Strong Chromatic Index in Subset Graphs

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Abstract

The strong chromatic index of a graph G, denoted $sq(G)$, is the minimum number of parts needed to partition the edges of G into induced matchings. The subset graph $B_m(k)$ is the bipartite graph whose vertices represent the elements and the k-subsets of an m element ground set where two vertices are adjacent if and only if the vertices are distinct and the element corresponding to one vertex is contained in the subset corresponding to the other. We show that $sq(B_m(k)) = \binom{m}{k-1}$ and that this satisfies the strong chromatic index conjecture by Brualdi and Quinn [3] for bipartite graphs.

1 Introduction

A *strong edge coloring* of a graph G is an assignment of colors to the edges so that edges of the same color form an induced matching in the graph. You can think of a strong edge coloring as an edge coloring in which there are at least 2 edges of different colors on the shortest path between each edge of the same color. The *strong chromatic index*, $sq(G)$, equals the smallest number of colors in a strong edge coloring.

Much work has been focused on bounding the strong chromatic index of a graph based on its maximum degree. A conjecture given by Erdős and Nešetřil [5] states that for a graph of maximum degree Δ its strong chromatic index will be less than or equal to

$$\begin{cases} \frac{5}{4}\Delta^2 & \text{if } \Delta \text{ is even.} \\ \frac{5}{4}\Delta^2 - \frac{1}{2}\Delta + \frac{1}{4} & \text{if } \Delta \text{ is odd.} \end{cases}$$

This was verified by Cameron [4] for chordal graphs and by Horák et al. [7] and Andersen [1] when $\Delta = 3$. Restricting our attention to bipartite graphs, Faudree et. al. [6] conjectured that a bipartite graph G of maximum degree Δ has strong chromatic index less than or equal to Δ^2. They verified the conjecture for trees, d-dimensional cubes, revolving door graphs, and graphs with all cycle lengths divisible by 4. Steger and Yu [9] verified the bipartite conjecture when $\Delta = 3$. Brualdi and Quinn [3] further conjectured that a bipartite graph with bipartition X and Y, where the maximum degree of a vertex in X is α and the maximum degree of a vertex in Y is β, has strong chromatic index less than or equal to $\alpha\beta$. This statement contains the Faudree et. al. conjecture since $\alpha, \beta \leq \Delta$, hence $\alpha\beta \leq \Delta^2$. Their conjecture is verified for bipartite graphs when $\alpha = 2$, β is arbitrary and no cycles are of length 4; when α and β are arbitrary and all cycle lengths are divisible by 4; and when the associated incidence matrix is from a projective plane, an affine plane, or is 2-totally unimodular (see e.g. [8].) In this paper, we verify the Brualdi-Quinn conjecture for an infinite family of graphs we call subset graphs. For integers k and m with $0 < k \leq m$, the *subset graph* $B_m(k)$ is the bipartite graph whose vertices represent the elements and the k-subsets of an m element ground set. Two distinct vertices are adjacent if and only if the element corresponding to one vertex is contained in the subset corresponding to the other. For $B_3(2)$, $X = \{1, 2, 3\}$, $Y = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$, and $E = \{\{1, \{1, 2\}\}, \{1, \{1, 3\}\}, \{2, \{1, 2\}\}, \{2, \{2, 3\}\}, \{3, \{1, 3\}\}, \{3, \{2, 3\}\}\}$. (See Figure 1.)

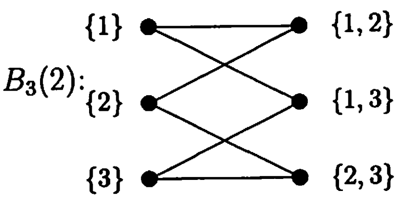

Figure 1: A representation of the subset graph $B_3(2)$.

We prove that $sq(B_m(k)) = \binom{m}{k-1}$ and verify that this satisfies the Brualdi–Quinn conjecture for the strong chromatic index of bipartite graphs.

2 Results

We present a simple lower bound for the strong chromatic index by considering a large set of edges which must be given different colors in any strong edge coloring of $B_m(k)$.

Theorem 2.1 $sq(B_m(k)) \geq \binom{m}{k-1}$.

Proof. Label the first k vertices in X: $x_1, x_2, \ldots, x_k$. Now consider the number of k-subsets of vertices from X that contain the vertex x_1. There will be $\binom{m-1}{k-1}$ of them because we can choose $k-1$ more elements to fill the k-subset from a set of $m-1$ elements. Associated with each of these $\binom{m-1}{k-1}$ subsets is an edge between each of these subsets and vertex x_1. Now consider the number of k-subsets that contain both vertices x_1 and x_2, there will be $\binom{m-2}{k-2}$ of these subsets. Associated with these $\binom{m-2}{k-2}$ subsets is an edge between each of these subsets and vertex x_2. There will be $\binom{m-3}{k-3}$ k-subsets that contain vertices x_1, x_2, and x_3, $\binom{m-4}{k-4}$ k-subsets that contain vertices x_1, x_2, x_3, and x_4, up to $\binom{m-k}{0} = 1$ k-subsets that contain vertices $x_1, x_2, \ldots, x_k$. Associated with these subsets are the corresponding edges to the vertices $x_3, x_4, \ldots, x_k$. Pairwise, these edges are at most 1 edge away from each other and hence must all be colored differently. So a strong edge coloring of $B_m(k)$ requires at least $\binom{m-1}{k-1} + \binom{m-2}{k-2} + \binom{m-3}{k-3} + \cdots + \binom{m-k}{0}$ colors. By repeated applications of Pascal's formula (see e.g. Brualdi [2]) this sum of binomial coefficients reduces to $\binom{m}{k-1}$. Hence $sq(B_m(k)) \geq \binom{m}{k-1}$. □

Before proving that equality holds in Theorem 2.1, we need to develop a special property of edge colorings. An edge coloring of a bipartite graph $G = (X, E, Y)$ using t colors is *k-distributed* if for every k-subset, A, of X, there is exactly one of the t colors which is not incident to some vertex of A. Further, each color is absent from at most one such k-subset of X. Since a color can be missing at most once, we see that there have to be at least as many colors as k-subsets. So $t \geq \binom{|X|}{k}$. If $t = \binom{|X|}{k}$ then there are exactly as many colors as k-subsets and each color is missing from any k-subset exactly once. The following theorem constructs a $(k-1)$-distributed strong edge coloring for $B_m(k)$.

Theorem 2.2 *There is a $(k-1)$-distributed strong edge coloring of $B_m(k)$ using $\binom{m}{k-1}$ colors.*

Proof. The proof proceeds by induction on $m + k$. Two families of subset graphs serve as the basis, $B_m(1)$ and $B_m(m)$. For any positive integer m, $B_m(1)$ is a matching with m edges. Assigning each edge the same color creates a 0-distributed strong edge coloring using $1 = \binom{m}{0}$ color. The graph $B_m(m)$ is isomorphic to the complete bipartite graph $K_{m,1}$. Assigning each edge a different color creates a $(m-1)$-distributed strong edge coloring using $m = \binom{m}{m-1}$ colors.

For $1 \leq k \leq m$ and $m + k < C$, assume that $B_m(k)$ has a $(k-1)$-distributed strong edge coloring using $\binom{m}{k-1}$ colors. Now consider $B_m(k)$ when $m + k = C$ and $1 < k < m$. Label the vertices in X: $x_1, x_2, \ldots, x_m$; and label the vertices in Y that are adjacent to x_m: $y_1, y_2, \ldots, y_{\binom{m-1}{k-1}}$.

Let H_1 be the subgraph of $B_m(k)$ which results from removing x_m and those vertices of Y adjacent to x_m. Then H_1 is isomorphic to $B_{m-1}(k)$, and by induction can be given a $(k-1)$-distributed strong edge coloring using $\binom{m-1}{k-1}$ colors. Let H_2 be the subgraph of $B_m(k)$ induced by the vertices $x_1, x_2, \ldots, x_{m-1}$ and $y_1, y_2, \ldots, y_{\binom{m-1}{k-1}}$. Then H_2 is isomorphic to $B_{m-1}(k-1)$, and by induction can be given a $(k-2)$-distributed strong edge coloring using $\binom{m-1}{k-2}$ additional colors. Hence, we have used $\binom{m-1}{k-1} + \binom{m-1}{k-2} = \binom{m}{k-1}$ colors so far. It remains to color the $\binom{m-1}{k-1}$ edges incident to x_m.

Since our coloring of H_1 is $(k-1)$-distributed, there is exactly 1 color missing from each $(k-1)$-subset of the vertices $x_1, x_2, \ldots, x_{m-1}$. Assign the edge $\{x_m, y_1\}$ the color that is missing from the other $(k-1)$ neighbors of y_1. Do the same for the edges $\{x_m, y_2\}$, $\{x_m, y_3\}, \ldots, \{x_m, y_{\binom{m-1}{k-1}}\}$. This requires no additional colors and gives us a valid strong edge coloring for $B_m(k)$ using $\binom{m}{k-1}$ colors.

Now we must show that the strong edge coloring is $(k-1)$-distributed. Any $(k-1)$-subsets taken from the set $\{x_1, x_2, \ldots, x_{m-1}\}$ will satisfy the necessary properties to be $(k-1)$-distributed, because they were $(k-1)$-distributed in the coloring of H_1, and the colors added to them by the coloring of H_2 were $(k-2)$-distributed (so any $(k-1)$-subset will contain all of the colors used in H_2.)

A $(k-1)$-subset that contains x_m is composed of x_m and a $(k-2)$-subset from $\{x_1, x_2, \ldots, x_{m-1}\}$. Now x_m is incident to all of the $\binom{m-1}{k-1}$ colors used in H_1, so the missing colors from these $(k-1)$-subsets must be from the colors used in H_2. Since we are selecting $(k-2)$-subsets from $\{x_1, x_2, \ldots, x_{m-1}\}$ and the coloring of H_2 is $(k-2)$-distributed, we know that there will be exactly 1 color missing from each $(k-1)$-subset containing x_m, and each color is missing exactly once. Hence the strong edge coloring of $B_m(k)$ is $(k-1)$-distributed. □

Corollary 2.3 *The strong chromatic index of the bipartite graph $B_m(k)$ equals $\binom{m}{k-1}$.*

Proof. Theorem 2.2 inductively constructs an $\binom{m}{k-1}$ strong edge coloring for $B_m(k)$. Theorem 2.1 requires a strong edge coloring to contain at least $\binom{m}{k-1}$ colors. Hence equality holds and $sq(B_m(k)) = \binom{m}{k-1}$. □

Finally, we verify that the strong chromatic index of $B_m(k)$ satisfies the Brualdi–Quinn conjecture. The degree of every vertex in Y is k and the degree of every vertex in X is $\binom{m-1}{k-1}$. To satisfy the conjecture, we must show that $\binom{m}{k-1} \leq \binom{m-1}{k-1}k$.

By assumption $1 \leq k \leq m$ so $0 \leq (m-k)(k-1)$. Hence $m \leq (m-k+1)k$ and

$$m \cdot \frac{(m-1)(m-2)\cdots(m-k+2)}{(k-1)!} \leq \frac{(m-1)(m-2)\cdots(m-k+2)}{(k-1)!} \cdot (m-k+1)k.$$

Thus $\binom{m}{k-1} \leq \binom{m-1}{k-1}k$. Hence the inequality always holds. Further $sq(B_m(k)) = \binom{m-1}{k-1}k$ if and only if $k = m$ or $k = 1$.

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