On Seed Graphs with Two Components

Lisa Markus and Douglas Rall
Department of Mathematics
Furman University
Greenville, SC 29613

December 28, 1995

Abstract

A graph H is called a *seed graph* if there exists a graph G such that the deletion of any closed neighborhood of G always results in H. In this paper we investigate disconnected seed graphs. By degree and order considerations we show that for certain pairs of connected graphs, H_1 and H_2, $H_1 \cup H_2$ cannot be a seed graph. Furthermore, for every connected graph H such that $K_1 \cup H$ is a seed graph we show that H can be obtained by a certain graph product of K_2 and H', where H' is itself a seed graph.

1 Introduction.

We shall follow the notation of [1], and so a graph G has vertex set $V(G)$ and edge set $E(G)$ which will be abbreviated to V and E, respectively, if the graph G is clear from the context. The order of G will be denoted by $|G|$. For $x \in V(G)$ the **neighborhood of** x **in** G is $N_G(x) = \{u \in V | xu \in E\}$ and the **closed neighborhood of** x **in** G is $N_G[x] = N_G(x) \cup \{x\}$. In what follows $N_G(x)$ and $N_G[x]$ may represent either just a set of vertices or the subgraph induced by that set of vertices with the meaning being understood from the context. We also drop the reference to G if no confusion will arise. We write $a \perp b$ if a is neither adjacent nor equal to b. If A and B are both subsets of $V(G)$ we write $A \perp B$ to indicate that $a \perp b$ for every $a \in A$ and every $b \in B$. For a vertex u of G and $X \subset V(G)$, X **survives** in $G - N[u]$ means that $\{u\} \perp X$.

ARS COMBINATORIA 49(1998), pp. 193-204

If A is a nonempty set of vertices in G the subgraph of G **induced** by A will be denoted by $\langle A\rangle_G$, or simply by $\langle A\rangle$ if G is clear from the context. The **complement** of a graph G is that graph $\overline{G}$ with vertex set $V(G)$ and edge set $\{ab|a, b \in V(G)$ and $ab \notin E(G)\}$. Whenever we form the union of two or more graphs we will always assume they are vertex disjoint. The **join** of a collection $G_1, G_2, \cdots, G_n$ of vertex disjoint graphs is the graph, $G_1 + G_2 + \ldots + G_n$, obtained from the union of these individual graphs by also including all edges of the form $u_r u_s, r \neq s$, where $u_i \in V(G_i)$ for $1 \leq i \leq n$.

In [2] a graph F is defined to be a graph with **constant neighborhood** if there is a graph K such that $N[x] \cong K$ for every vertex x of F. Any regular, triangle-free graph is an example of a constant neighborhood graph. A graph H is called a **seed graph** if there exists a graph G such that for every vertex v of G, $G - N[v] \cong H$. As in [3], we call the graph G an **isomorphic survivor graph** with seed H. For example, for $n \geq 4$ the path P_{n-3} is a seed graph arising from the isomorphic survivor graph C_n. Further, any vertex transitive graph is both a constant neighborhood graph and an isomorphic survivor graph. These two classes of graphs are closely related as can be seen from the graph equation $\overline{G - N_G[x]} = N_{\overline{G}}(x)$. It follows that the graph G is an isomorphic survivor graph if and only if $\overline{G}$ is a constant neighborhood graph.

The problem we shall be interested in here is that of characterizing those graphs which can be seed graphs of isomorphic survivor graphs. Gunther and Hartnell [3] observed that K_3 is the seed graph of an isomorphic survivor graph G which is the complement of a 3-regular, triangle-free graph. In [4] Hartnell and Kocay determined which larger cycles can be seed graphs. This is summarized in the following theorem.

Theorem 1.1 *Suppose the n-cycle, $n \geq 4$, is a seed graph of the isomorphic survivor graph G. Then one of the following holds:*

1. *$n = 4$ and $\overline{G}$ is the line graph of a 3-regular, triangle-free graph.*
2. *$n = 5$ and each connected component of $\overline{G}$ is isomorphic to the graph of the icosahedron.*
3. *$n = 6$ and each connected component of $\overline{G}$ is isomorphic to the line graph of K_5, the complete graph of order 5.*

In particular they showed that a cycle of order 7 or more cannot be a seed graph. Gunther and Hartnell [3] observed that every isomorphic survivor graph is regular and then proved the following characterization of cubic seed graphs.

Theorem 1.2 *If H is a 3-regular seed graph, then H must be one of the following: K_4, the Cartesian product of K_3 and K_2, the Cartesian product of C_4 and K_2, or the Petersen graph.*

In addition they derived several necessary conditions involving the diameter, the girth and the order of any k-regular seed graph.

We will focus our attention on seed graphs which are the disjoint union of two connected components. In Section 2 we derive general properties of such graphs and also consider some specific situations where the two components come from several well-know classes of graphs. Section 3 is devoted to seed graphs where the smaller of the two components is an isolated vertex.

2 General Seed Graphs with Two Components.

Several of the main results of this section will show that for certain pairs of connected graphs H_1 and H_2, the graph $H_1 \cup H_2$ cannot be a seed graph for any isomorphic survivor graph (hereinafter abbreviated to IS-graph as in [3]) G. As proved in [3] if G_1 and G_2 are both IS-graphs having the same seed H then so is their join $G_1 + G_2$. Thus, if convenient, we may assume a particular IS-graph is connected.

We begin this section with an example which contains some of the arguments typical of those in the section. Suppose that G is a connected IS-graph having seed $K_{1,2} \cup K_3$. As mentioned in the introduction G must be regular since for every vertex v in G, $G - N[v]$ has order six. Let x be an arbitrary vertex of G and assume $V(G - N[x]) = A \cup B$ where $\langle A \rangle \cong K_{1,2}$ and $\langle B \rangle \cong K_3$. Note that $A \perp B$. Let $R = N(x) \cap N(a)$ where a is the vertex of degree two in $\langle A \rangle$. R must be nonempty since G is connected and regular of degree at least three. Let $S = N(x) - R$ and let b be a vertex from B. B induces a complete graph of order three in $G - N[a]$ so $|S| = 2$, say $S = \{u, v\}$, with $u \perp v$. Since $\{x, u, v\}$ and B are distinct components of $G - N[a]$, it follows that $\{u, v\} \perp B$. But now in $G - N[b]$, A survives as does $\{x, u, v\}$. But $\langle A \rangle \cong \langle \{x, u, v\} \rangle \cong K_{1,2}$, which is a contradiction since the subgraph induced by this set contains no K_3. It follows that $K_{1,2} \cup K_3$ is not a seed graph.

Similar to this example we have the following lemma.

Lemma 2.1 *Let H_1 and H_2 be connected graphs of orders m and n, respectively, with $m < n$. If $\Delta(H_1) = m - 1$ and $\Delta(H_2) = n - 1$, then $H_1 \cup H_2$ is not a seed graph.*

Proof. Assume that $H_1 \cup H_2$ is a seed graph of some connected IS-graph G. Recall that G is regular. For a fixed vertex x of G, let $V(G - N[x]) = A \cup B$ where $\langle A \rangle \cong H_1$, $\langle B \rangle \cong H_2$ and $A \perp B$. Suppose $a \in A$ has degree $m-1$ in $\langle A \rangle$ and $b \in B$ has degree $n-1$ in $\langle B \rangle$. Let $R = N_G(a) \cap N_G(x)$ and let $S = N_G(x) - R$. R must be nonempty since G is regular and $\deg_{H_2}(b) > \deg_{H_1}(a)$. B and S survive in $G - N[a]$, and so it follows that $\langle S \cup \{x\} \rangle \cong H_1$ and $S \cap N_G(B) = \emptyset$. If there exists $r \in R$ such that $b \notin N_G(r)$, then $G - N[b]$ is a single component. Thus $N_G(b) \cap N_G(x) = R$, and so $\deg(a) = m - 1 + |R|$ and $\deg(b) = n - 1 + |R|$, which contradicts the regularity of G. Therefore $H_1 \cup H_2$ is not a seed graph. □

For any positive integers k and n, the disjoint union of k copies of the graph K_n is an IS-graph having as a seed the disjoint union of $k-1$ copies of K_n. (If a connected IS-graph having the same seed is desired one only need to form the join of two copies of the original IS-graph.) However, the above lemma yields the following corollary.

Corollary 2.2 *If r and s are distinct positive integers, then $K_r \cup K_s$ is not a seed graph. Equivalently, there does not exist a constant neighborhood graph G all of whose (open) neighborhoods are isomorphic to $K_{r,s}$.*

Lemma 2.3 *If H_1 and H_2 are connected graphs such that $|H_1| \leq |H_2|$ and $H_1 \cup H_2$ is a seed graph, then H_1 is an induced subgraph of H_2.*

Proof. Let G be a connected IS-graph with $G - N[v] \cong H_1 \cup H_2$ for every $v \in V(G)$. Fix $x \in V(G)$ and assume that H_1 is not an induced subgraph of H_2. $V(G - N[x]) = A \cup B$, where A induces H_1 and B induces H_2 in G. For $a \in A$ let $R_a = N_G(a) \cap N_G(x)$. Since $A \perp B$, B survives in $G - N[a]$. From the assumption that $|H_1| \leq |H_2|$ it now follows that for every $b \in B$, $N_G(b) \cap N_G(x) \subseteq R_a$. In fact, since A induces H_1 in G and H_1 is not an induced subgraph of H_2, $\langle A \rangle$ is a component of $G - N[b]$ for every $b \in B$. Thus for every vertex a in A and every vertex b in B, $N(b) \cap N(x) = N(a) \cap N(x)$.

If H_2 is not a complete subgraph, then for distinct nonadjacent vertices b and c in B it follows that c, x and A lie in distinct components of $G - N[b]$, a contradiction. Thus H_2 is complete. Similarly, if H_1 is not complete then for $a, d \in A$ with $a \perp d$, $N(a) \cap N(x) = N(d) \cap N(x)$ and so $G - N[a]$ contains at least three components again contradicting the assumption that $H_1 \cup H_2$ is the seed. But then H_1 is also complete and so by using the result of Corollary 2.2 it follows that $H_1 \cong H_2$. This contradiction shows that H_1 is an induced subgraph of H_2. □

In addition to being an induced subgraph of the larger of the components, the smaller component has maximum degree which is bounded above by the minimum degree of the larger component.

Lemma 2.4 *If $H_1 \cup H_2$ is a seed graph where H_1 and H_2 are connected and $|H_1| \leq |H_2|$, then $\Delta(H_1) \leq \delta(H_2)$.*

Proof. Suppose G is a k-regular IS-graph with seed $H_1 \cup H_2$. Fix $x \in V(G)$ and let $V(G-N[x]) = A \cup B$ such that $\langle A \rangle \cong H_1$ and $\langle B \rangle \cong H_2$. As in the proof of previous lemmas, for any vertex a in A, B is a component of $G - N[a]$. That is, for every $a \in A$ and every $b \in B$, $N(b) \cap N(x) \subseteq N(a) \cap N(x)$. Let u be a vertex from A with $\deg_{\langle A \rangle}(u) = \Delta(H_1)$ and let v be a vertex from B with $\deg_{\langle B \rangle}(v) = \delta(H_2)$. Then

$$k - \delta(H_2) = |N(v) \cap N(x)| \leq |N(u) \cap N(x)| = k - \Delta(H_1).$$

Thus $\Delta(H_1) \leq \delta(H_2)$. □

When the two surviving components are of the same order much more can be concluded.

Theorem 2.5 *Suppose H_1 and H_2 are connected graphs both of order n. $H_1 \cup H_2$ is a seed graph if and only if $H_1 \cong K_n \cong H_2$.*

Proof. Assume $H_1 \cup H_2$ is a seed graph where H_1 and H_2 are connected. By Lemma 2.4, $\Delta(H_1) \leq \delta(H_2)$ because $|H_1| \leq |H_2|$, and similarly $\Delta(H_2) \leq \delta(H_1)$. Thus H_1 and H_2 are regular of the same degree. But now by Lemma 2.3 it follows that $H_1 \cong H_2$. Let $x \in V(G)$ and assume that $V(G - N[x]) = A \cup B$ where $\langle A \rangle \cong H_1$ and $\langle B \rangle \cong H_1$. $A \perp B$ and for every $a, d \in A$ and $b \in B$ it follows that $N(a) \cap N(x) = N(d) \cap N(x) = N(b) \cap N(x)$. Now if H_1 is not a complete graph, then for any $a \in A$, $A - N[a] \neq \emptyset$ and survives as a component in $G - N[a]$ because no $d \in A$ is adjacent to other vertices of $N[x]$ than the ones in $N(x) \cap N(a)$. But $A - N[a]$ has order smaller than $|H_1|$. This contradiction implies that $H_1 \cong K_n \cong H_2$.

Let $F = K_n \cup K_n \cup K_n$. Then $F + F$ is a connected IS-graph with seed $K_n \cup K_n$. □

Although we have no characterization of those pairs of connected graphs H_1, H_2 such that $H_1 \cup H_2$ is a seed graph, we conclude this section with several cases based on the minimum degree of the large component. Since the case $|H_1| = |H_2|$ is covered by Theorem 2.5 we assume in what is to follow that the components of the seed graph have different order. First we have the following lemma which will also be useful in Section 3.

Lemma 2.6 *Suppose H is a connected graph of order at least three such that $K_1 \cup H$ is a seed graph. Then H has even order and for every vertex a of H there is a vertex a' of H such that $N_H(a) = N_H(a')$. In addition, every vertex of H belongs to a 4-cycle in H.*

Proof. Assume G is an IS-graph with seed $K_1 \cup H$. Let $x \in V(G)$ and assume $V(G - N[x]) = \{y\} \cup A$, where $\langle A \rangle \cong H$ and $y \notin N(A)$. Thus $N_G(x) = N_G(y)$. Let a be an arbitrary vertex from A. Since $K_1 \cup H$ is a seed of G it follows that $V(G - N[a]) = \{a'\} \cup B$ where $\langle B \rangle \cong H$ and $a' \notin N_G(B)$. Since every vertex from $N_G(x)$ which survives in $G - N[a]$ is adjacent to both x and y it follows that $a' \in A$. Because a' is isolated in $G - N[a]$ and G is regular, $N_G(a) = N_G(a')$. Also a' is the only vertex in A with this property, since no other vertex is isolated in $G - N[a]$.

Let $b \in A - \{a, a'\}$. As above there exists a vertex b' in $A - \{a, a'\}$ with $N_G(b) = N_G(b')$. Continuing this process gives a pairing of the vertices of A. Thus H has even order. For $u, v \in A$ such that $N_G(u) = N_G(v)$ it also follows that $N_{\langle A \rangle}(u) = N_{\langle A \rangle}(v)$ and hence u and v belong to a common 4-cycle in H. □

Theorem 2.7 *Suppose H_1 and H_2 are connected graphs with $|H_1| < |H_2|$ and suppose H_2 has minimum degree at most 2. If $H_1 \cup H_2$ is a seed graph, then $\delta(H_2) = 2$, and either $H_1 = K_1$, or $H_1 \cong P_{n-3}$ and $H_2 \cong C_n$ for some $n \geq 5$.*

Proof. Suppose G is an isomorphic survivor graph which is regular of degree k and which has seed $H_1 \cup H_2$, where H_1 and H_2 are as in the statement of the theorem. Assume first that $\delta(H_2) = 1$. By Lemma 2.4, H_1 is isomorphic to either K_1 or K_2. The case $H_1 \cong K_1$ is impossible by Lemma 2.6, and so $H_1 \cong K_2$. Let $x \in V(G)$ and suppose $V(G - N[x]) = \{a, c\} \cup B$ where $ac \in E(G)$ and $\langle B \rangle \cong H_2$.

Since $|N(a) \cap N(x)| = k - 1$, there exists a unique vertex y in $N(x) - N(a)$. Also because $G - N[a] \cong H_1 \cup H_2$, $y \perp B$. See Figure 1. If c is adjacent to y, then there exists a unique vertex $z \in N(x)$ such that $c \perp z$. But now it follows that $z \perp B$ since B is a component of $G - N[c]$. If u is a leaf in $\langle B \rangle$, then u must have exactly $k - 1$ neighbors in $N(x)$. This contradicts $y \perp B$ and $z \perp B$.

Hence $c \perp y$, and so $N(a) \cap N(x) = N(c) \cap N(x)$. Because $y \perp B$, y must be adjacent to every other vertex of $N(x)$. Again if u is a leaf in $\langle B \rangle$, then by regularity it follows that u is adjacent to every vertex of $N(x) - \{y\}$. However, at least three components survive in $G - N[u]$, namely $\{x, y\}$, $\{a, c\}$ and $H_2 - N[u]$. This contradiction shows that it is not possible for H_2 to have minimum degree one.

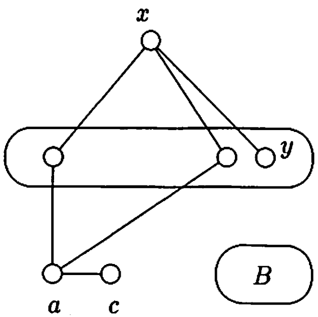

Figure 1.

Therefore, assume that $\delta(H_2) = 2$. By Lemma 2.3 and Lemma 2.4 it follows that H_1 must be one of the following: K_1, $P_m(m \geq 2)$, or $C_m(m \geq 3)$.

H_1 cannot be C_m for any $m \geq 3$. For let $x \in V(G)$ and assume $V(G - N[x]) = A \cup B$ such that $\langle A \rangle \cong C_m$ and $\langle B \rangle \cong H_2$. Let $b \in B$ have degree 2 in H_2. Since B survives in $G - N[t]$ for every $t \in A$, it follows from the regularity of G that $N(b) \cap N(x) = N(t) \cap N(x)$, for every $t \in A$. In particular, for every pair of vertices $t_1, t_2 \in A$, $N(t_1) \cap N(x) = N(t_2) \cap N(x)$. Fix $a \in A$. There exists a pair $a_1, a_2 \in N(x) - N(a)$. Assume first that $m \geq 4$. $B \perp A$ and so B survives in $G - N[a] \cong C_m \cup H_2$. But then $\{x, a_1, a_2\}$, B and $A - N[a]$ are three distinct components in $G - N[a]$. Therefore, H_1 cannot be a cycle of order at least 4. Similarly, by considering $G - N[b]$, it follows that $H_1 \ncong C_3$.

Assume then that $H_1 \cong P_m$ for some $m \geq 2$. We first consider the case where $m = 2$. Let $A = \{a, c\}$ and let b be a vertex of degree two in B. Since G is regular, there is a single vertex $a_1 \in N(x) - N(a)$. $A \perp B$ and so $N(b) \cap N(x) \subseteq N(a) \cap N(c) \cap N(x)$ otherwise $G - N[a]$ or $G - N[c]$ contains a component larger than B, which is impossible. Let $\{a_1, a_2\} = N(x) - N(b)$. The vertex c is adjacent to exactly one of a_1 or a_2 (if it is adjacent to both a_1 and a_2, then there is a vertex $a_3 \in N(b) - N(c)$, which is impossible as shown above). By considering these two cases and the graph $G - N[b]$ it follows that $H_2 = \langle \{x, a_1, a_2, a, c\} \rangle$ is either a 5-cycle or is isomorphic to one of G_1 or G_2 of Figure 2. If H_2 is not a 5-cycle, then it has a vertex of degree at least 3. Let $v \in B$ such that $\deg_{\langle B \rangle}(v) \geq 3$. But then in $G - N[v]$ there exists a vertex $w \in N(x) - \{a_1, a_2\}$ such that w, a, c, a_1, a_2, x all belong to the same component in $G - N[v]$, a contradiction to the fact that $|H_2| = 5$. Therefore, if $H_1 \cong P_2$, then $H_2 \cong C_5$.

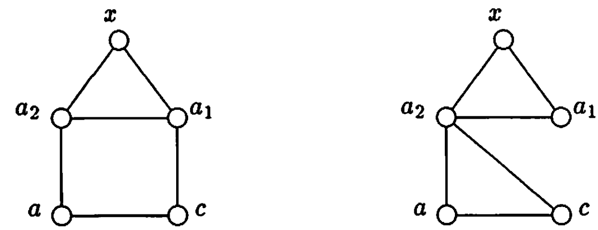

Figure 2.

Now assume $m \geq 3$. Let u and v be the leaves of the path $\langle A \rangle$ and let a be the vertex of A adjacent to u. Suppose $b \in B$ has degree two in $\langle B \rangle$, say $N_{\langle B \rangle}(b) = \{y, z\}$, and suppose $N(x) - N(a) = \{a_1, a_2\}$. Since B survives in $G - N[a]$ it follows that no vertex of B is adjacent to either of a_1, a_2 (otherwise $G - N[a]$ contains a connected graph $\langle B \cup \{x, a_1, a_2\} \rangle$, which is impossible) and the two components of $G - N[b]$ are induced by $B - \{b, y, z\}$ and $A \cup \{x, a_1, a_2\}$. Hence $|H_2| = m + 3$. If $w \in A$ and $\deg_{\langle A \rangle}(w) = 2$, then $N(b) \cap N(x)$ is a subset of $N(w) \cap N(x)$ and they have the same order. Therefore, $N(b) \cap N(x) = N(w) \cap N(x)$, and so neither of a_1 or a_2 is adjacent to w. Also, v is adjacent to exactly one of a_1 or a_2, and similarly for u. $\{x, a_1, a_2\}$ is a part of the path of order m in $G - N[a]$, and so $a_1 a_2 \notin E(G)$. If, say a_1, is adjacent to both of u and v, then a_2 has degree one in $G - N[b] \cong H_2$, a contradiction. Thus assume without loss of generality that $a_1 u \in E(G)$ and $a_2 v \in E(G)$. Now by considering $G - N[b]$ we see that $H_2 \cong \langle A \cup \{x, a_1, a_2\} \rangle \cong C_{m+3}$. □

Note that for $n \geq 4$, $P_{n-3} \cup C_n$ is a seed graph. $G = C_n + C_n$ will serve as an appropriate isomorphic survivor graph having this seed. There are many connected graphs other than C_4 whose union with $K_1 = P_1$ forms a seed graph. We delay their consideration until Section 3. While Theorem 2.7 deals with two-component seed graphs where the larger component has small minimum degree, the following result considers the other extreme. The proof is similar to many of the results in this section, and we shall therefore omit it.

Theorem 2.8 *Suppose H_1 and H_2 are connected graphs such that $|H_1| \leq |H_2|$. If $H_1 \cup H_2$ is a seed graph and $\delta(H_2) = |H_2| - 2$, then $H_1 \cong K_1$.*

3 Seeds With An Isolated Vertex.

If H_2 is an isomorphic survivor graph whose seed graph is connected, say $H_2 - N[u] \cong H_1$ for every $u \in V(H_2)$, then $H_1 \cup H_2$ is also a seed graph. Indeed, if F is the disjoint union of two copies of H_2, then $G = F + F$ is a connected isomorphic survivor graph whose seed is $H_1 \cup H_2$. Any vertex transitive graph can be used for H_2. For example, if H_2 is an even clique, say K_{2r}, with a perfect matching removed then this construction shows that $K_1 \cup H_2$ is a seed graph. This particular H_2 can be obtained from K_r by "splitting" each vertex into two nonadjacent vertices each having the same neighborhood in the resulting graph. See Figure 3 for an example of this process for $r = 3$.

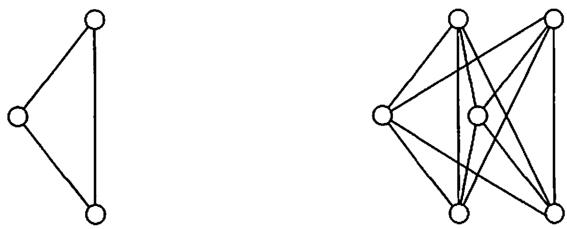

Figure 3

We will show in this section that, in fact, for every connected graph H such that $K_1 \cup H$ is a seed graph, H is the "split" of a graph which is itself a seed graph. We first begin with an example and then introduce a more formal setting which will make the results easier to present.

Let G be the cycle of order 14 with vertex set $V(G) = \{v_0, v_1, \ldots, v_{13}\}$ and edge set consisting of all edges of the form v_iv_{i+1}, v_iv_{i+6} or v_iv_{i+8}, where the subscripts are taken modulo 14. Note that for every vertex x of G, $G - N[x] \cong K_1 \cup H$ as shown in Figure 4. As it is drawn there, the graph H can be seen to be two copies of the path P_4 with each vertex w_i having the same neighborhood as u_i. We formalize this splitting of vertices by using the following graph product.

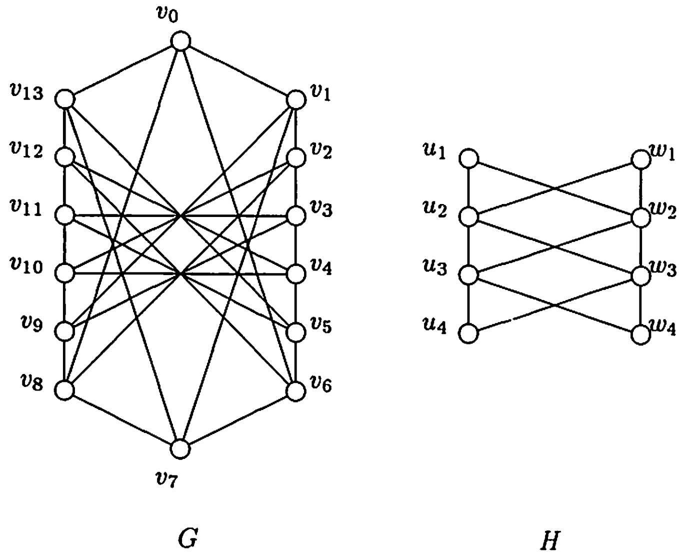

Figure 4

For graphs G_1 and G_2, let $G_1 \oplus G_2$ be the graph whose vertex set is the Cartesian product $V(G_1) \times V(G_2)$. Two vertices (a, b) and (c, d) are adjacent in $G_1 \oplus G_2$ precisely when either $ac \in E(G_1)$ and $bd \in E(G_2)$, or $a = c$ and $bd \in E(G_2)$. The graph $G_1 \oplus G_2$ is defined for any two graphs G_1 and G_2, but when G_1 is a complete graph of order two we get the splitting of G_2 as mentioned above. The graph H of Figure 4 is isomorphic to $K_2 \oplus P_4$. The proof of the following theorem is simply a matter of applying the definition of the above product.

Theorem 3.1 *If G is an isomorphic survivor graph with seed H and $r \geq 2$ is a positive integer, then $K_r \oplus G$ is an isomorphic survivor graph with seed $(r-1)K_1 \cup (K_r \oplus H)$. In particular, $K_1 \cup (K_2 \oplus H)$ is a seed graph.*

We now establish the converse of Theorem 3.1 in the case $r = 2$. This reduces the problem of determining the connected graphs whose union with an isolated vertex forms a seed graph to that of finding all connected graphs which are themselves seed graphs. We will need the following special case of a "cancellation lemma" for the graph product $\oplus$.

Lemma 3.2 *If C and D are graphs such that $K_2 \oplus C \cong K_2 \oplus D$, then $C \cong D$.*

Proof. Assume $V(K_2) = \{1, 2\}$. For $i = 1, 2$ let $C_i = \{(i, x)|x \in V(C)\}$ and let $D_i = \{(i, x)|x \in V(D)\}$. For an isomorphism $g : K_2 \oplus C \rightarrow K_2 \oplus D$ let $A_g = g(C_1) \cap D_1$ and let $B_g = g(C_1) \cap D_2$. From among all isomorphisms from $K_2 \oplus C$ to $K_2 \oplus D$ choose f to be the one with $|A_f|$ a maximum. Note that $|A_f| + |B_f| = |C| = |D|$. We will show for this f that $A_f = D_1$ and thus that C and D are isomorphic.

Project A_f and B_f onto D. That is, let $A = \{d \in V(D)|(1, d) \in A_f\}$ and let $B = \{d \in V(D)|(2, d) \in B_f\}$. If there exists a vertex $d \in B - A$, let $x \in V(C)$ be such that $f(1, x) = (2, d)$. Hence $(1, d) \notin A_f$, but f is surjective and so there exists $c \in V(C)$ such that $f(2, c) = (1, d)$. Since $N((1, d)) = N((2, d))$ in $K_2 \oplus D$ and since f^{-1} is also an isomorphism, it follows that $N((1, x)) = N((2, c))$. But then define $f' : K_2 \oplus C \rightarrow K_2 \oplus D$ by

$$f'(i, u) = \begin{cases} f(2, c) = (1, d) & \text{if } (i, u) = (1, x) \\ f(1, x) = (2, d) & \text{if } (i, u) = (2, c) \\ f(i, u) & \text{otherwise} \end{cases} .$$

But now $|A_{f'}| = |A_f| + 1$. This contradiction proves that $B \subseteq A$.

Suppose $B \neq \emptyset$ and let $b \in B$. Then there exists $c_1 \in V(C)$ such that $f(1, c_1) = (2, b)$. Let $L = \{(2, d)|d \in V(D) - A\}$, $M = \{(2, d)|d \in A - B\}$ and $N = \{(1, d)|d \in V(D) - A\}$. If $f(2, c_1)$ belongs to either N (or L) then as above we can switch $f(1, c_1)$ and $f(2, c_1)$ and contradict the choice of f (the fact that $B \subseteq A$). Therefore, for some $d_1 \in V(D)$, $f(2, c_1) = (2, d_1) \in M$, and $(1, d_1) = f(1, c_2)$ for some $c_2 \in V(C)$. As above $f(2, c_2) \in M$, say $f(2, c_2) = (2, d_2)$. Since $N((1, d_1)) = N((2, d_2))$ then $N((1, c_2)) = N((2, c_1))$ and $N(c_1) = N(c_2)$. This similarly yields $N((2, c_2)) = N((2, c_1))$ and $N((2, d_2)) = N((2, d_1))$. It now follows that $N(d_1) = N(d_2)$.

Now $(1, d_2) \in A_f$ so $(1, d_2) = f(1, c_3)$ for some $c_3 \in V(C)$. But then, since $N(d_1) = N(d_2)$, it follows that $N(c_3) = N(c_2) = N(c_1)$. Continuing in this manner we can show that for every pair of vertices u and v in $V(C)$, $N(u) = N(v)$. That is, C is an independent set and so D is as well. Therefore, either $C \cong \overline{K_n} \cong D$ or else $B = \emptyset$. However, when $B = \emptyset$, $A = V(D)$ and so $f(C_1) = D_1$, and this implies that $C \cong D$ as well. □

We are now prepared to prove the main theorem of this section.

Theorem 3.3 *Suppose H is a connected graph of order at least two such that $K_1 \cup H$ is a seed graph. There exists a connected seed graph H_1 such that $H \cong (K_2 \oplus H_1)$.*

Proof. Assume that G is a graph such that for every $x \in V(G)$, $G - N[x] \cong K_1 \cup H$. For each vertex x of G let $f(x)$ denote the unique vertex in G such that $N(x) = N(f(x))$. Note that $f(f(x)) = x$ and the degree of regularity of G is even. Fix a vertex $a \in V(G)$. Then $V(G-N[a]) = \{f(a)\} \cup A$, where $f(a) \perp A$ and $\langle A \rangle_G \cong H$. If $u \in N(a)$ then $f(u) \in N(a)$ since u and $f(u)$ have the same open neighborhood. In a similar way it follows that for any $v \in A$, $f(v) \in A$. Thus there exist $x_1, \ldots, x_r, g_1, \ldots, g_s \in V(G)$ such that $N(a) = \{x_1, \ldots, x_r, f(x_1), \ldots, f(x_r)\}$ and $A = \{g_1, \ldots, g_s, f(g_1), \ldots, f(g_s)\}$. Let $W = \{a, x_1, \ldots, x_r, g_1, \ldots, g_s\}$ and let $X = \{g_1, \ldots, g_s\}$. Denote by G_1 the subgraph of G induced by the set W, and let $H_1 = \langle X \rangle_G = \langle X \rangle_{G_1}$.

Since $N_G(g_i) = N_G(f(g_i))$, it follows by construction that $H \cong \langle A \rangle_G \cong (K_2 \oplus H_1)$. Note also that $V(G_1 - N_{G_1}[a]) = X$ and so $G_1 - N_{G_1}[a] = H_1$. What remains to be shown is that for every $u \in V(G_1) = W$, $G_1 - N_{G_1}[u] \cong H_1$. Consider first the case $u \in N_{G_1}(a)$. Assume without loss of generality that $u = x_1$ and $N_{G_1}[u] = \{a, x_1, \ldots, x_i, g_1, \ldots, g_j\}$ for some i and j. But then $V(G - N_G[u]) = \{f(x_1)\} \cup (R \cup S)$, where $R = \{x_{i+1}, \ldots, x_r, g_{j+1}, \ldots, g_s\} \subseteq V(G_1)$, and $S = \{f(x_{i+1}), \ldots, f(x_r), f(g_{j+1}), \ldots, f(g_s)\}$. Now $H \cong \langle R \cup S \rangle_G$ and because of the definition of f it is also the case that $H \cong (K_2 \oplus \langle R \rangle_G)$. Since $\langle A \rangle_G \cong H \cong K_2 \oplus H_1$, by Lemma 3.2 it follows that $H_1 \cong \langle R \rangle_G \cong \langle R \rangle_{G_1}$, and so $G_1 - N_{G_1}[u] \cong H_1$. The proof for $u \in X$ is similar. Therefore, H_1 is a seed graph of G_1 and $H \cong (K_2 \oplus H_1)$. □

References

[1] M. Behzad, G. Chartrand & L. Lesniak-Foster, *Graphs and Digraphs*, Prindle, Weber & Schmidt International Series, 1979.

[2] D. Fronček (1993) Neighbourhood-symmetric graphs, *Graph Theory, Combinatorics and Applications, Proceedings of the Seventh Quadrenial International Conference on the Theory and Applications of Graphs*, (Y. Alavi and A. Schwenk, Eds.), Wiley Interscience, 1995, 413–424.

[3] G. Gunther & B. Hartnell (1990) Graphs with isomorphic survivor graphs, *Congressus Numerantium* **79**, 69–77.

[4] B. Hartnell & W. Kocay (1991) On minimal neighbourhood-connected graphs, *Discrete Mathematics* **92**, 95–105.