(\ldots, i_1, \ldots, i_4, \ldots, i_5, \ldots, i_2, \ldots, i_3, \ldots, i_6, \ldots)$.

If $q_1, q_2, \ldots, q_n$ are the points of the convex polygon Q numbered clockwise, then the following is proved in [4]:

Theorem 1. The orders in which the points of Q can be visited in any simple polygon, are permutations that do not contain a six point star and verify the subdivision property.

2. RECURRENCE FORMULAS

For the particular configuration previously explained, we will give recurrence formulas to count the number of simple polygonal paths that visit all the points of P and Q, and that are of one of the following types (see figure 2).

Type 1: paths beginning in p_1 and finishing in p_m. We will denote by $g_1(m,n)$ the number of paths of this type.

Type 2: paths beginning in p_1 and finishing in q_1. Their number will be denoted by $g_2(m,n)$. By symetry, $g_2(m,n)$ also is the number of paths from q_n to p_m.

Type 3: paths from p_1 to q_n. Their number will be denoted by $g_3(m,n)$, which is the same as the one of paths from q_1 to p_m.

Type 4: this is a subclass of type 1, formed by paths in which the points q_1 and q_n are directly connected, and furthermore, a q_i is visited before q_1, and a q_j is visited after q_n. Their number will be denoted by $g_4(m,n)$.

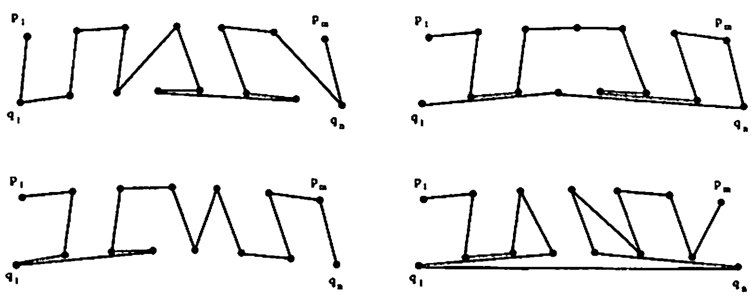

Figure 2

Notice that, for any of these paths, points $p_1, p_2, \ldots, p_m$ must be visited in this order and, if an edge $p_i q_j$ exists, then an edge $p_{i'} q_{j'}$, with $i' > i$ and $j' < j$, cannot exist. Also, as a consequence of this last result, for paths of type 2 (type 3), if q_1 (q_n) is visited directly from q_j, then points $\{q_2, \ldots, q_{j-1}\}$ ($\{q_1, \ldots, q_{j-1}\}$) are visited before $\{q_j, \ldots, q_n\}$ ($\{q_j, \ldots, q_{n-1}\}$).

On the other hand, the quantities $g_i(m,n), i = 1,2,3,4$ are well defined; they only depend on m and n, and not on the exact position of the points on P or Q, or on the election of P and Q while these lines verify the conditions explained in the previous section. Recurrence formulas for $g_i(m,n), i = 1,2,3,4$ can be obtained because each path of any of the previous types is formed by shorter paths of the same types. We will use the following lemma.

Lemma 1. Given a path of type 1, there exists $j < n$ such that the points $\{q_1, \ldots, q_j\}$ are visited before $\{q_{j+1}, \ldots, q_n\}$ and also q_1 is the last visited point among $\{q_1, \ldots, q_j\}$.

Proof:

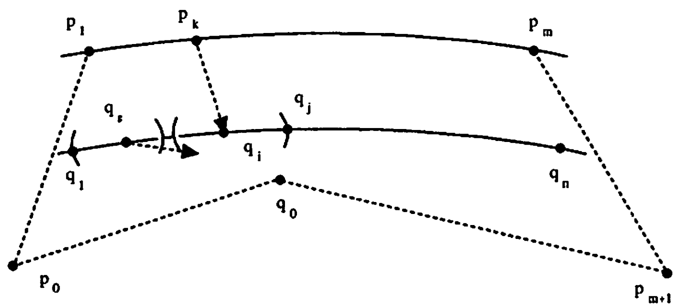

Figure 3

We will add three points p_{m+1}, p_0 and q_0 such that $p_0, \ldots, p_{m+1}$ form a convex polygon and $q_0, \ldots, q_n$ form another convex polygon inside the first (see figure 3). Any path of type 1 can be transformed into a cycle by adding the edges $p_m p_{m+1}$, $p_{m+1} q_0$, $q_0 p_0$ and $p_0 p_1$ (see figure 3). Then, $q_1, \ldots, q_n$ are visited consecutively in the cycle and, by the subdivision property, $j < n$ must exist such that, beginning in p_1, first $\{q_1, \ldots, q_j\}$ are visited and later $\{q_{j+1}, \ldots, q_n\}$. We will choose this j as the minimun of

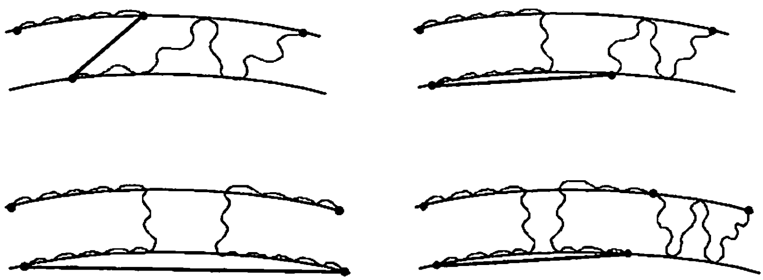

Figure 4

indices such that $\{q_1, \ldots, q_j\}$ are visited first.

If $j = 1$, the result is reached. If $j \neq 1$, then, the first visited point q_i $(i \neq 1)$ from $\{q_1, \ldots, q_j\}$ will be visited after a point p_k. Let us assume that $i = j$. If in $\{q_1, \ldots, q_j\}$ there are three points q_{i_1}, q_{i_2} and q_{i_3}, with $i_1 < i_2 < i_3$, and q_{i_1} and q_{i_3} directly joined, then q_{i_2} must be joined with a point $p_{k'}$ $(k' > k)$. But this is not possible because a link $q_{i_2}p_{k'}$ (or $p_{k'}q_{i_2}$) crosses $p_k q_i$. Hence, the points $\{q_1, \ldots, q_j\}$ are visited consecutively in counter-clockwise direction in Q.

Finally, let us assume that $i \neq j$. If q_s is the last visited point from $\{q_1, \ldots, q_j\}$, then, by the minimality of j and the subdivision property, the order of points in Q is $q_1, q_s, q_i, q_j, q_n, q_0$ and they are visited in the cycle in the order $q_i, q_j, q_1, q_s, q_n, q_0$ forming a six point star (see figure 3). Therefore, necessarily $q_1 = q_s$ so that the six point star does not appear. #

Let us see the recurrence formulas for $g_i(m,n), i = 1, 2, 3, 4$.

Type 1)

We will assume $m \geq 2$ and $n \geq 1$. By applying lemma 1, we have the following possibilities (see figure 4):

a) If the first point visited of Q is q_1, then the path continues with a path of type 3 and the possibilities are:

$$\sum_{k \in [1, m-1]} g_3(k, n)$$

b) If points $\{q_2, \ldots, q_j\}$ are visited before q_1, and then q_{j+1} is visited, the number of different possibilities for $n \geq 3$ is:

$$\sum_{k\in[1,m-1],j\in[2,n-1]} g_2(k,j)g_3(m-k,n-j)$$

c) If, after q_1, the point q_n is visited, then for $n \geq 4$, the possibilities are:

$$g_4(m,n)$$

d) Finally, if after q_1, a point $q_l, j+1 < l < n$ is visited, then, for $n \geq 5$ and $m \geq 3$, the number of continuations is:

$$\sum_{\substack{k\in[2,m-1]\\ l\in[4,n-1]}} g_4(k,l)g_1(m-k+1,n-l) - \sum_{\substack{k\in[2,m-1]\\ l\in[4,n-1]}} g_4(k-1,l)g_1(m-k+1,n-l)$$

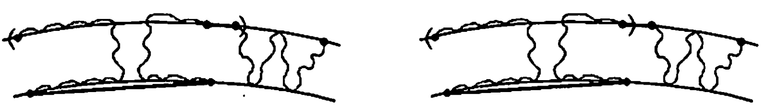

Figure 5

In this case, we must note that, if in one of these paths point p_k is visited directly after p_{k-1} , the first term of the previous formula counts these paths twice: once with the term $g_4(k,l)g_1(m-k+1,n-l)$ and the other with $g_4(k-1,l)g_1(m-k+2,n-l)$ (see figure 5).

Therefore, only paths of type $g_4(k,l)$ such that the point p_k is visited from a point of Q are considered so as not to count the same path several times (see figure 6). There are $g_4(k,l) - g_4(k-1,l)$ of such paths and this originates the subtracting term in the previous formula.

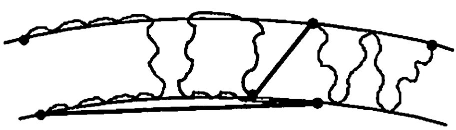

Figure 6

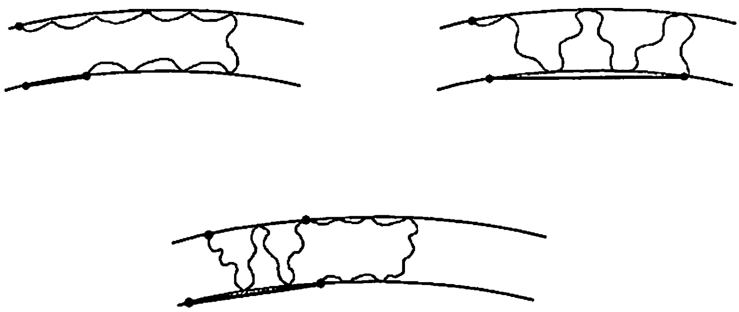

Figure 7

For the validity of the above formulas, the initial conditions must be:

$$g_1(m,0)=0 \quad \forall m \geq 2$$
$$g_1(0,n)=0 \quad \forall n$$
$$g_1(1,n)=0 \quad \forall n$$

Type 2)

The curve starts at p_1 and finishes at q_1 and we will assume $m \geq 1$ and $n \geq 2$. Then, we have the possibilities shown in figure 7:

a) If we arrive at point q_1 from q_2, then a term of type $g_2(m,n-1)$ appears.

b) If we arrive at point q_1 from point q_n, then a term of type $g_3(m,n-1)$ appears if $n \geq 3$.

c) If we arrive at point q_1 from q_j ($j \neq 2$ and $j \neq n$), then, for $n \geq 4$ and $m \geq 2$, a term of the following type appears:

$$\sum_{\substack{k\in[2,m]\\ l\in[1,n-3]}} g_1(k,l)g_2(m-k+1,n-l-1) - \sum_{\substack{k\in[2,m]\\ l\in[1,n-3]}} g_1(k-1,l)g_2(m-k+1,n-l-1)$$

As before, if p_k is visited directly after p_{k-1}, the first term of the previous formula counts these paths twice, therefore, only paths of type $g_1(k,l)$ such that point p_k is visited from a point of Q are considered and their number is $g_1(k,l)-g_1(k-1,l)$.

The initial conditions must be:

$$g_2(0,n) = 0 \quad \forall n$$
$$g_2(m,0) = 0 \quad \forall m \geq 1$$
$$g_2(m,1) = 1 \quad \forall m \geq 1$$

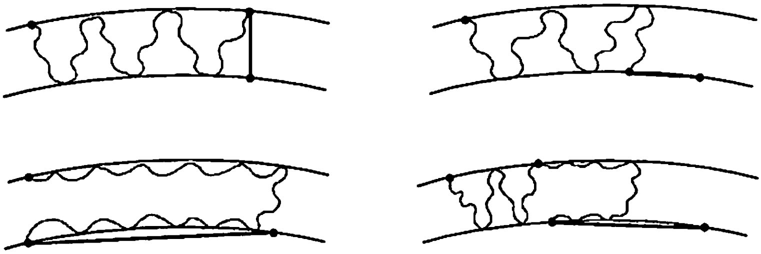

Figure 8

Type 3)

The path starts at p_1 and finishes at q_n and we will assume $m \geq 1$ and $n \geq 2$. Then, we have the following cases (see figure 8):

a) If we arrive at q_n from p_m, then a term of type $g_1(m, n-1)$ appears when $m \geq 2$.

b) If we arrive at q_n from q_{n-1}, then a term of type $g_3(m, n-1)$ appears.

c) If we arrive at q_n from q_1, then a term of type $g_2(m, n-1)$ appears when $n \geq 3$.

d) Finally, if we arrive at q_n from q_j ($1 < j < n-1$) then, for $n \geq 4$ and $m \geq 2$, a term of the following type appears:

$$\sum_{\substack{k\in[2,m]\\ l\in[1,n-3]}} g_1(k,l)g_2(m-k+1, n-l-1) - \sum_{\substack{k\in[2,m]\\ l\in[1,n-3]}} g_1(k-1,l)g_2(m-k+1, n-l-1)$$

The initial conditions must be:

$$g_3(0,n) = 0 \quad \forall n$$
$$g_3(m,0) = 0 \quad \forall m \geq 1$$
$$g_3(m,1) = 1 \quad \forall m \geq 1$$

Type 4)

If $m < 2$ and $n < 4$, $g_4(m,n) = 0$. For $g_4(m,n)$ with $m \geq 2$ and $n \geq 4$ we have:

$$g_4(m,n) = \sum_{k\in[1,m-1],l\in[2,n-2]} g_2(k,l)g_2(m-k,n-l)$$

In conclusion, by simplifying the above formulas we obtain:

$$\begin{aligned} g_1(m,n) = &\sum_{k\in[0,m-1]} g_3(k,n) + g_4(m,n) + \sum_{\substack{k\in[0,m]\\ l\in[0,n]}} g_2(k,l)g_3(m-k,n-l) \\ &- \sum_{k\in[0,m-1]} g_3(k,n-1) + \sum_{\substack{k\in[0,m+1]\\ l\in[0,n]}} g_1(k,l)g_4(m-k+1,n-l) \\ &- \sum_{\substack{k\in[0,m]\\ l\in[0,n]}} g_1(k,l)g_4(m-k,n-l) \qquad m \geq 1, n \geq 1 \end{aligned} \tag{1}$$

$$\begin{aligned} g_2(m,n) = &\ g_2(m,n-1) + g_3(m,n-1) - g_1(m,n-2) \\ &+ \sum_{\substack{k\in[0,m+1]\\ l\in[0,n-1]}} g_1(k,l)g_2(m-k+1,n-l-1) \\ &- \sum_{\substack{k\in[0,m]\\ l\in[0,n-1]}} g_1(k,l)g_2(m-k,n-l-1) \qquad m \geq 1, n \geq 3 \end{aligned} \tag{2}$$

$$\begin{aligned} g_3(m,n) = &\ g_1(m,n-1) + g_2(m,n-1) + g_3(m,n-1) - g_1(m,n-2) \\ &+ \sum_{\substack{k\in[0,m+1]\\ l\in[0,n-1]}} g_1(k,l)g_2(m-k+1,n-l-1) \\ &- \sum_{\substack{k\in[0,m]\\ l\in[0,n-1]}} g_1(k,l)g_2(m-k,n-l-1) \qquad m \geq 1, n \geq 3 \end{aligned} \tag{3}$$

$$\begin{aligned} g_4(m,n) = &\sum_{k\in[0,m],l\in[0,n]} g_2(k,l)g_2(m-k,n-l) \\ &- 2\sum_{k\in[0,m-1]} g_2(k,n-1) \qquad m \geq 1, n \geq 3 \end{aligned} \tag{4}$$

	m=5	m=10	m=15	m=20
n=1	1	1	1	1
n=5	104	434	1114	2269
n=10	82546	1353295	8575904	35524423
n=15	41242712	2366615340	32864853312	246076961668
n=20	14013817844	2682887569136	78191775842203	1018897823215759

Table 1

with the initial conditions:

$$
\begin{aligned}
g_1(0,n) &= 0 \quad \forall n \\
g_1(m,0) &= 0 \quad \forall m \geq 1 \\
g_2(0,n) &= 0 \quad \forall n \\
g_2(m,0) &= 0 \quad \forall m \geq 1 \\
g_2(m,1) &= 1 \quad \forall m \geq 1 \\
g_2(m,2) &= g_2(m,1) = 1 \quad \forall m \geq 1 \\
g_3(0,n) &= 0 \quad \forall n \\
g_3(m,0) &= 0 \quad \forall m \geq 1 \\
g_3(m,1) &= 1 \quad \forall m \geq 1 \\
g_3(m,2) &= g_1(m,1) + g_3(m,1) = m \quad \forall m \geq 1 \\
g_4(0,n) &= 0 \quad \forall n \\
g_4(m,0) &= g_4(m,1) = g_4(m,2) = 0 \quad \forall m \geq 1
\end{aligned} \tag{5}
$$

These recurrence formulas allow us to calculate $g_1(m,n)$, $g_2(m,n)$, $g_3(m,n)$ and $g_4(m,n)$ for every m and n. Table 1 shows $g_2(m,n)$ for some values of m and n.

3. GENERATING FUNCTION OF $g_2(m,n)$

From a path of type 2, a simple polygon can be formed joining the two extremes of this path. Consequently, $\Phi(m+n) > g_2(m,n)$, and if we calculate the asymptotic value of $g_2(m,n)$, we will obtain a lower bound for $\Phi(m+n)$. We will calculate in this section the generating function of $g_2(m,n)$.

From (2) and (3) we obtain:

$$g_3(m,n) = g_2(m,n) + g_1(m,n-1) \qquad m \geq 0, n \geq 1$$

and then, we have for (1) and (2) the following formulas:

$$\begin{aligned} g_1(m,n) = & \sum_{k\in[0,m-1]} g_2(k,n) + \sum_{k\in[0,m-1]} g_1(k,n-1) + g_4(m,n) \\ & + \sum_{k\in[0,m],l\in[0,n]} g_2(k,l)g_2(m-k,n-l) \\ & + \sum_{k\in[0,m],l\in[0,n-1]} g_2(k,l)g_1(m-k,n-l-1) \\ & - \sum_{k\in[0,m-1]} g_2(k,n-1) - \sum_{k\in[0,m-1]} g_1(k,n-2) \\ & + \sum_{k\in[0,m+1],l\in[0,n]} g_1(k,l)g_4(m-k+1,n-l) \\ & - \sum_{k\in[0,m],l\in[0,n]} g_1(k,l)g_4(m-k,n-l) \qquad m \geq 1, n \geq 2 \end{aligned} \tag{6}$$

$$\begin{aligned} g_2(m,n) = & \; 2g_2(m,n-1) + \sum_{\substack{k\in[0,m+1] \\ l\in[0,n-1]}} g_1(k,l)g_2(m-k+1,n-l-1) \\ & - \sum_{\substack{k\in[0,m] \\ l\in[0,n-1]}} g_1(k,l)g_2(m-k,n-l-1) \qquad m \geq 1, n \geq 3 \end{aligned} \tag{7}$$

Let $G1(x,y) = \sum_{m\geq 0}\sum_{n\geq 0} g_1(m,n)x^m y^n$ be the generating function of $g_1(m,n)$. Analogously, we will define $G2(x,y)$ and $G4(x,y)$ the generating functions of $g_2(m,n)$ and $g_4(m,n)$, respectively.

We have for $G2(x,y)G2(x,y)$:

$$G2(x,y)G2(x,y) = \sum_{\substack{m\geq 0\\ n\geq 0}} x^m y^n \left(\sum_{\substack{k\in[0,m]\\ l\in[0,n]}} g_2(k,l) g_2(m-k,n-l) \right) =$$

$$= \sum_{\substack{m\geq 1\\ n=2}} x^m y^2 \left(\sum_{\substack{k\in[0,m]\\ l=1}} g_2(k,1) g_2(m-k,1) \right)$$

$$+ \sum_{\substack{m\geq 1\\ n\geq 3}} x^m y^n \left(\sum_{\substack{k\in[0,m]\\ l\in[0,n]}} g_2(k,l) g_2(m-k,n-l) \right)$$

As $g_2(m,1) = 1, m \geq 1$ and using (4) we have:

$$G2(x,y)^2 = \sum_{m\geq 2} x^m y^2 (m-1) + \sum_{\substack{m\geq 1\\ n\geq 3}} x^m y^n \left(g_4(m,n) + 2 \sum_{k\in[0,m-1]} g_2(k,n-1) \right)$$

$$= \sum_{m\geq 2} x^m y^2 (m-1) + G4(x,y)$$

$$+ 2 \left(\sum_{k\geq 1} x^k \right) \left(\sum_{m\geq 0, n\geq 3} x^m y^n g_2(m,n-1) \right)$$

$$= y^2 \left(\sum_{m\geq 2} m x^m - \sum_{m\geq 2} x^m \right) + G4(x,y)$$

$$+ 2 \left(\sum_{k\geq 1} x^k \right) y \left(\sum_{m\geq 0, n\geq 2} x^m y^n g_2(m,n) \right)$$

$$= y^2 \left(\sum_{m\geq 2} m x^m - \sum_{m\geq 2} x^m \right) + G4(x,y)$$

$$+ 2 \left(\sum_{k\geq 1} x^k \right) y \left(G2(x,y) - \sum_{\substack{m\geq 0\\ n=0}} x^m g_2(m,0) - \sum_{\substack{m\geq 0\\ n=1}} x^m y g_2(m,1) \right)$$

Using $\sum_{m\geq 0} x^m = \frac{1}{1-x}$ and $\sum_{m\geq 1} m x^m = \frac{x}{(1-x)^2}$ for $|x| < 1$ we obtain:

$$G2(x,y)G2(x,y) = G4(x,y) + \frac{2xy}{1-x}G2(x,y) - \frac{x^2y^2}{(1-x)^2} \tag{8}$$

Similarly, we obtain for $G1(x,y)G2(x,y)$ and $G1(x,y)G4(x,y)$ from (6) and (7) respectively:

$$G1(x,y)G2(x,y) = \frac{x(1-2y)}{(1-x)y}G2(x,y) + \frac{x^2(y-1)}{(1-x)^2} \tag{9}$$

$$\begin{aligned} G1(x,y)G4(x,y) = {} & \frac{x}{1-x}\left(\frac{xy(y-1)}{1-x}+1\right)G1(x,y) + \frac{x^2(y-2)}{(1-x)^2}G2(x,y) \\ & - \frac{2x}{1-x}G4(x,y) + \frac{x^3y}{(1-x)^3} \end{aligned} \tag{10}$$

From (8), (9) and (10) we obtain the equation of third degree for $G2(x,y)$:

$$\begin{aligned} & G2(x,y)^3 - \frac{xy}{1-x}G2(x,y)^2 + \frac{xy(1-x)-x(1-y)(1-x-xy)}{(1-x)^2}G2(x,y) + \\ & + \frac{x^2y(1-y)(1-x-xy)}{(1-x)^3} = 0 \end{aligned} \tag{11}$$

4. ASYMPTOTIC VALUE OF $g_2(m,n)$

For fixed complexes x and y the equation (11) will have three solutions, and the series

$$\sum g_2(m,n)x^my^n \tag{12}$$

will converge to one of these solutions for values of x and y in a determined region of C^2. We must then solve two problems: which is the convergence region of the double series (12) and which of the roots of (11) coincides with the value of the series in that region.

The following theorem solves both problems:

Theorem 2. The absolute convergence region of series (12) is the region of $(x, y) \in C^2$ so that $0 < |x| < 1$, $0 < |y| < \frac{1}{2}$ and $D(|x|, |y|) < 0$, where $D(x, y)$ is the discriminant of the equation $z^3 + az + b = 0$ with $a = x^2(1 - y - \frac{4}{3}y^2) + x(2y - 1)$ and $b = -\frac{x^2 y}{3}\left(x(2 + y - \frac{16}{9}y^2) + y - 2\right)$. Furthermore, for real values of x and y in the interior of that region, equation (11) holds three real roots $v_1 \leq v_2 \leq v_3$ and the intermediate root v_2 gives the value of the series.

Proof:

For complexes x' and y', if $\sum g_2(m, n)|x'|^m|y'|^n$ converges, then series (12) is absolutely convergent. Given real values $x > 0$ and $y > 0$, if (12) converges for x and y, then (12) is absolutely convergent for x' and y' complexes, such that $|x'| = x$ and $|y'| = y$. Thus, henceforth we will only consider positive real values of x and y.

Multiplying the equation (11) by $(1-x)^3$ and substituting $(1-x)G2(x, y)$ $-\frac{xy}{3}$ for z, equation (11) yields:

$$z^3 + az + b = 0 \tag{13}$$

with $a = x^2(1-y-\frac{4}{3}y^2)+x(2y-1)$ and $b = -\frac{x^2 y}{3}\left(x(2 + y - \frac{16}{9}y^2) + y - 2\right)$.

For each value of y, equation (13) represents a 3-form algebraic function, whose only singularities can be the zeroes of the discriminant of the equation. Any branch of the function will be analytical in any region without singularities, and consequently, in a neighborhood of the origin with radius less than the minimum among the absolute values of the singularities. In addition, as $g_2(m, n) > 0 \ \forall \ m, n$, the singularity of the root of (11) that coincides with (12), must be real.

The discriminant of (13) is:

$$D(x, y) = \frac{b^2}{4} + \frac{a^3}{27} = \frac{x^3}{108}\left(x^3 c_0(y) + x^2 c_1(y) + x c_2(y) + c_3(y)\right) \tag{14}$$

where:

$$c_0(y) = -32y^5 - 13y^4 + 40y^3 + 8y^2 - 12y + 4$$
$$c_1(y) = 32y^5 + 70y^4 - 72y^3 - 52y^2 + 48y - 12$$
$$c_2(y) = -61y^4 + 4y^3 + 92y^2 - 60y + 12$$
$$c_3(y) = 32y^3 - 48y^2 + 24y - 4 = 32\left(y - \frac{1}{2}\right)^3$$

On the other hand, for an equation of type (13) with real coefficients, there are three real roots if $D(x,y) < 0$. If $D(x,y) > 0$, there is only a real root.

Let y be a fixed real value $0 < y < 1/2$. In addition to $x = 0$, which is not a singularity, the other zeroes of (14) are the solutions of the cubic equation $\Delta(x,y) = 0$, where

$$\Delta(x,y) = x^3 c_0(y) + x^2 c_1(y) + x c_2(y) + c_3(y)$$

Dividing by $c_0(x)$ and substituting $x + \frac{c_1(y)}{3c_0(y)}$ for w this equation becomes:

$$w^3 + a_1 w + b_1 = 0 \tag{15}$$

Its discriminant is given by $D_1(y) = \frac{a_1^3}{27} + \frac{b_1^2}{4}$, and on operating:

$$D_1(y) = \frac{4}{27} \frac{y^{12}(y-1)(3y^2-3y+1)(20y-19)^3}{c_0(y)^4}$$

For the considered values of y the discriminant $D_1(y)$ is positive , hence (15) has only one real root, that we will denote by $w(y)$. Therefore, series (12) is convergent for real y if $0 < y < 1/2$ and real x with $0 < x < w(y)$.

For $x = 0$ the three roots of equation (13) are $z_1 = -\sqrt{x(1-x)}, z_2 = 0, z_3 = +\sqrt{x(1-x)}$. Then, the solutions of (11) are: $v_1 = -\sqrt{\frac{x}{1-x}}, v_2 = 0, v_3 = \sqrt{\frac{x}{1-x}}$ and the solution that corresponds to value of the series is the intermediate v_2. By continuity, while x is real and $x < w(y)$, equation (13) has three real roots (because $D(x,y) < 0$) and the corrresponding root to series (12) is the intermediate one. But if $x > w(y)$ then $D(x,y) > 0$ and this intermediate root becomes complex, so series (12) cannot be convergent for values of x outside the above region (except if $y = 0$) and $w(y)$ is the singularity of this intermediate root for fixed y. #

Figure 9 shows the convergence zone of $G2(x,y)$ as a function of $|x|$ and $|y|$.

Let r_1, r_2 denote the associated radii of convergence of series (12). They are real numbers with $0 < r_1 < 1$, $0 < r_2 < 1/2$, $\Delta(r_1, r_2) = 0$. Then, for any r_1, r_2,we can apply the formula of Cauchy-Hadamard in C^2:

$$\limsup_{N\to\infty} \max_{n+m=N} (g_2(m,n) r_1^m r_2^n)^{1/N} = 1$$

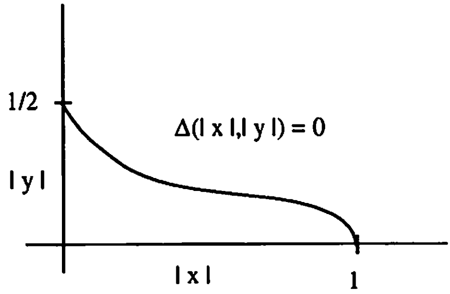

Figure 9

If we choose the radii of convergence $\bar{r}_1$ and $\bar{r}_2$ such that $\bar{r}_1 = \bar{r}_2$, the above formula becomes:

$$\limsup_{N\to\infty} \max_{n+m=N} (g_2(m,n))^{1/N} = \frac{1}{\bar{r}_1}$$

By solving the equation

$$\Delta(r_1, r_1) = 0$$

with $0 < r_1 < 1$, we obtain the value $\bar{r}_1 = 0.27339098$, so $\frac{1}{\bar{r}_1} = 3.60501960$.

Therefore, we have proved the following result:

Theorem 3.

$$\limsup_{N\to\infty} \Phi(N)^{1/N} \geq \limsup_{N\to\infty} \max_{n+m=N} (g_2(m,n))^{1/N} = 3.60501960$$

5. BIBLIOGRAPHY

[1] M. Ajtai, V. Chvátal, M.M. Newborn, E. Szemerédi, Crossing-free subgraphs, *Ann. of Disc. Math.* 12 (1982), 9-12.

[2] S. Akl, A lower bound on the maximum number of crossing-free hamiltonian cycles in a rectilinear drawing of K_n, *Ars Combin.* 7 (1979), 7-18.

[3] B.A. Fucks, *Introduction to the Theory of Analytical Functions of Several Complex Variables, vol.* 8, American Mathematical Society, Providence, Rhode Island, 1963.

[4] A. García, J. Tejel, The order of points on the second convex hull of a simple polygon, *Disc. and Comput. Geom*, to appear.
[5] R.B. Hayward, A lower bound for the optimal crossing-free hamiltonian cycle problem, *Disc. and Comput. Geom.* 2 (1987), 327-343.
[6] A. Markushevich, *Teoría de las Funciones Analíticas. Tomo II*, Mir, Moscow, 1978.
[7] M.M. Newborn, W.O.J. Moser, Optimal crossing-free Hamiltonian circuit drawings of K_n, *J. of Comb. Theory, Series B* 29 (1980), 13-26.
[8] V.I. Smirnov, *A Course of Higher Mathematics. Vol I and III*, Pergamon Press Ltd., Edinburgh, 1964.
[9] W.D. Smith. *Studies in Computational Geometry motivated by mesh generation.* Ph.D. Thesis, Princeton University, 1989.