Score Vectors and Tournaments with Cyclic Chromatic Number 1 or 2

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Abstract

The cyclic chromatic number is the smallest number of colours needed to colour the nodes of a tournament so that no cyclic triple is monochromatic. Bagga, Beineke, and Harary [1] conjectured that every tournament score vector belongs to a tournament with cyclic chromatic number 1 or 2. In this paper, we prove this conjecture and derive some other results.

1 Introduction

An *n-tournament* T_n consists of n nodes $p_1, p_2, \ldots, p_n$ such that each pair of distinct nodes p_i and p_j is joined by one and only one of the oriented arcs $\overrightarrow{p_ip_j}$ or $\overrightarrow{p_jp_i}$. If the arc $\overrightarrow{p_ip_j}$ is in T_n, then we say that p_i *dominates* p_j, denoted by $p_i \rightarrow p_j$. The *score* of p_i is the number s_i of nodes that p_i dominates. The *score vector* of T_n is the ordered n-tuple $(s_1, s_2, \ldots, s_n)$, where the nodes have been labelled so that $s_1 \leq s_2 \leq \ldots \leq s_n$. We denote the set of all nodes of T_n by $V(T_n)$, and the set of all arcs of T_n by $A(T_n)$. For any subset $W \subseteq V(T_n)$, the *subtournament* T_r $(r = |W|)$ of T_n *induced by* W is the r-tournament with $V(T_r) = W$ and $A(T_r) \subseteq A(T_n)$. We denote by $T_n[W]$ the subtournament of T_n induced by W. A tournament is transitive if whenever $p \to q$ and $q \to r$ then $p \to r$ also. We denote by L_r the transitive r-tournament, and by C_3 the cyclic 3-tournament.

For any nontrivial graph F, the *F-chromatic number* of a graph G is the smallest number of colours needed to colour the vertices so that no copy of F in G has all of its vertices the same colour. Thus, the usual chromatic number $\chi(G)$ is the K_2-chromatic number of G. In [1], Bagga, Beineke, and Harary consider such colourings of tournaments by focusing on forbidding as monochromatic subtournaments L_3 and C_3. The *cyclic chromatic number* $\chi_c(T)$ of a tournament T is the minimum number of colours with which the nodes of T can be coloured so that no C_3 is monochromatic, that is, $\chi_c(T)$ is the C_3-chromatic number of T. The maximum cyclic chromatic number among all tournaments on n vertices was shown in [2] to be of order $\frac{n}{\log_2 n}$.

Bagga et al conjectured in [1] that every tournament score vector belongs to a tournament with cyclic chromatic number 1 or 2. Here, we prove that this conjecture is true.

We list two results which are used in the proof of our main result.

The following theorem was first proved by Landau [3] (1953), and later (1964) Ryser [5] gave another proof (see [4]).

Theorem 1 *A set of integers $(s_1, s_2, \ldots, s_n)$ with $s_1 \leq s_2 \leq \cdots \leq s_n$ is the score vector of some tournament T_n if and only if*

$$\sum_{i=1}^{k} s_i \geq \binom{k}{2},$$

for $k = 1, 2, \ldots, n$ with equality holding when $k = n$.

The next lemma is easily proved with the help of Theorem 1.

Lemma 1 *(Landau [3] 1953) If the scores $s_1, s_2, \ldots, s_n$ of a tournament T_n are in nondecreasing order, then*

$$\frac{i-1}{2} \leq s_i \leq \frac{n+i-2}{2},$$

for $i = 1, 2, \ldots, n$.

2 Proof of the conjecture

For any score vector $S = (s_1, s_2, \ldots, s_n)$, we show that there is a tournament T_n with S as its score vector such that $V(T_n) = V_1 + V_2$, and that $T_n[V_1]$ and $T_n[V_2]$ are transitive subtournaments of T_n. In other words, we can partition $V(T_n)$ into two subsets V_1 and V_2 such that the subtournaments $T_n[V_1]$ and $T_n[V_2]$ are both transitive. Hence, we can colour V_1 with one colour and V_2 with another colour, giving $\chi_c(T_n) \leq 2$, and the Bagga-Beineke-Harary's conjecture follows.

NOTE: If one of V_1 and V_2 is empty, then T_n itself is transitive, and thus, $\chi_c(T_n) = 1$.

Now, we state and prove our main result.

Theorem 2 *For any score vector $S = (s_1, s_2, \ldots, s_n)$ with $s_1 \leq s_2 \leq \ldots \leq s_n$, there is a tournament T_n with S as its score vector such that $V(T_n) = V_1 + V_2$, and that $T_n[V_1]$ and $T_n[V_2]$ are transitive subtournaments of T_n, where $V(T_n) = \{p_1, p_2, \ldots, p_n\}$, $V_1 = \{p_i \mid i \text{ is odd}\}$ with $|V_1| = \lceil \frac{n}{2} \rceil$, and $V_2 = \{p_i \mid i \text{ is even}\}$ with $|V_2| = \lfloor \frac{n}{2} \rfloor$.*

Proof: By induction on n.

The theorem is trivial when $n = 1, 2$.

Consider the case $n \geq 3$ and suppose the theorem is true for all smaller n.

Let $S = (s_1, s_2, \ldots, s_n)$ be any score vector with $s_1 \leq s_2 \leq \ldots \leq s_n$. Then there is some i with $1 \leq i \leq n-1$ such that $s_{i+1} = s_i$ or $s_{i+1} = s_i + 1$ (since for each $1 \leq j \leq n$, s_j can only have n values: $0, 1, \ldots, n-1$). Let k be the largest such i, and let $w = s_k + s_{k+1} + 1 - k$. Then $w \geq 1$ from Lemma 1 and the choice of k. Consider s_w. Let r and t be the smallest and largest indices less than k such that $s_r = s_w = s_t$. Let

$$q = k - 1 - s_k - s_{k+1} + (t + r) = (t + k) - (s_k + s_{k+1}) + (r - 1).$$

Define the set of integers $(s'_1, s'_2, \ldots, s'_{n-2})$ as follows:

$$\begin{aligned} s'_i &= s_i, && \text{if } i = 1, 2, \ldots, r-1 \text{ or } i = q+1, q+2, \ldots, t; \\ s'_i &= s_i - 1, && \text{if } i = r, r+1, \ldots, q \text{ or } i = t+1, t+2, \ldots, k-1; \\ s'_i &= s_{i+2} - 2, && \text{if } i = k, k+1, \ldots, n-2. \end{aligned}$$

From this definition, it follows that $s'_1 \leq s'_2 \leq \ldots \leq s'_{n-2}$, $s'_i = s_i$ for $(s_k + s_{k+1}) - k$ values of i, $s'_i = s_i - 1$ for $2k - (s_k + s_{k+1} + 1)$ values of i, and $s'_i = s_{i+2} - 2$ for $(n - k - 1)$ values of i.

Consequently,

$$\sum_{i=1}^{n-2} s'_i = \sum_{i=1}^{n} s_i - (s_k + s_{k+1}) - (2k - s_k - s_{k+1} - 1) - 2(n - k - 1)$$

$$= \sum_{i=1}^{n} s_i - (2n-3) = \binom{n}{2} - (2n-3) = \binom{n-2}{2}.$$

Therefore, by Theorem 1, we need only show that the inequality

$$\sum_{i=1}^{h} s_i' < \binom{h}{2}$$

is impossible for every integer $h = 2, 3, \ldots, n-3$ in order to complete the proof that $(s_1', s_2', \ldots, s_{n-2}')$ is the score vector of some tournament T_{n-2}. We follow Ryser's proof of Theorem 1 (see [4]).

First, consider the smallest value of h $(1 < h \leq k-1)$ for which inequality

$$\sum_{i=1}^{h} s_i' < \binom{h}{2}$$

holds (if it ever holds). Since

$$\sum_{i=1}^{h-1} s_i' \geq \binom{h-1}{2},$$

it follows that $s_h \leq h$. Furthermore, $r \leq h$ since the first $r-1$ scores were unchanged. Hence,

$$s_h = s_{h+1} = \cdots = s_f,$$

where

$$f = \max(h, t).$$

Let m denote the number of values of i not exceeding h such that $s_i' = s_i - 1$. Then it must be that

$$w \leq f + 1 - m,$$

and hence

$$s_k + s_{k+1} - k = w - 1 \leq f - m.$$

So,

$$s_j + s_{j+1} \leq f + k$$

for all $j \leq k$.

Therefore,

$$\binom{k+1}{2} \leq \sum_{i=1}^{k+1} s_i = \sum_{i=1}^{h} s_i' + m + \sum_{i=h+1}^{f} s_i + \sum_{i=f+1}^{k+1} s_i$$

$$< \binom{h}{2} + (f-h)s_h + \sum_{i=f+1}^{k-1} s_i + s_k + s_{k+1} + m$$

$$< \binom{f}{2} + \frac{k-1-f}{2}(f+k) + (f+k)$$
$$= \frac{f(f-1)}{2} + \frac{(k-f)(k+f)+(f+k)}{2}$$
$$= \frac{k^2+k}{2} = \binom{k+1}{2},$$

a contradiction. So,

$$\sum_{i=1}^{h} s'_i \geq \binom{h}{2}$$

for all $1 < h \leq k-1$.

Now consider $k \leq h < n-2$. We have, by the definition of the s_i's, that

$$\sum_{i=1}^{h} s'_i = \sum_{i=1}^{h+2} s_i - (s_k + s_{k+1}) - (2h - s_k - s_{k+1} - 1) - 2[(h+2) - k - 1]$$
$$= \sum_{i=1}^{h+2} s_i - 2h - 1 \geq \binom{h+2}{2} - 2h - 1$$
$$= \binom{h+2}{2} - \binom{h+1}{1} - \binom{h}{1} = \binom{h}{2}.$$

That is,

$$\sum_{i=1}^{h} s'_i \geq \binom{h}{2}$$

for all $k \leq h < n-2$.

Therefore, $S' = (s'_1, s'_2, \ldots, s'_{n-2})$ is indeed a tournament score vector. Thus by the induction hypothesis, there is an $(n-2)$-tournament T_{n-2} with S' as its score vector such that $V(T_{n-2}) = V'_1 + V'_2$, and that $T_{n-2}[V'_1]$ and $T_{n-2}[V'_2]$ are transitive subtournaments of T_{n-2}, where $V(T_{n-2}) = \{p'_1, p'_2, \ldots, p'_{n-2}\}$, $V'_1 = \{p'_i \mid i \text{ is odd }\}$ with $|V'_1| = \lceil \frac{n-2}{2} \rceil = \lceil \frac{n}{2} \rceil - 1$, and $V'_2 = \{p'_i \mid i \text{ is even }\}$ with $|V'_2| = \lfloor \frac{n-2}{2} \rfloor = \lfloor \frac{n}{2} \rfloor - 1$.

Now, we construct an n-tournament T_n with T_{n-2} as its subtournament as follows:

$$V(T_n) = \{p_1, p_2, \ldots, p_n\},$$

where

$$p_i = p'_i, \quad \text{if} \quad i = 1, 2, \ldots, k-1;$$
$$p_i = p'_{i-2}, \quad \text{if} \quad i = k+2, k+3, \ldots, n.$$

$$A(T_n) = A(T_{n-2}) \cup A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5,$$

where

$A_1 = \{\overrightarrow{p_i p_k},\ \overrightarrow{p_i p_{k+1}} \mid k+2 \le i \le n\}$;
$A_2 = \{\overrightarrow{p_k p_i},\ \overrightarrow{p_{k+1} p_j} \mid 1 \le i \le k-1$ and $k \equiv i (mod\ 2)$, $1 \le j \le k-1$ and $k+1 \equiv j (mod\ 2).\}$;
$A_3 = \{\overrightarrow{p_k p_i},\ \overrightarrow{p_{k+1} p_j} \mid 1 \le i \le k-1$ and $k - i \equiv 1 (mod\ 2)$ and $s'_i = s_i$, $1 \le j \le k-1$ and $k \equiv j (mod\ 2)$ and $s'_j = s_j.\}$;
$A_4 = \{\overrightarrow{p_i p_k},\ \overrightarrow{p_j p_{k+1}} \mid 1 \le i \le k-1$ and $k - i \equiv 1 (mod\ 2)$ and $s'_i = s_i - 1$, $1 \le j \le k-1$ and $k \equiv j (mod\ 2)$ and $s'_j = s_j - 1.\}$;

and $A_5 = \{$ either $\overrightarrow{p_{k+1} p_k}$ or $\overrightarrow{p_k p_{k+1}}\}$ can be decided from the following two cases:

Case 1: $s_{k+1} = s_k$.
Then $k - w = 2k - s_k - s_{k+1} - 1 = 2(k - s_k) - 1$ is odd, that is, $(q-r+1)+(k-1-t) = k-w$ is odd. If $(q-r+1)$ is odd and $q \equiv k (mod\ 2)$, then let $p_{k+1} \to p_k$. Otherwise, let $p_k \to p_{k+1}$.
Case 2: $s_{k+1} = s_k + 1$.
Then $k - w = 2k - s_k - s_{k+1} - 1 = 2(k - s_k - 1)$ is even, that is, $(q-r+1)+(k-1-t) = k-w$ is even. If both $(q-r+1)$ and $(k-1-t)$ are odd and $q - k \equiv 1 (mod\ 2)$, then let $p_k \to p_{k+1}$. Otherwise, let $p_{k+1} \to p_k$.
From the above construction, we can see that T_n has score vector $S = (s_1, s_2, \ldots, s_n)$ and has the following properties:

(i) $V(T_n) = V_1 + V_2$, where $V_1 = \{p_i \mid i \text{ is odd }\}$ with $|V_1| = |V'_1| + 1 = \lceil \frac{n}{2} \rceil$, and $V_2 = \{p_i \mid i \text{ is even }\}$ with $|V_2| = |V'_2| + 1 = \lfloor \frac{n}{2} \rfloor$.

(ii) $T_n[V_1]$ and $T_n[V_2]$ are transitive subtournaments of T_n.

Hence, the proof is complete. ∎

Corollary 1 *Every tournament score vector belongs to a tournament with cyclic chromatic number* 1 *or* 2.

Proof: By Theorem 2, for any tournament score vector $S = (s_1, s_2, \ldots, s_n)$, there is an n-tournament T_n with S as its score vector such that $\chi_c(T_n) \le 2$.
∎

It is easy to see that Theorem 2 implies the following result.

Corollary 2 *For any score vector* $S = (s_1, s_2, \ldots, s_n)$ *with* $s_1 \leq s_2 \leq \ldots \leq s_n$*, if* $m = \lfloor \frac{n}{2} \rfloor$*,* $R_1 = (s_1, s_3 - 1, \ldots, s_{2i-1} - (i-1), \ldots, s_{2m+1} - m)$*, and* $R_2 = (s_2, s_4 - 1, \ldots, s_{2i} - (i-1), \ldots, s_{2m} - (m-1))$*, then* R_1 *and* R_2 *form the score vectors of some bipartite tournament.*

References

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