Counting Balanced Ternary Designs

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Abstract

An inductive process is used to find formulae for the number of 3-block configurations in BTD's with parameters $(\mathcal{V};\mathcal{B};\mathcal{R},\rho_1,\rho_2;3;2)$. In the process, a generating set of size nine is produced for the formulae. Because BIBD's can be viewed as BTD's with $\rho_2 = 0$, once found, the BTD formulae yield the 3-block configuration formulae for BIBD's with parameters $(v,b,r,3,2)$.

Key words: balanced incomplete block designs, configurations, designs, ternary designs

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1 Introduction

A balanced ternary design, BTD, with parameters $(\mathcal{V},\mathcal{B},\mathcal{R},\mathcal{K},\Lambda)$ is a collection of $\mathcal{B}$ blocks on $\mathcal{V}$ elements such that each element occurs $\mathcal{R}$ times in the design; each block contains $\mathcal{K}$ elements, where an element may occur 0, 1 or 2 times in a block (i.e. a block is a collection of elements rather than a set of elements); and, each pair of distinct elements occurs Λ times in the design. BTD's are regular in the sense that every design element occurs singly in ρ_1 blocks and doubly in ρ_2 blocks, where $\mathcal{R} = \rho_1 + 2\rho_2$ [1]. Because of this regularity, BTD parameters are most often specified as $(\mathcal{V};\mathcal{B};\mathcal{R},\rho_1,\rho_2;\mathcal{K};\Lambda)$. If ρ_2 is zero (i.e. no element appears doubly in a block), the block design reduces to what is called a balanced incomplete block design, BIBD [2, Chapter 8] and [3]. BIBD parameters are given as (v,b,r,k,λ).

An n-block design configuration is a collection of any n distinct blocks of the design. The set of n-block configurations of a design can be partitioned into configuration classes where the classes are based on element dependency patterns. Each class of the partition (i.e. each n-block configuration pattern) is represented by a template that describes the inter-block and intra-block element relationships shared by the configurations in that particular class. For example, the set

$$\left\{\begin{array}{lll} \{126,126,114\}, & \{126,126,225\}, & \{126,126,664\}, \\ \{234,234,225\}, & \{234,234,336\}, & \{234,234,445\}, \\ \{315,315,336\}, & \{315,315,114\}, & \{315,315,556\} \end{array}\right\}$$

is a 3-block configuration class in the BTD(6;12;4,1;3;2) with blocks $\{114, 225, 336, 445, 556, 664, 126, 126, 234, 234, 135, 135\}$. The class template is *abc abc aad.*

Configuration classes have no particular structure. However, it is possible to determine formulae that specify the number of configurations contained in the different classes. These formulae are classified as constant or variable. A configuration formula is labeled constant if it is stated solely in terms of the design parameters. A formula that is not constant is said to be variable. Let G be a set of design configuration templates where each template represents an m-block configuration class for some $m \leq n$. Let G' be the set of configuration formulae associated with the templates of G. If every n- block configuration formula of the design can be written in terms of the design parameters and the configuration formulae of G', we say G is a generating set for the formulae. A generating set that is minimal is called a basis.

Several recent papers [4, 5, 6, 7] investigate the templates, formulae and generating sets of n-block design configurations. Grannell et al. concentrate on configurations in BIBD's with parameters $(v, b, r, 3, 1)$. In their sentinel paper [6], formulae for all 1, 2, 3 and 4-block BIBD $(v, b, r, 3, 1)$ configurations are presented. The formulae establish four as the smallest n for which a variable n-block BIBD $(v, b, r, 3, 1)$ configuration formula exists. Basis size for the 4-block BIBD $(v, b, r, 3, 1)$ configuration formulae is shown to be one. Francel and Sarvate [5] consider configuration templates, formulae and generating sets in BTD$(\mathcal{V}; \mathcal{B}; \mathcal{R}, \rho_1, \rho_2; 3; 2)$. Their work shows that variable BTD configuration formulae first appear for 2-block configurations. Here, basis size is shown to be two.

The purpose of this paper is to further the study of BTD configuration templates, formulae and generating sets. In particular, we establish 3-block BTD $(\mathcal{V}; \mathcal{B}; \mathcal{R}, \rho_1, \rho_2; 3; 2)$ configuration templates, formulae and generating sets. In Section 2 we list the templates of the 3-block configuration classes and explain the process used to find the desired configuration formulae. In section 3, we present several examples that illustrate the details of the process. In section 4, the system of equations and the basis used in finding a solution are discussed. As stated above, a BTD with ρ_2 equal to zero is a BIBD. Because of this relationship between BTD's and BIBD's, 3-block BTD $(\mathcal{V}; \mathcal{B}; \mathcal{R}, \rho_1, \rho_2; 3; 2)$ configuration formulae can be used to produce 3-block BIBD $(\mathcal{V}, \mathcal{B}, \mathcal{R}, 3, 2)$ configuration formulae. We do this in Section 5.

Unless otherwise noted, throughout the paper distinct letters will be used to indicate distinct elements. For example the pair of blocks *aab bbc* contains three distinct elements; *a* appearing twice in the first block, *b* appearing once in block one and twice in block two, and *c* appearing once in block two.

0-1	abc abc axy	0-6	abc abd cdx	0-10	abc ade bdx
0-2	abc abc xyz	0-7	abc abd cxy	0-11	abc ade bxy
0-3	abc abd acd	0-8	abc abd xyz	0-12	abc ade xyz
0-4	abc abd acx	0-9	abc ade axy	0-13	abc def xyz
0-5	abc abd axy				

Single-Single-Single templates

1-1	aab acd acd	1-8	aab acd cxy	1-14	aab bcd cxy
1-2	aab acd acx	1-9	aab acd xyz	1-15	aab bcd xyz
1-3	aab acd axy	1-10	aab bcd bcd	1-16	aab cde cde
1-4	aab acd bcd	1-11	aab bcd bcx	1-17	aab cde cdx
1-5	aab acd bcx	1-12	aab bcd bxy	1-18	aab cde cxy
1-6	aab acd bxy	1-13	aab bcd cdx	1-19	aab cde xyz
1-7	aab acd cdx				

Double-Single-Single templates

2-1	aab aac axy	2-8	aab bbc cxy	2-14	aab ccd acx
2-2	aab aac bcx	2-9	aab bbc xyz	2-15	aab ccd adx
2-3	aab aac bxy	2-10	aab bcc acx	2-16	aab ccd axy
2-4	aab aac xyz	2-11	aab bcc axy	2-17	aab ccd bdx
2-5	aab bbc acx	2-12	aab bcc bxy	2-18	aab ccd bxy
2-6	aab bbc axy	2-13	aab bcc xyz	2-19	aab ccd xyz
2-7	aab bbc bxy				

Double-Double-Single templates

3-1	aab aac aax	3-6	aab aac xxy	3-11	aab bbc xxy
3-2	aab aac axx	3-7	aab bbc acc	3-12	aab bcc bxx
3-3	aab aac bbc	3-8	aab bbc axx	3-13	aab bcc xxy
3-4	aab aac bbx	3-9	aab bbc bxx	3-14	aab ccd xxy
3-5	aab aac bxx	3-10	aab bbc cxx		

Double-Double-Double templates

Table 1: 3-block BTD $(\mathcal{V};\mathcal{B};\mathcal{R},\rho_1,\rho_2;3;2)$ configuration templates

Template	Formula	Reference
abc abc	c_1 Basic	fs-1
abc abd	$c_2 = \mathcal{V}\frac{(\rho_1-\rho_2)}{2} - 3c_1$	fs-2
abc ade	$c_3 = 3\,c_1 + 2\,\mathcal{V}\,\rho_2 - \frac{3}{2}\,\mathcal{V}\,\rho_1 - \mathcal{V}\,\rho_2\,\rho_1 + \frac{1}{2}\,\rho_1^2\,\mathcal{V} + c_9$	fs-3
abc def	$c_4 = -c_1 - \frac{4}{3}\,\mathcal{V}\,\rho_2 + \frac{5}{6}\,\mathcal{V}\,\rho_1 + \mathcal{V}\,\rho_2\,\rho_1 - \frac{1}{2}\,\rho_1^2\,\mathcal{V} - c_9 + \frac{1}{18}\,\mathcal{V}^2\,\rho_2^2 - \frac{1}{9}\,\mathcal{V}^2\,\rho_2\,\rho_1 + \frac{1}{18}\,\mathcal{V}^2\,\rho_1^2$	fs-4
....	....	....
aab aab	$c_5 = 0$ Does not exist for $\Lambda = 2$	fs-5
aab aac	$c_6 = \frac{\mathcal{V}\rho_2(\rho_2-1)}{2}$	fs-6
aab bba	$c_7 = 0$ Does not exist for $\Lambda = 2$	fs-7
aab bbc	$c_8 = \mathcal{V}\rho_2^2$	fs-8
....	....	....
aab ccb	c_9 Basic	fs-9
aab ccd	$c_{10} = \frac{\mathcal{V}\rho_2[(\mathcal{V}-3)\rho_2-\rho_1+1]}{2}$	fs-10
aab abc	$c_{11} = 0$ Does not exist for $\Lambda = 2$	fs-11
aab acd	$c_{12} = \mathcal{V}\rho_2(\rho_1 - \rho_2)$	fs-12
....	....	....
aab bcd	$c_{13} = -2\,c_9 + \mathcal{V}\,\rho_2\,\rho_1 - \mathcal{V}\,\rho_2$	fs-13
aab cde	$c_{14} = -2\,\mathcal{V}\,\rho_2\,\rho_1 + \mathcal{V}\,\rho_2^2 + 2\,c_9 + \mathcal{V}\,\rho_2 - \frac{1}{3}\,\mathcal{V}^2\,\rho_2^2 + \frac{1}{3}\,\mathcal{V}^2\,\rho_2\,\rho_1$	fs-14

Table 2: 2-block BTD $(\mathcal{V};\mathcal{B};\mathcal{R},\rho_1,\rho_2;3;2)$ configuration templates and formulae

2 The strategy

As stated in the introduction, our goal is to partition, by pattern, the 3-block configurations of a BTD with parameters $(\mathcal{V};\mathcal{B};\mathcal{R},\rho_1,\rho_2;3;2)$ and to find the templates and formulae corresponding to the various partition classes. Although labor intensive and time consuming, finding the 3-block configuration templates is a straightforward process. Thus without any further explanation of the process, we list the 3-block configuration templates in Table 1. In the table, the 3-block configuration templates are grouped by pattern characteristics within the defining blocks. Those templates in the "Single-Single-Single" group have the property that all the defining blocks contain three distinct elements. Those templates where exactly one block contains a repeated element are labeled as "Double-Single-Single". Similarly, the remaining groups are referred to as "Double-Double-Single" and "Double-Double-Double". The groups provide a natural way to list equations and solutions later in the paper.

Table 1 attaches a reference number, n-m, to each template. The variable $a_{n\text{-}m}$ is used to represent the size of the corresponding configuration class. For example, checking Table 1 we see that 0-10 refers to the 3-block configuration template *abc ade bdx*. Thus $a_{0\text{-}10}$ represents the number of the configurations with template *abc ade bdx*.

To find the 3-block configuration formulae, we use the technique first discussed in Danziger et al. [4]. That is, we establish a system of linear equations that relate the configuration class size variables, $a_{n\text{-}m}$. This system is solved to produce the desired formulae. During the process, a generating set for the formulae is created.

Each of the equations in the linear system is produced by counting a certain list of 3-block configurations in two different ways. Each of the lists is produced by combining a class of 2-block configurations with blocks from the design that meet certain criteria. Because all configurations in a configuration class satisfy the same element relationships, if one element from a partition configuration class is part of a list, then all configurations from that class will be part of the list. Also since 2-block configurations play a major role in our constructions, it is not surprising that the 2-block configuration formulae appear in the system of linear equations that is produced. The 2-block configuration formulae were determined by Francel and Sarvate in [5]. For the convenience of the reader, we include a listing of these in Table 2. Note that the variables c_i represent the sizes of the 2-block configuration classes, and each 2-block configuration template has a label of the form "fs-i".

3 A sampling of equation construction

The combinatorial arguments used to establish the individual equations of the system of equations discussed in the previous section are based on the relationships that exist between the 2-block and 3-block configuration templates. Thus, the counting arguments used to establish the equations provide insight into the relationships between the 3-block configuration classes. Hence, they are an important part of this paper. Although it is not possible to present all eighty-three constructions in this paper, we present five interesting, yet representative examples.

Example A: The equation associated with the 3-block configuration class whose template is *aab ccd adx* (2-15), Equation 2-15 in Table 7.

For each 2-block configuration in the class with template *aab ccd* (fs-10), create four 3-block configurations. Do this by first finding the two appearances of the pair *ad* and the two appearances of the pair *cb* in the design. For each of these four appearances, add the block containing the appearance to the 2-block configuration under consideration. Since there exist c_{10} 2-block configurations of the form *aab ccd*, the above method generates a list of size $4c_{10}$.

The number of 3-block configurations in the above list can be counted in a second way. The 3-block configurations can be partitioned by pattern. Three patterns appear. Those are represented by the templates *aab ccd adx* (2-15), *aab ccd aad* (3-5) and *aab ccd add* (3-8). Each configuration associated with the second and third pattern was introduced into the list twice. For example, there are two occurrences of *aab cdd add* in the list since the pair *ad* appears in the same block twice, namely block *aad*. Hence, the block *aad* is used twice. Thus, $4c_{10} = a_{2\text{-}15} + 2a_{3\text{-}5} + 2a_{3\text{-}8}$.

Example B: The equation associated with the 3-block configuration class whose template is *aab aac bbx* (3-4), Equation (3-4) in Table 8.

For each 2-block configuration in the class with template *aab aac* (fs-6), create $2\rho_2$ 3-block configurations. Do this by finding in the design the ρ_2 blocks containing the pair *bb*, and the ρ_2 blocks containing the pair *cc*. Add each of these $2\rho_2$ blocks to the 2-block configuration under consideration. Since there exist c_6 2-block configurations of the form *aab aac*, the total number of 3-block configurations in the list is $2\rho_2 c_6$.

The number of 3-block configurations in the above list can be counted in a second way. The 3-block configurations can again be partitioned by pattern. Two patterns occur, *aab aac bbc* (3-3) and *aab aac bbx* (3-4). Thus, $2c_6\rho_2 = a_{3\text{-}3} + a_{3\text{-}4}$.

Example C: The equation associated with the 3-block configuration class whose template is *aab bcd cdx* (1-13), Equation (1-13) in Table 5.

The templates *aab bcd cdx* (1-13) and *abc abd cxx* represent the same 3-block configuration class. For the purposes of this example the latter form will be used. As in the previous two examples, a list of 3-block configurations will be constructed and then the number of entries will be enumerated in two ways, which will yield an equation.

To each 2-block configuration with template *abc abd* (fs-2) add all blocks, different from *abc* and *abd*, that contain at least one of *c* or *d* singly. If a block contains both *c* and *d* singly add the block in twice. Since every element appears singly in ρ_1 blocks of the design, and since there exist c_2 2-block configurations with template *abc abd*, the list will consist of $2c_2(\rho_1 - 1)$ 3-block configurations.

The 3-block configurations come from the classes represented by the templates *abc abd acd* (0-3), *abc abd acx* (0-4), *abc abd cdx* (0-6), *abc abd cxy*

(0-7), *aab acd bcd* (1-4) and *aab bcd cdx* (1-13). Each 0-3 configuration appears in the list six times since any two blocks of the 0-3 configuration forms a 2-block configuration with template *xyz xyw* (fs-2) and the third block always contains two elements that appear singly in the block and singly in the *xyz xyw* template. Each of the 0-4 and 0-6 configurations appear in the list twice since, like above, the third block contains both single elements from the *xyz xyw* template. Thus $2c_2(\rho_1 - 1) = 6a_{0\text{-}3} + 2a_{0\text{-}4} + 2a_{0\text{-}6} + a_{0\text{-}7} + a_{1\text{-}4} + a_{1\text{-}13}$.

Example D: The equation associated with the 3-block configuration class whose template is *aab cde cde* (1-16), Equation (1-16) in Table 6.

This template can be rewritten as *abc abc xxy*. So to each 2-block configuration with template *abc abc* (fs-1) add all double blocks where the element that appears doubly is not *a*, *b* or *c*. Since there exists c_1 2-block configurations with form *abc abc* and since each element of the design appears doubly in ρ_2 blocks, the list will contain $c_1(\mathcal{V} - 3)\rho_2$ elements.

Now partition the list according to configuration classes. The only two classes that contribute to the list are *aab bcd bcd* (1-10) and *aab cde cde* (1-16). Further, each configuration only appears once. Thus, $c_1(\mathcal{V}-3)\rho_2 = a_{1\text{-}10} + a_{1\text{-}16}$.

Example E: The equation associated with the 3-block configuration class whose template is *aab ccd xyz* (2-19), Equation (2-19) in Table 7.

The number of pairs of blocks associated with *aab ccd* (fs-10) is c_{10}. The 1-block template *xyz* represents a block in the BTD with no duplicates, and the number of these is $\mathcal{B} - \rho_2\mathcal{V}$. Thus, the number of 3-block configurations in the list naively constructed by adjoining 2-block configurations from the class defined by the template *aab ccd* with blocks *xyz* is $c_{10}(\mathcal{B} - \rho_2\mathcal{V})$.

Only the 3-block configurations *aab ccd acx* (2-14), *aab ccd adx* (2-15), *aab ccd axy* (2-16), *aab ccd bdx* (2-17), *aab ccd bxy* (2-18) and *aab ccd xyz* (2-19) appear in the list. All the 3-block configurations in the classes defined by the patterns as templates occur uniquely in the list. This gives the equation $c_{10}(\mathcal{B} - \rho_2\mathcal{V}) = a_{2\text{-}14} + a_{2\text{-}15} + a_{2\text{-}16} + a_{2\text{-}17} + a_{2\text{-}18} + a_{2\text{-}19}$.

4 The system and its solution

Although only five equations are derived in Section 3, the methods illustrated are suggestive of the methods used in producing all of the equations in the system. Thus, because of space constraints, we omit the arguments used to establish the remaining equations in the system. However, the complete system of equations is listed in Appendix A.

Using the algebraic symbolic manipulation system Maple[1], one finds the dimension of the solution space of the system to be nine. So before

[1]Maple is a registered trademark of Waterloo Maple Software

solving the system, a basis of size nine must be specified. The variables c_1, c_9, $a_{0\text{-}3}$, $a_{1\text{-}4}$, $a_{1\text{-}10}$, $a_{1\text{-}14}$, $a_{3\text{-}3}$, $a_{3\text{-}7}$ and $a_{3\text{-}12}$ are made basic. From Francel and Sarvate [5], we know that c_1 and c_9 can be used as a basis for the 2-block configurations. Thus, since all 2-block configuration formulae appear in the 3-block formulae, we include these as part of our basis. The remaining variables are chosen to be part of the basis because of their degree of independence and their minimality with respect to distinct element containment.

The system solutions, as found by Maple, are listed in Appendix B. The formulae derived from the system can be categorized according to their dependencies. The constant formulae are: c_6, c_8, c_{12}, $a_{2\text{-}1}$, $a_{3\text{-}1}$ and $a_{3\text{-}2}$. Those formulae, excluding the c_i's, that depend only on the BTD parameters and the 2-configuration values are: $a_{1\text{-}1}$, $a_{1\text{-}2}$, $a_{1\text{-}3}$, $a_{2\text{-}7}$ and $a_{3\text{-}9}$. All remaining 56 formulae are variable: 6 depend only on c_1, c_9 and the 2-configuration formulae; 16 depend on one of the $a_{i\text{-}j}$; 8 depend on two of the $a_{i\text{-}j}$; 6 depend on three of the $a_{i\text{-}j}$; 2 on four; 8 on five; 9 on six; and, 1 on seven.

The technique used here and introduced in [4] can certainly be applied in general. Unfortunately, it is a brute force technique that does not reveal any possible combinatorial structure that might be present. However, with the availability of symbolic manipulation programs, once the system of equations is developed, the solution to the system is readily available (at least for small systems).

5 BIBD

Grannell, Griggs and Mendelsohn suggest in their paper [6] that configuration formulae in designs with parameters other than $(v, b, r, 3, 1)$ would also be of interest to researchers. To date, no such results appear in print. All published results concerning n-block BIBD configuration formulae are restricted to BIBD's with parameters $(v, b, r, 3, 1)$.

So in this section we include one, two and three-block configuration BIBD formulae for BIBD with parameters $(v, b, r, 3, 2)$. These results are obtained directly from the formulae of Section 4 and Appendix B, which include the formulae for 2-block BTD configurations given by Francel and Sarvate [5]. We present a complete listing of the BIBD results in Table 3. A basis for each set of formulae is derived from the corresponding BTD basis. The basis for the 2-configuration formulae is $\{c_1\}$, and that for the 3-configuration formulae is $\{c_1, a_{0\text{-}3}\}$. Thus, these too are listed in Table 3 and are indicated by a †.

These formulae are especially interesting in that they show the rarity of constant configurations counts. In BIBD's with parameters $(v, b, r, 3, 1)$,

n is required to be four before any of the n-block configuration formulae became variable. And even then, five of the sixteen counts are still constant. However, for BIBD with parameters $(v, b, r, 3, 2)$ quite the opposite is true. Our results show that when n is as small as two all n-block configurations counts are variable. This result suggests that constant BIBD configuration counts might be unique to BIBD with parameters $(v, b, r, 3, 1)$.

We also note that beginning with $n = 2$, the size of a basis for BIBD $(v, b, r, 3, 2)$ n-block configurations is larger than the size of a basis for the BIBD $(v, b, r, 3, 1)$ n-configurations. This seems natural given, that the former contains more variable configurations than the latter.

n	n-configuration template	n-configuration formula
1	abc	$\frac{\mathcal{V}(\mathcal{V}-1)}{3}$
2	$abc\ abc\dagger$	c_1
	$abc\ abd$	$\frac{\mathcal{V}\rho_1}{2} - 3c_1$
	$abc\ ade$	$\mathcal{V}\rho_1(\rho_1 - 3) + 6c_1$
	$abc\ def$	$\mathcal{V}\rho_1(\mathcal{V}\rho_1 - 9\rho_1 + 15) - c_1$
3	$abc\ abc\ axy$	$3\,c_1\,\rho_1 - 6\,c_1$
	$abc\ abc\ xyz$	$-\frac{8}{3}\,c_1\,\rho_1 + 4\,c_1 + \frac{1}{3}\,c_1\,\rho_1^2$
	$abc\ abd\ acx$	$-6\,c_1 + \rho_1 + \rho_1^2 - 3\,a_{0\text{-}3}$
	$abc\ abd\ axy$	$24\,c_1 - 4\,\rho_1 - 3\,\rho_1^2 + \rho_1^3 - 6\,c_1\,\rho_1 + 3\,a_{0\text{-}3}$
	....	....
	$abc\ abd\ acd\dagger$	$a_{0\text{-}3}$
	$abc\ abd\ cdx$	$-6\,c_1 + \rho_1 + \rho_1^2 - 3\,a_{0\text{-}3}$
	$abc\ abd\ cxy$	$30\,c_1 - 5\,\rho_1 - 4\,\rho_1^2 + \rho_1^3 - 6\,c_1\,\rho_1 + 6\,a_{0\text{-}3}$
	$abc\ abd\ xyz$	$5\,\rho_1 - 30\,c_1 - 3\,a_{0\text{-}3} + 11\,c_1\,\rho_1 + \frac{19}{6}\,\rho_1^2 - c_1\,\rho_1^2 - \frac{5}{3}\,\rho_1^3 + \frac{1}{6}\,\rho_1^4$
	$abc\ ade\ axy$	$3\,c_1\,\rho_1 + \frac{11}{6}\,\rho_1^2 - \frac{4}{3}\,\rho_1^3 + \frac{1}{6}\,\rho_1^4 - 12\,c_1 + \frac{10}{3}\,\rho_1 - a_{0\text{-}3}$
	....	....
	$abc\ ade\ bdx$	$18\,c_1 - \frac{17}{3}\,\rho_1 - \frac{13}{3}\,\rho_1^2 + \frac{4}{3}\,\rho_1^3 + 5\,a_{0\text{-}3}$
	$abc\ ade\ bxy$	$15\,\rho_1^2 - 9\,\rho_1^3 + \rho_1^4 - 78\,c_1 + 25\,\rho_1 - 12\,a_{0\text{-}3} + 12\,c_1\,\rho_1$
	$abc\ ade\ xyz$	$-30\,\rho_1 + 84\,c_1 + 9\,a_{0\text{-}3} - 20\,c_1\,\rho_1 - 14\,\rho_1^2 + c_1\,\rho_1^2 + \frac{79}{6}\,\rho_1^3 - \frac{8}{3}\,\rho_1^4 + \frac{1}{6}\,\rho_1^5$
	$abc\ def\ xyz$	$\frac{85}{9}\,\rho_1 - 22\,c_1 - 2\,a_{0\text{-}3} + \frac{17}{3}\,c_1\,\rho_1 + \frac{61}{18}\,\rho_1^2 - \frac{1}{3}\,c_1\,\rho_1^2 - \frac{373}{81}\,\rho_1^3 + \frac{35}{27}\,\rho_1^4 + \frac{1}{162}\,\rho_1^6 - \frac{4}{27}\,\rho_1^5$

Table 3: BIBD $(v, b, r, 3, 2)$ configurations and formulae († indicates an n-configuration template which corresponds to a basic formulae)

A Equations

Tables 4 - 8 contain the complete set of eighty-three equations derived for finding the 3-block configuration formulae for the BTD $(\mathcal{V};\mathcal{B};\mathcal{R},\rho_1,\rho_2;3;2)$ 3-block configurations.

$$
\begin{aligned}
c_6 &= \mathcal{V}\rho_2\frac{(\rho_2-1)}{2} \\
c_8 &= \mathcal{V}\rho_2^2 \\
c_{12} &= \mathcal{V}\rho_2(\rho_1-\rho_2) \\
2c_2+6c_1 &= 3(\mathcal{B}-\mathcal{V}\rho_2) \\
2c_9+c_{13} &= \mathcal{V}\rho_2(\rho_1-1) \\
2c_{10}+2c_9+c_8 &= \mathcal{V}\rho_2(\mathcal{V}-2)\rho_2 \\
6c_1+4c_2+2c_3 &= (\mathcal{B}-\mathcal{V}\rho_2)3(\rho_1-1)-c_{13} \\
c_1+c_2+c_3+c_4 &= (\mathcal{B}-\mathcal{V}\rho_2)\frac{(\mathcal{B}-\mathcal{V}\rho_2-1)}{2} \\
c_{12}+c_{13}+c_{14} &= \mathcal{V}\rho_2(\mathcal{B}-\mathcal{V}\rho_2) \\
3\mathcal{B} &= \mathcal{V}(2\rho_2+\rho_1)
\end{aligned}
$$

Table 4: 2-configuration equations

$$
\begin{aligned}
c_1 3(\rho_1 - 2) &= a_{0\text{-}1} + a_{1\text{-}10} \\
c_1(\mathcal{B} - \mathcal{V}\rho_2 - 2) &= a_{0\text{-}1} + a_{0\text{-}2} \\
4c_2 &= 6a_{0\text{-}3} + 2a_{0\text{-}4} \\
2c_2(\rho_1 - 2) &= 3a_{0\text{-}3} + 2a_{0\text{-}4} + a_{0\text{-}5} + a_{1\text{-}11} \\
2c_2 &= 3a_{0\text{-}3} + a_{0\text{-}6} + 2a_{1\text{-}4} \\
2c_2(\rho_1 - 1) &= 6a_{0\text{-}3} + 2a_{0\text{-}4} + 2a_{0\text{-}6} \qquad \text{(Eq. 1-13)} \\
&\quad + a_{0\text{-}7} + a_{1\text{-}4} + a_{1\text{-}13} \\
c_2(\mathcal{B} - \rho_2\mathcal{V} - 2) &= 3a_{0\text{-}3} + 2a_{0\text{-}4} + a_{0\text{-}5} + a_{0\text{-}6} + a_{0\text{-}7} + a_{0\text{-}8} \\
c_3(\rho_1 - 2) &= 2a_{0\text{-}1} + a_{0\text{-}4} + 2a_{0\text{-}5} + 3a_{0\text{-}9} + a_{1\text{-}12} \\
8c_3 &= a_{0\text{-}4} + 4a_{0\text{-}6} + 3a_{0\text{-}10} + 2a_{1\text{-}5} \\
4c_3(\rho_1 - 1) &= 4a_{0\text{-}1} + 2a_{0\text{-}4} + 2a_{0\text{-}5} + 6a_{0\text{-}6} + 2a_{0\text{-}7} \\
&\quad + a_{1\text{-}5} + 6a_{0\text{-}10} + a_{1\text{-}14} + 2a_{0\text{-}11} \\
c_3(\mathcal{B} - \rho_2\mathcal{V} - 2) &= 2a_{0\text{-}1} + a_{0\text{-}4} + 2a_{0\text{-}5} + 2a_{0\text{-}6} + a_{0\text{-}7} \\
&\quad + 3a_{0\text{-}9} + 3a_{0\text{-}10} + 2a_{0\text{-}11} + a_{0\text{-}12} \\
c_4(\mathcal{B} - \rho_2\mathcal{V} - 2) &= 2a_{0\text{-}2} + a_{0\text{-}7} + 2a_{0\text{-}8} + a_{0\text{-}11} + 2a_{0\text{-}12} + 3a_{0\text{-}13} \\
4c_3 &= 4a_{0\text{-}1} + 2a_{0\text{-}4} + 2a_{0\text{-}5} \\
2c_3 &= 2a_{0\text{-}1} + 2a_{0\text{-}6} + a_{0\text{-}7} \\
6c_4 &= 6a_{0\text{-}2} + a_{0\text{-}7} + 2a_{0\text{-}8}
\end{aligned}
$$

Table 5: Single-single-single equations

$$
\begin{aligned}
3\rho_2 c_1 &= a_{1\text{-}1} \\
2c_{12} &= 4a_{1\text{-}1} + 2a_{1\text{-}2} \\
c_{12}(\rho_1 - 1) &= 2a_{1\text{-}1} + 2a_{1\text{-}2} + 2a_{1\text{-}3} + a_{2\text{-}7} \\
4c_{12} &= 2a_{1\text{-}4} + a_{1\text{-}5} + 2a_{2\text{-}5} + 4a_{2\text{-}10} \\
c_{12}(\rho_1 - 1) &= a_{1\text{-}4} + a_{1\text{-}5} + a_{1\text{-}6} + 2a_{2\text{-}10} + a_{2\text{-}11} \\
c_{12} &= 2a_{1\text{-}1} + a_{1\text{-}4} + a_{1\text{-}7} \\
2c_{12}(\rho_1 - 1) &= 4a_{1\text{-}1} + 2a_{1\text{-}2} + 2a_{1\text{-}4} + a_{1\text{-}5} + 2a_{1\text{-}7} \\
&\quad + a_{1\text{-}8} + a_{2\text{-}5} + a_{2\text{-}15} \\
c_{12}(\mathcal{B} - \rho_2\mathcal{V} - 1) &= 2a_{1\text{-}1} + 2a_{1\text{-}2} + 2a_{1\text{-}3} + a_{1\text{-}4} + a_{1\text{-}5} \\
&\quad + a_{1\text{-}6} + a_{1\text{-}7} + a_{1\text{-}8} + a_{1\text{-}9} \\
3c_1(\rho_1 - 2) &= a_{0\text{-}1} + a_{1\text{-}10} \\
2c_2(\rho_1 - 2) &= 3a_{0\text{-}3} + 2a_{0\text{-}4} + a_{0\text{-}5} + a_{1\text{-}11} \\
c_{13}(\rho_1 - 2) &= 2a_{1\text{-}10} + 2a_{1\text{-}11} + 2a_{1\text{-}12} + 2a_{2\text{-}12} \\
2c_2(\rho_1 - 1) &= 6a_{0\text{-}3} + 2a_{0\text{-}4} + 2a_{0\text{-}6} + a_{0\text{-}7} + a_{1\text{-}4} + a_{1\text{-}13} \\
2c_{13}(\rho_1 - 1) &= 2a_{1\text{-}4} + a_{1\text{-}5} + 4a_{1\text{-}10} + 2a_{1\text{-}11} + 2a_{1\text{-}13} + a_{1\text{-}14} \\
&\quad + 2a_{2\text{-}2} + 2a_{2\text{-}17} \\
c_{13}(\mathcal{B} - \rho_2\mathcal{V} - 1) &= a_{1\text{-}4} + a_{1\text{-}5} + a_{1\text{-}6} + 2a_{1\text{-}10} + 2a_{1\text{-}11} \\
&\quad + 2a_{1\text{-}12} + a_{1\text{-}13} + a_{1\text{-}14} + a_{1\text{-}15} \\
c_1(\mathcal{V} - 3)\rho_2 &= a_{1\text{-}10} + a_{1\text{-}16} \qquad \text{(Eq. 1-16)} \\
3c_{14} &= a_{1\text{-}7} + a_{1\text{-}13} + 2a_{1\text{-}17} + 6a_{1\text{-}16} \\
3c_{14}(\rho_1 - 1) &= 2a_{1\text{-}7} + a_{1\text{-}8} + 2a_{1\text{-}13} + a_{1\text{-}14} + 6a_{1\text{-}16} + 4a_{1\text{-}17} \\
&\quad + 2a_{1\text{-}18} + a_{2\text{-}3} + a_{2\text{-}8} + a_{2\text{-}18} \\
c_{14}(\mathcal{B} - \rho_2\mathcal{V} - 1) &= a_{1\text{-}7} + a_{1\text{-}8} + a_{1\text{-}9} + a_{1\text{-}13} + a_{1\text{-}14} + a_{1\text{-}15} \\
&\quad + 2a_{1\text{-}16} + 2a_{1\text{-}17} + 2a_{1\text{-}18} + 2a_{1\text{-}19} \\
4c_{13} &= 2a_{2\text{-}5} + 4a_{2\text{-}2} + 2a_{1\text{-}4} + a_{1\text{-}5}
\end{aligned}
$$

Table 6: Double-single-single equations

$$
\begin{aligned}
c_6\rho_1 &= a_{2\text{-}1} + a_{3\text{-}2} \\
2c_6 &= a_{2\text{-}2} + 2a_{3\text{-}3} \\
2c_6(\rho_1 - 1) &= 2a_{2\text{-}2} + a_{2\text{-}3} + a_{3\text{-}3} + a_{3\text{-}5} \\
c_6(\mathcal{B} - \rho_2\mathcal{V}) &= a_{2\text{-}1} + a_{2\text{-}2} + a_{2\text{-}3} + a_{2\text{-}4} \\
2c_8 &= a_{2\text{-}5} + 2a_{3\text{-}3} + 6a_{3\text{-}7} \\
c_8\rho_1 &= a_{2\text{-}5} + a_{2\text{-}6} + 3a_{3\text{-}7} + a_{3\text{-}8} \\
c_8(\rho_1 - 1) &= a_{2\text{-}7} + 2a_{3\text{-}9} \\
c_8(\rho_1 - 1) &= a_{2\text{-}5} + a_{2\text{-}8} + a_{3\text{-}3} + a_{3\text{-}10} \\
c_8(\mathcal{B} - \rho_2\mathcal{V}) &= a_{2\text{-}5} + a_{2\text{-}6} + a_{2\text{-}7} + a_{2\text{-}8} + a_{2\text{-}9} \\
2c_9 &= a_{2\text{-}10} + 2a_{3\text{-}3} \\
2c_9\rho_1 &= 2a_{2\text{-}10} + a_{2\text{-}11} + a_{3\text{-}3} + a_{3\text{-}10} \\
c_9(\rho_1 - 2) &= a_{2\text{-}12} + 3a_{3\text{-}12} \\
c_9(\mathcal{B} - \rho_2\mathcal{V}) &= a_{2\text{-}10} + a_{2\text{-}11} + a_{2\text{-}12} + a_{2\text{-}13} \\
2c_{10} &= 2a_{3\text{-}4} + a_{2\text{-}14} \\
4c_{10} &= a_{2\text{-}15} + 2a_{3\text{-}5} + 2a_{3\text{-}8} && \text{(Eq. 2-15)} \\
2c_{10}\rho_1 &= 2a_{2\text{-}14} + a_{2\text{-}15} + a_{2\text{-}16} + a_{3\text{-}4} + a_{3\text{-}8} + a_{3\text{-}11} \\
2c_{10} &= a_{2\text{-}17} + 2a_{3\text{-}10} \\
2c_{10}(\rho_1 - 1) &= a_{2\text{-}15} + 2a_{2\text{-}17} + a_{2\text{-}18} + a_{3\text{-}5} + a_{3\text{-}10} + 2a_{3\text{-}13} \\
c_{10}(\mathcal{B} - \rho_2\mathcal{V}) &= a_{2\text{-}14} + a_{2\text{-}15} + a_{2\text{-}16} && \text{(Eq. 2-19)} \\
&\quad + a_{2\text{-}17} + a_{2\text{-}18} + a_{2\text{-}2} \\
c_{12}(\rho_2 - 1) &= 2a_{2\text{-}1} \\
c_{13}(\rho_2 - 1) &= 2a_{2\text{-}2} + a_{2\text{-}3} \\
c_{13}\rho_2 &= a_{2\text{-}7} \\
c_{13}\rho_1 &= a_{1\text{-}4} + a_{1\text{-}5} + a_{2\text{-}5} + a_{1\text{-}6} + a_{2\text{-}8}
\end{aligned}
$$

Table 7: Double-double-single equations

$$
\begin{aligned}
c_6(\rho_2 - 2) &= 3a_{3\text{-}1} \\
c_6\rho_1 &= a_{3\text{-}2} + a_{2\text{-}1} \\
2c_6\rho_2 &= a_{3\text{-}3} + a_{3\text{-}4} \\
2c_6(\rho_1 - 1) &= 2a_{2\text{-}2} + a_{2\text{-}3} + a_{3\text{-}3} + a_{3\text{-}5} \\
c_6(\mathcal{V} - 3)\rho_2 &= a_{3\text{-}2} + a_{3\text{-}5} + a_{3\text{-}6} \\
c_8\rho_1 &= a_{2\text{-}5} + a_{2\text{-}6} + 3a_{3\text{-}7} + a_{3\text{-}8} \\
c_8(\rho_1 - 1) &= a_{2\text{-}7} + 2a_{3\text{-}9} \\
c_8\rho_2 &= 3a_{3\text{-}7} + a_{3\text{-}8} \\
c_8(\rho_1 - 1) &= a_{2\text{-}5} + a_{2\text{-}8} + a_{3\text{-}3} + a_{3\text{-}10} \\
c_8(\mathcal{V} - 3)\rho_1 &= a_{2\text{-}5} + 2a_{2\text{-}6} + 2a_{2\text{-}7} + 2a_{2\text{-}8} + 3a_{2\text{-}9} \\
&\quad + 2a_{3\text{-}2} + a_{3\text{-}4} + a_{3\text{-}8} + a_{3\text{-}11} \\
2c_9\rho_1 &= 2a_{2\text{-}10} + a_{2\text{-}11} + a_{3\text{-}3} + a_{3\text{-}10} \\
c_9(\rho_1 - 2) &= a_{2\text{-}12} + 3a_{3\text{-}12} \\
c_9(\mathcal{V} - 3)\rho_2 &= 3a_{3\text{-}12} + a_{3\text{-}13} + a_{3\text{-}10} \\
c_{10}\rho_2(\mathcal{V} - 4) &= a_{3\text{-}11} + 2a_{3\text{-}13} + 3a_{3\text{-}14} \\
2a_{3\text{-}2} &= c_8(\rho_2 - 1) \\
c_9\rho_2 &= a_{3\text{-}9}
\end{aligned}
\qquad \text{(Eq. 3-4)}
$$

Table 8: Double-double-double equations

B Solutions

Tables 9 - 16 contain the complete set of solutions to the system presented in Appendix A. There are 9 formulae related to the c_i's and there are 58 associated with the $a_{i\text{-}j}$. The basis elements are: c_1, c_9, $a_{0\text{-}3}$, $a_{1\text{-}4}$, $a_{1\text{-}10}$, $a_{1\text{-}14}$, $a_{3\text{-}3}$, $a_{3\text{-}7}$ and $a_{3\text{-}12}$.

$$
\begin{aligned}
c_2 &= -3\,c_1 - \frac{1}{2}\mathcal{V}\,\rho_2 + \frac{1}{2}\mathcal{V}\,\rho_1 \\
c_3 &= 3\,c_1 + 2\,\mathcal{V}\,\rho_2 - \frac{3}{2}\mathcal{V}\,\rho_1 - \mathcal{V}\,\rho_2\,\rho_1 + \frac{1}{2}\rho_1^2\,\mathcal{V} + c_9 \\
c_4 &= -c_1 - \frac{4}{3}\mathcal{V}\,\rho_2 + \frac{5}{6}\mathcal{V}\,\rho_1 + \mathcal{V}\,\rho_2\,\rho_1 - \frac{1}{2}\rho_1^2\,\mathcal{V} - c_9 \\
&\quad + \frac{1}{18}\mathcal{V}^2\,\rho_2^2 - \frac{1}{9}\mathcal{V}^2\,\rho_2\,\rho_1 + \frac{1}{18}\mathcal{V}^2\,\rho_1^2 \\
c_6 &= \frac{1}{2}\mathcal{V}\,\rho_2^2 - \frac{1}{2}\mathcal{V}\,\rho_2 \\
c_8 &= \mathcal{V}\,\rho_2^2 \\
c_{10} &= -c_9 - \frac{3}{2}\mathcal{V}\,\rho_2^2 + \frac{1}{2}\mathcal{V}^2\,\rho_2^2 \\
c_{12} &= \mathcal{V}\,\rho_2\,\rho_1 - \mathcal{V}\,\rho_2^2 \\
c_{13} &= -2\,c_9 + \mathcal{V}\,\rho_2\,\rho_1 - \mathcal{V}\,\rho_2 \\
c_{14} &= -2\,\mathcal{V}\,\rho_2\,\rho_1 + \mathcal{V}\,\rho_2^2 + 2\,c_9 + \mathcal{V}\,\rho_2 - \frac{1}{3}\mathcal{V}^2\,\rho_2^2 + \frac{1}{3}\mathcal{V}^2\,\rho_2\,\rho_1 \\
\mathcal{B} &= \frac{2}{3}\mathcal{V}\,\rho_2 + \frac{1}{3}\mathcal{V}\,\rho_1
\end{aligned}
$$

Table 9: Solutions for the 2-configurations

$$
\begin{aligned}
a_{0\text{-}1} &= 3\,c_1\,\rho_1 - 6\,c_1 - a_{1\text{-}10} \\
a_{0\text{-}2} &= -\frac{1}{3}\,c_1\,\mathcal{V}\,\rho_2 + \frac{1}{3}\,c_1\,\mathcal{V}\,\rho_1 + 4\,c_1 - 3\,c_1\,\rho_1 + a_{1\text{-}10} \\
a_{0\text{-}4} &= -6\,c_1 - \mathcal{V}\,\rho_2 + \mathcal{V}\,\rho_1 - 3\,a_{0\text{-}3} \\
a_{0\text{-}5} &= 24\,c_1 + 5\,\mathcal{V}\,\rho_2 - 4\,\mathcal{V}\,\rho_1 - 2\,\mathcal{V}\,\rho_2\,\rho_1 + \rho_1^2\,\mathcal{V} + 2\,c_9 - 6\,c_1\,\rho_1 \\
&\quad + 2\,a_{1\text{-}10} + 3\,a_{0\text{-}3} \\
a_{0\text{-}6} &= -6\,c_1 - \mathcal{V}\,\rho_2 + \mathcal{V}\,\rho_1 - 3\,a_{0\text{-}3} - 2\,a_{1\text{-}4} \\
a_{0\text{-}7} &= 30\,c_1 + 6\,\mathcal{V}\,\rho_2 - 5\,\mathcal{V}\,\rho_1 - 2\,\mathcal{V}\,\rho_2\,\rho_1 + \rho_1^2\,\mathcal{V} + 2\,c_9 - 6\,c_1\,\rho_1 \\
&\quad + 2\,a_{1\text{-}10} + 6\,a_{0\text{-}3} + 4\,a_{1\text{-}4} \\
a_{0\text{-}8} &= -30\,c_1 - 7\,\mathcal{V}\,\rho_2 + 5\,\mathcal{V}\,\rho_1 + 4\,\mathcal{V}\,\rho_2\,\rho_1 - 2\,\rho_1^2\,\mathcal{V} \\
&\quad - 4\,c_9 + 12\,c_1\,\rho_1 - 4\,a_{1\text{-}10} - 3\,a_{0\text{-}3} - 2\,a_{1\text{-}4} + c_1\,\mathcal{V}\,\rho_2 \\
&\quad + \frac{1}{6}\,\mathcal{V}^2\,\rho_2^2 - \frac{1}{3}\,\mathcal{V}^2\,\rho_2\,\rho_1 - c_1\,\mathcal{V}\,\rho_1 + \frac{1}{6}\,\mathcal{V}^2\,\rho_1^2 \\
a_{0\text{-}9} &= c_9\,\rho_1 - \frac{1}{2}\,\mathcal{V}\,\rho_2\,\rho_1^2 + \frac{7}{2}\,\mathcal{V}\,\rho_2\,\rho_1 - 4\,c_9 - 5\,\mathcal{V}\,\rho_2 - a_{1\text{-}10} - a_{3\text{-}12} \\
&\quad - 12\,c_1 + \frac{10}{3}\,\mathcal{V}\,\rho_1 - \frac{3}{2}\,\rho_1^2\,\mathcal{V} + 3\,c_1\,\rho_1 - a_{0\text{-}3} + \frac{1}{6}\,\rho_1^3\,\mathcal{V} \\
a_{0\text{-}10} &= 18\,c_1 + 7\,\mathcal{V}\,\rho_2 - \frac{17}{3}\,\mathcal{V}\,\rho_1 - \frac{16}{3}\,\mathcal{V}\,\rho_2\,\rho_1 + \frac{4}{3}\,\rho_1^2\,\mathcal{V} + 8\,c_9 \\
&\quad + 5\,a_{0\text{-}3} + 4\,a_{1\text{-}4} + \frac{16}{3}\,\mathcal{V}\,\rho_2^2 - 8\,a_{3\text{-}3} - 8\,a_{3\text{-}7}
\end{aligned}
$$

Table 10: Solutions for the single-single-single 3-configurations (Sheet 1 of 2)

$$
\begin{aligned}
a_{0\text{-}11} &= \rho_1^3 \mathcal{V} - 12\,\mathcal{V}\,\rho_2^2 - 2\,a_{1\text{-}10} - \frac{1}{2}\,a_{1\text{-}14} - 78\,c_1 - 32\,\mathcal{V}\,\rho_2 - 26\,c_9 \\
&\quad - 12\,a_{0\text{-}3} - 9\,a_{1\text{-}4} + 2\,c_9\,\rho_1 + 18\,a_{3\text{-}3} + 18\,a_{3\text{-}7} + 25\,\mathcal{V}\,\rho_1 \\
&\quad + 12\,c_1\,\rho_1 + 24\,\mathcal{V}\,\rho_2\,\rho_1 - 10\,\rho_1^2\,V - 2\,\mathcal{V}\,\rho_2\,\rho_1^2 \\
a_{0\text{-}12} &= -\frac{5}{2}\,\rho_1^3\,\mathcal{V} + \frac{1}{3}\,\mathcal{V}^2\,\rho_2^2\,\rho_1 + \frac{1}{6}\,\mathcal{V}^2\,\rho_1^3 + 8\,\mathcal{V}\,\rho_2^2 + 3\,a_{1\text{-}10} + a_{1\text{-}14} + 3\,a_{3\text{-}12} \\
&\quad + 84\,c_1 + 41\,\mathcal{V}\,\rho_2 + 32\,c_9 + 9\,a_{0\text{-}3} + 6\,a_{1\text{-}4} - 7\,c_9\,\rho_1 - 12\,a_{3\text{-}3} \\
&\quad - 12\,a_{3\text{-}7} - \frac{1}{2}\,\mathcal{V}^2\,\rho_2\,\rho_1^2 - 30\,\mathcal{V}\,\rho_1 - 21\,c_1\,\rho_1 - \frac{2}{3}\,\mathcal{V}^2\,\rho_2^2 \\
&\quad + \frac{7}{6}\,\mathcal{V}^2\,\rho_2\,\rho_1 + c_1\,\mathcal{V}\,\rho_1 - \frac{69}{2}\,\mathcal{V}\,\rho_2\,\rho_1 - c_1\,\mathcal{V}\,\rho_2 - \frac{1}{3}\,c_9\,\mathcal{V}\,\rho_2 \\
&\quad + \frac{1}{3}\,c_9\,\mathcal{V}\,\rho_1 + \frac{33}{2}\,\rho_1^2\,\mathcal{V} - \frac{1}{2}\,\mathcal{V}^2\,\rho_1^2 + \frac{11}{2}\,\mathcal{V}\,\rho_2\,\rho_1^2 \\
a_{0\text{-}13} &= \frac{4}{3}\,\rho_1^3\,\mathcal{V} - \frac{1}{3}\,\mathcal{V}^2\,\rho_2^2\,\rho_1 - \frac{1}{162}\,\mathcal{V}^3\,\rho_2^3 - \frac{1}{6}\,\mathcal{V}^2\,\rho_1^3 - \frac{4}{3}\,\mathcal{V}\,\rho_2^2 \\
&\quad - \frac{1}{2}\,a_{1\text{-}14} - 2\,a_{3\text{-}12} - 22\,c_1 - \frac{118}{9}\,\mathcal{V}\,\rho_2 - 10\,c_9 - 2\,a_{0\text{-}3} - a_{1\text{-}4} \\
&\quad + 4\,c_9\,\rho_1 + 2\,a_{3\text{-}3} + 2\,a_{3\text{-}7} + \frac{1}{2}\,\mathcal{V}^2\,\rho_2\,\rho_1^2 + \frac{85}{9}\,\mathcal{V}\,\rho_1 + 6\,c_1\,\rho_1 \\
&\quad + \frac{4}{9}\,\mathcal{V}^2\,\rho_2^2 - \frac{13}{18}\,\mathcal{V}^2\,\rho_2\,\rho_1 - \frac{1}{3}\,c_1\,\mathcal{V}\,\rho_1 + \frac{37}{3}\,\mathcal{V}\,\rho_2\,\rho_1 + \frac{1}{3}\,c_1\,\mathcal{V}\,\rho_2 \\
&\quad + \frac{1}{3}\,c_9\,\mathcal{V}\,\rho_2 - \frac{1}{3}\,c_9\,\mathcal{V}\,\rho_1 - \frac{19}{3}\,\rho_1^2\,\mathcal{V} + \frac{1}{54}\,\mathcal{V}^3\,\rho_2^2\,\rho_1 \\
&\quad + \frac{5}{18}\,\mathcal{V}^2\,\rho_1^2 - \frac{1}{54}\,\mathcal{V}^3\,\rho_1^2\,\rho_2 - 3\,\mathcal{V}\,\rho_2\,\rho_1^2 + \frac{1}{162}\,\mathcal{V}^3\,\rho_1^3
\end{aligned}
$$

Table 11: Solutions for the single-single-single 3-configurations (Sheet 2 of 2)

$$
\begin{aligned}
a_{1\text{-}1} &= 3\,\rho_2\,c_1 \\
a_{1\text{-}2} &= \mathcal{V}\,\rho_2\,\rho_1 - \mathcal{V}\,\rho_2^2 - 6\,\rho_2\,c_1 \\
a_{1\text{-}3} &= \frac{1}{2}\,\mathcal{V}\,\rho_2\,\rho_1^2 - \rho_1\,\mathcal{V}\,\rho_2^2 - \frac{3}{2}\,\mathcal{V}\,\rho_2\,\rho_1 + 2\,\mathcal{V}\,\rho_2^2 + 3\,\rho_2\,c_1 + c_9\,\rho_2 \\
a_{1\text{-}5} &= 4\,\mathcal{V}\,\rho_2\,\rho_1 - 8\,\mathcal{V}\,\rho_2^2 - 2\,a_{1\text{-}4} + 12\,a_{3\text{-}3} + 12\,a_{3\text{-}7} - 8\,c_9 \\
a_{1\text{-}6} &= \frac{1}{2}\,\mathcal{V}\,\rho_2\,\rho_1^2 - \rho_1\,\mathcal{V}\,\rho_2^2 - 2\,\mathcal{V}\,\rho_2\,\rho_1 + 6\,\mathcal{V}\,\rho_2^2 + \frac{1}{2}\,a_{1\text{-}4} - 9\,a_{3\text{-}3} \\
&\quad - 9\,a_{3\text{-}7} + 2\,c_9 - c_9\,\rho_1 - 2\,\mathcal{V}\,\rho_2 - a_{1\text{-}10} + \frac{1}{4}\,a_{1\text{-}14} + \frac{1}{2}\,\mathcal{V}^2\,\rho_2^2 \\
a_{1\text{-}7} &= \mathcal{V}\,\rho_2\,\rho_1 - \mathcal{V}\,\rho_2^2 - 6\,\rho_2\,c_1 - a_{1\text{-}4} \\
a_{1\text{-}8} &= 2\,\mathcal{V}\,\rho_2\,\rho_1^2 - 2\,\rho_1\,\mathcal{V}\,\rho_2^2 - 10\,\mathcal{V}\,\rho_2\,\rho_1 + 18\,\mathcal{V}\,\rho_2^2 + 2\,a_{1\text{-}4} - 12\,a_{3\text{-}3} \\
&\quad - 12\,a_{3\text{-}7} + 8\,c_9 + 12\,\rho_2\,c_1 - 2\,\mathcal{V}^2\,\rho_2^2 + 4\,c_9\,\rho_2 + 2\,\mathcal{V}\,\rho_2^3 \\
a_{1\text{-}9} &= -\frac{2}{3}\,\mathcal{V}^2\,\rho_2^2\,\rho_1 - 16\,\mathcal{V}\,\rho_2^2 - 6\,\rho_2\,c_1 - \frac{7}{2}\,\mathcal{V}\,\rho_2\,\rho_1^2 + 5\,\rho_1\,\mathcal{V}\,\rho_2^2 \\
&\quad + 7\,\mathcal{V}\,\rho_2\,\rho_1 - \frac{1}{2}\,a_{1\text{-}4} + 9\,a_{3\text{-}3} + 9\,a_{3\text{-}7} - 2\,c_9 + c_9\,\rho_1 + 2\,\mathcal{V}\,\rho_2 \\
&\quad + a_{1\text{-}10} - \frac{1}{4}\,a_{1\text{-}14} + \frac{3}{2}\,\mathcal{V}^2\,\rho_2^2 - 6\,c_9\,\rho_2 + \frac{1}{3}\,\mathcal{V}^2\,\rho_2^3 - 2\,\mathcal{V}\,\rho_2^3 \\
&\quad + \frac{1}{3}\,\mathcal{V}^2\,\rho_2\,\rho_1^2 \\
a_{1\text{-}11} &= \mathcal{V}\,\rho_2\,\rho_1 - \mathcal{V}\,\rho_2 - 2\,c_9 - 2\,a_{1\text{-}10}
\end{aligned}
$$

Table 12: Solutions for the double-single-single 3-configurations (Sheet 1 of 2)

$$
\begin{aligned}
a_{1\text{-}12} &= -2\,c_9\,\rho_1 + \frac{1}{2}\,\mathcal{V}\,\rho_2\,\rho_1^2 - \frac{5}{2}\,\mathcal{V}\,\rho_2\,\rho_1 + 6\,c_9 + 2\,\mathcal{V}\,\rho_2 + a_{1\text{-}10} \\
&\quad + 3\,a_{3\text{-}12} \\
a_{1\text{-}13} &= -\mathcal{V}\,\rho_2 + \mathcal{V}\,\rho_2\,\rho_1 - 2\,c_9 - 2\,a_{1\text{-}10} - a_{1\text{-}4} \\
a_{1\text{-}15} &= -\frac{1}{3}\,\mathcal{V}^2\,\rho_2^2\,\rho_1 + 2\,\mathcal{V}\,\rho_2^2 + 3\,a_{1\text{-}10} - \frac{5}{4}\,a_{1\text{-}14} - 6\,a_{3\text{-}12} + 2\,\mathcal{V}\,\rho_2 + 2\,c_9 \\
&\quad + \frac{3}{2}\,a_{1\text{-}4} + 5\,c_9\,\rho_1 - 3\,a_{3\text{-}3} - 3\,a_{3\text{-}7} + \frac{1}{3}\,\mathcal{V}^2\,\rho_2\,\rho_1^2 - \frac{1}{6}\,\mathcal{V}^2\,\rho_2^2 \\
&\quad - \frac{1}{3}\,\mathcal{V}^2\,\rho_2\,\rho_1 - \mathcal{V}\,\rho_2\,\rho_1 + \frac{2}{3}\,c_9\,\mathcal{V}\,\rho_2 + \rho_1\,\mathcal{V}\,\rho_2^2 - \frac{2}{3}\,c_9\,\mathcal{V}\,\rho_1 \\
&\quad - \frac{3}{2}\,\mathcal{V}\,\rho_2\,\rho_1^2 \\
a_{1\text{-}16} &= c_1\,\mathcal{V}\,\rho_2 - 3\,\rho_2\,c_1 - a_{1\text{-}10} \\
a_{1\text{-}17} &= -4\,\mathcal{V}\,\rho_2\,\rho_1 + 2\,\mathcal{V}\,\rho_2^2 + 4\,c_9 + 2\,\mathcal{V}\,\rho_2 - \frac{1}{2}\,\mathcal{V}^2\,\rho_2^2 + \frac{1}{2}\,\mathcal{V}^2\,\rho_2\,\rho_1 \\
&\quad + 12\,\rho_2\,c_1 + a_{1\text{-}4} + 4\,a_{1\text{-}10} - 3\,c_1\,\mathcal{V}\,\rho_2 \\
a_{1\text{-}18} &= -\mathcal{V}^2\,\rho_2^2\,\rho_1 - 12\,\mathcal{V}\,\rho_2^2 - a_{1\text{-}10} - a_{1\text{-}14} - 3\,a_{3\text{-}12} - 2\,\mathcal{V}\,\rho_2 - 6\,c_9 \\
&\quad - 15\,\rho_2\,c_1 + 2\,c_9\,\rho_1 - 5\,c_9\,\rho_2 + \frac{1}{2}\,\mathcal{V}^2\,\rho_2\,\rho_1^2 + 4\,\mathcal{V}^2\,\rho_2^2 \\
&\quad - \frac{3}{2}\,\mathcal{V}^2\,\rho_2\,\rho_1 + 10\,\mathcal{V}\,\rho_2\,\rho_1 + 3\,c_1\,\mathcal{V}\,\rho_2 + c_9\,\mathcal{V}\,\rho_2 + 3\,\rho_1\,\mathcal{V}\,\rho_2^2 \\
&\quad - 2\,\mathcal{V}\,\rho_2^3 - 3\,\mathcal{V}\,\rho_2\,\rho_1^2 \\
a_{1\text{-}19} &= 2\,\mathcal{V}^2\,\rho_2^2\,\rho_1 + \frac{1}{18}\,\mathcal{V}^3\,\rho_2^3 + 8\,\mathcal{V}\,\rho_2^2 - 3\,a_{1\text{-}10} + \frac{5}{4}\,a_{1\text{-}14} + 6\,a_{3\text{-}12} - 2\,\mathcal{V}\,\rho_2 \\
&\quad - 2\,c_9 - \frac{3}{2}\,a_{1\text{-}4} + 6\,\rho_2\,c_1 - 5\,c_9\,\rho_1 + 3\,a_{3\text{-}3} + 3\,a_{3\text{-}7} + 6\,c_9\,\rho_2 \\
&\quad - \frac{7}{6}\,\mathcal{V}^2\,\rho_2\,\rho_1^2 - \frac{19}{6}\,\mathcal{V}^2\,\rho_2^2 + \frac{7}{6}\,\mathcal{V}^2\,\rho_2\,\rho_1 - 4\,\mathcal{V}\,\rho_2\,\rho_1 - c_1\,\mathcal{V}\,\rho_2 \\
&\quad - \frac{5}{3}\,c_9\,\mathcal{V}\,\rho_2 - 5\,\rho_1\,\mathcal{V}\,\rho_2^2 + \frac{2}{3}\,c_9\,\mathcal{V}\,\rho_1 - \frac{1}{9}\,\mathcal{V}^3\,\rho_2^2\,\rho_1 - \frac{1}{3}\,\mathcal{V}^2\,\rho_2^3 \\
&\quad + 2\,\mathcal{V}\,\rho_2^3 + \frac{1}{18}\,\mathcal{V}^3\,\rho_1^2\,\rho_2 + \frac{9}{2}\,\mathcal{V}\,\rho_2\,\rho_1^2
\end{aligned}
$$

Table 13: Solution for double-single-single 3-configurations (Sheet 2 of 2)

$$
\begin{aligned}
a_{2\text{-}1} &= \frac{1}{2}\rho_1\,\mathcal{V}\,\rho_2^2 - \frac{1}{2}\mathcal{V}\,\rho_2\,\rho_1 - \frac{1}{2}\mathcal{V}\,\rho_2^3 + \frac{1}{2}\mathcal{V}\,\rho_2^2 \\
a_{2\text{-}2} &= \mathcal{V}\,\rho_2^2 - \mathcal{V}\,\rho_2 - 2\,a_{3\text{-}3} \\
a_{2\text{-}3} &= \rho_1\,\mathcal{V}\,\rho_2^2 - 3\,\mathcal{V}\,\rho_2^2 - 2\,c_9\,\rho_2 + 2\,c_9 - \mathcal{V}\,\rho_2\,\rho_1 + 3\,\mathcal{V}\,\rho_2 + 4\,a_{3\text{-}3} \\
a_{2\text{-}4} &= -\frac{3}{2}\rho_1\,\mathcal{V}\,\rho_2^2 + \frac{3}{2}\mathcal{V}\,\rho_2^2 + 2\,c_9\,\rho_2 - 2\,c_9 + \frac{3}{2}\mathcal{V}\,\rho_2\,\rho_1 - 2\,\mathcal{V}\,\rho_2 \\
&\quad - 2\,a_{3\text{-}3} - \frac{1}{6}\mathcal{V}^2\,\rho_2^3 + \frac{1}{6}\mathcal{V}^2\,\rho_2^2\,\rho_1 + \frac{1}{6}\mathcal{V}^2\,\rho_2^2 \\
&\quad - \frac{1}{6}\mathcal{V}^2\,\rho_2\,\rho_1 + \frac{1}{2}\mathcal{V}\,\rho_2^3 \\
a_{2\text{-}5} &= 2\,\mathcal{V}\,\rho_2^2 - 2\,a_{3\text{-}3} - 6\,a_{3\text{-}7} \\
a_{2\text{-}6} &= \rho_1\,\mathcal{V}\,\rho_2^2 - 2\,\mathcal{V}\,\rho_2^2 + 2\,a_{3\text{-}3} + 6\,a_{3\text{-}7} - \mathcal{V}\,\rho_2^3 \\
a_{2\text{-}7} &= \rho_1\,\mathcal{V}\,\rho_2^2 - \mathcal{V}\,\rho_2^2 - 2\,c_9\,\rho_2 \\
a_{2\text{-}8} &= \rho_1\,\mathcal{V}\,\rho_2^2 - a_{3\text{-}3} + 3\,a_{3\text{-}7} - c_9\,\rho_1 + \frac{1}{2}\mathcal{V}\,\rho_2\,\rho_1^2 - 3\,\mathcal{V}\,\rho_2\,\rho_1 + 6\,c_9 \\
&\quad + 2\,\mathcal{V}\,\rho_2 + a_{1\text{-}10} + \frac{1}{2}a_{1\text{-}4} - \frac{1}{4}a_{1\text{-}14} - \frac{1}{2}\mathcal{V}^2\,\rho_2^2 \\
a_{2\text{-}9} &= -\frac{1}{3}\mathcal{V}^2\,\rho_2^3 + \frac{1}{3}\mathcal{V}^2\,\rho_2^2\,\rho_1 + \mathcal{V}\,\rho_2^2 - 3\,\rho_1\,\mathcal{V}\,\rho_2^2 + \mathcal{V}\,\rho_2^3 + 2\,c_9\,\rho_2 \\
&\quad + a_{3\text{-}3} - 3\,a_{3\text{-}7} + c_9\,\rho_1 - \frac{1}{2}\mathcal{V}\,\rho_2\,\rho_1^2 + 3\,V\,\rho_2\,\rho_1 - 6\,c_9 \\
&\quad - 2\,\mathcal{V}\,\rho_2 - a_{1\text{-}10} - \frac{1}{2}a_{1\text{-}4} + \frac{1}{4}a_{1\text{-}14} + \frac{1}{2}\mathcal{V}^2\,\rho_2^2 \\
a_{2\text{-}10} &= 2\,c_9 - 2\,a_{3\text{-}3}
\end{aligned}
$$

Table 14: Solution for double-double-single 3-configurations (Sheet 1 of 2)

$$
\begin{aligned}
a_{2\text{-}11} &= c_9\,\rho_1 + 2\,c_9 + a_{3\text{-}3} + \frac{1}{2}\,\mathcal{V}\,\rho_2\,\rho_1^2 - 3\,V\,\rho_2\,\rho_1 + 2\,\mathcal{V}\,\rho_2 + 3\,\mathcal{V}\,\rho_2^2 \\
&\quad - 3\,a_{3\text{-}7} + a_{1\text{-}10} + \frac{1}{2}\,a_{1\text{-}4} - \frac{1}{4}\,a_{1\text{-}14} - \frac{1}{2}\,\mathcal{V}^2\,\rho_2^2 \\
a_{2\text{-}12} &= c_9\,\rho_1 - 2\,c_9 - 3\,a_{3\text{-}12} \\
a_{2\text{-}13} &= -\frac{1}{3}\,c_9\,\mathcal{V}\,\rho_2 + \frac{1}{3}\,c_9\,\mathcal{V}\,\rho_1 + a_{3\text{-}3} - 2\,c_9\,\rho_1 - 2\,c_9 - \frac{1}{2}\,\mathcal{V}\,\rho_2\,\rho_1^2 \\
&\quad + 3\,\mathcal{V}\,\rho_2\,\rho_1 - 2\,\mathcal{V}\,\rho_2 - 3\,\mathcal{V}\,\rho_2^2 + 3\,a_{3\text{-}7} - a_{1\text{-}10} - \frac{1}{2}\,a_{1\text{-}4} + \frac{1}{4}\,a_{1\text{-}14} \\
&\quad + \frac{1}{2}\,\mathcal{V}^2\,\rho_2^2 + 3\,a_{3\text{-}12} \\
a_{2\text{-}14} &= -2\,c_9 - \mathcal{V}\,\rho_2^2 + \mathcal{V}^2\,\rho_2^2 - 2\,\mathcal{V}\,\rho_2^3 + 2\,a_{3\text{-}3} \\
a_{2\text{-}15} &= -6\,\mathcal{V}\,\rho_2^2 + 2\,\mathcal{V}^2\,\rho_2^2 + 2\,a_{3\text{-}3} - 4\,c_9\,\rho_2 - 2\,\mathcal{V}\,\rho_2^3 + 6\,a_{3\text{-}7} \\
a_{2\text{-}16} &= -c_9\,\rho_1 - 3\,\rho_1\,\mathcal{V}\,\rho_2^2 + \mathcal{V}^2\,\rho_2^2\,\rho_1 - 2\,c_9 + 6\,\mathcal{V}\,\rho_2^2 - \frac{7}{2}\,\mathcal{V}^2\,\rho_2^2 \\
&\quad + 8\,\mathcal{V}\,\rho_2^3 - 3\,a_{3\text{-}3} + 6\,c_9\,\rho_2 - 3\,a_{3\text{-}7} - \mathcal{V}^2\,\rho_2^3 - \frac{1}{2}\,\mathcal{V}\,\rho_2\,\rho_1^2 \\
&\quad + 3\,\mathcal{V}\,\rho_2\,\rho_1 - 2\,\mathcal{V}\,\rho_2 - a_{1\text{-}10} - \frac{1}{2}\,a_{1\text{-}4} + \frac{1}{4}\,a_{1\text{-}14} \\
a_{2\text{-}17} &= 10\,c_9 + 3\,\mathcal{V}\,\rho_2^2 - 2\,c_9\,\rho_1 + \mathcal{V}\,\rho_2\,\rho_1^2 - 6\,\mathcal{V}\,\rho_2\,\rho_1 + 4\,\mathcal{V}\,\rho_2 - 4\,a_{3\text{-}3} \\
&\quad - 6\,a_{3\text{-}7} + 2\,a_{1\text{-}10} + a_{1\text{-}4} - \frac{1}{2}\,a_{1\text{-}14} \\
a_{2\text{-}18} &= -\frac{5}{2}\,\mathcal{V}^2\,\rho_2^2 + 9\,a_{3\text{-}3} + 8\,c_9\,\rho_2 + 2\,\mathcal{V}\,\rho_2^3 + 9\,a_{3\text{-}7} + 3\,c_9\,\rho_1 \\
&\quad - 3\,\rho_1\,\mathcal{V}\,\rho_2^2 + \mathcal{V}^2\,\rho_2^2\,\rho_1 - 22\,c_9 - \frac{5}{2}\,\mathcal{V}\,\rho_2\,\rho_1^2 + 15\,V\,\rho_2\,\rho_1 \\
&\quad - 10\,\mathcal{V}\,\rho_2 - 5\,a_{1\text{-}10} - \frac{5}{2}\,a_{1\text{-}4} + \frac{5}{4}\,a_{1\text{-}14} - 2\,c_9\,\mathcal{V}\,\rho_2 + 6\,a_{3\text{-}12} \\
a_{2\text{-}19} &= -\frac{5}{2}\,\mathcal{V}^2\,\rho_2^2\,\rho_1 - \frac{1}{6}\,\mathcal{V}^3\,\rho_2^3 - 2\,\mathcal{V}\,\rho_2^2 + 4\,a_{1\text{-}10} - a_{1\text{-}14} - 6\,a_{3\text{-}12} + 8\,\mathcal{V}\,\rho_2 \\
&\quad + 16\,c_9 + 2\,a_{1\text{-}4} - 6\,a_{3\text{-}3} - 6\,a_{3\text{-}7} - 10\,c_9\,\rho_2 + 3\,\mathcal{V}^2\,\rho_2^2 \\
&\quad - 12\,\mathcal{V}\,\rho_2\,\rho_1 + \frac{7}{3}\,c_9\,\mathcal{V}\,\rho_2 + 6\,\rho_1\,\mathcal{V}\,\rho_2^2 - \frac{1}{3}\,c_9\,\mathcal{V}\,\rho_1 + \frac{1}{6}\,\mathcal{V}^3\,\rho_2^2\,\rho_1 \\
&\quad + \frac{3}{2}\,\mathcal{V}^2\,\rho_2^3 - 6\,\mathcal{V}\,\rho_2^3 + 2\,\mathcal{V}\,\rho_2\,\rho_1^2
\end{aligned}
$$

Table 15: Solutions for double-double-single 3-configurations (Sheet 2 of 2)

$$
\begin{aligned}
a_{3\text{-}1} &= \frac{1}{6}\mathcal{V}\rho_2^3 - \frac{1}{2}\mathcal{V}\rho_2^2 + \frac{1}{3}\mathcal{V}\rho_2 \\
a_{3\text{-}2} &= \frac{1}{2}\mathcal{V}\rho_2^3 - \frac{1}{2}\mathcal{V}\rho_2^2 \\
a_{3\text{-}4} &= \mathcal{V}\rho_2^3 - \mathcal{V}\rho_2^2 - a_{3\text{-}3} \\
a_{3\text{-}5} &= -a_{3\text{-}3} + 2c_9\rho_2 - 2c_9 \\
a_{3\text{-}6} &= \frac{1}{2}\mathcal{V}^2\rho_2^3 - \frac{1}{2}\mathcal{V}^2\rho_2^2 - 2\mathcal{V}\rho_2^3 + 2\mathcal{V}\rho_2^2 + a_{3\text{-}3} - 2c_9\rho_2 + 2c_9 \\
a_{3\text{-}8} &= \mathcal{V}\rho_2^3 - 3a_{3\text{-}7} \\
a_{3\text{-}9} &= c_9\rho_2 \\
a_{3\text{-}10} &= c_9\rho_1 - \frac{1}{2}\mathcal{V}\rho_2\rho_1^2 + 3\mathcal{V}\rho_2\rho_1 - 6c_9 - 2\mathcal{V}\rho_2 - 3\mathcal{V}\rho_2^2 + 2a_{3\text{-}3} \\
&\quad + 3a_{3\text{-}7} - a_{1\text{-}10} - \frac{1}{2}a_{1\text{-}4} + \frac{1}{4}a_{1\text{-}14} + \frac{1}{2}\mathcal{V}^2\rho_2^2 \\
a_{3\text{-}11} &= \mathcal{V}^2\rho_2^3 + 3\mathcal{V}\rho_2^2 - 4\mathcal{V}\rho_2^3 - 2c_9\rho_2 - 2a_{3\text{-}3} - c_9\rho_1 + \frac{1}{2}\mathcal{V}\rho_2\rho_1^2 \\
&\quad - 3\mathcal{V}\rho_2\rho_1 + 6c_9 + 2\mathcal{V}\rho_2 + a_{1\text{-}10} + \frac{1}{2}a_{1\text{-}4} - \frac{1}{4}a_{1\text{-}14} - \frac{1}{2}\mathcal{V}^2\rho_2^2 \\
a_{3\text{-}13} &= c_9\mathcal{V}\rho_2 - 3c_9\rho_2 - 3a_{3\text{-}12} - c_9\rho_1 + \frac{1}{2}\mathcal{V}\rho_2\rho_1^2 - 3\mathcal{V}\rho_2\rho_1 \\
&\quad + 6c_9 + 2\mathcal{V}\rho_2 + 3\mathcal{V}\rho_2^2 - 2a_{3\text{-}3} - 3a_{3\text{-}7} + a_{1\text{-}10} + \frac{1}{2}a_{1\text{-}4} \\
&\quad - \frac{1}{4}a_{1\text{-}14} - \frac{1}{2}\mathcal{V}^2\rho_2^2 \\
a_{3\text{-}14} &= -\frac{3}{2}\mathcal{V}^2\rho_2^3 + \frac{10}{3}\mathcal{V}\rho_2^3 + 4c_9\rho_2 + 2a_{3\text{-}7} + c_9\rho_1 - \frac{1}{2}\mathcal{V}\rho_2\rho_1^2 \\
&\quad + 3\mathcal{V}\rho_2\rho_1 - 6c_9 - 2\mathcal{V}\rho_2 - 3\mathcal{V}\rho_2^2 + 2a_{3\text{-}3} - a_{1\text{-}10} - \frac{1}{2}a_{1\text{-}4} \\
&\quad + \frac{1}{4}a_{1\text{-}14} + \frac{1}{2}\mathcal{V}^2\rho_2^2 - c_9\mathcal{V}\rho_2 + \frac{1}{6}\mathcal{V}^3\rho_2^3 + 2a_{3\text{-}12}
\end{aligned}
$$

Table 16: Solution for double-double-double 3-configurations

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