On weak domination in graphs

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Abstract

Sampathkumar and Pushpa Latha (see [3]) conjectured that the independent domination number, $i(T)$, of a tree T is less than or equal to its weak domination number, $\gamma_w(T)$. We show that this conjecture is true, prove that $\gamma_w(T) \leq \beta(T)$ for a tree T, exhibit an infinite class of trees in which the differences $\gamma_w - i$ and $\beta - \gamma_w$ can be made arbitrarily large, and show that the decision problem corresponding to the computation of $\gamma_w(G)$ is NP-complete, even for bipartite graphs. Lastly, we provide a linear algorithm to compute $\gamma_w(T)$ for a tree T.

1 Introduction

Let $G = (V, E)$ be a graph. A set $S \subseteq V$ is a *dominating set* of G if for every $u \in V - S$, there exists a $v \in S$ such that $uv \in E$. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. That the domination number γ is a well-studied parameter, is clear from the extensive bibliography on domination (see [2]). The *independent domination number* of G, denoted by $i(G)$, is the minimum cardinality of an independent dominating set in G. The *independence number* of G, denoted by $\beta(G)$, is the maximum cardinality of an independent set in G. A set $S \subseteq V$ is a *weak dominating set* of G if for every u in $V - S$, there exists a $v \in S$ such that $uv \in E$ and $deg(u) \geq deg(v)$. The *weak domination number* of G, denoted by $\gamma_w(G)$, is the minimum cardinality of a weak dominating set of G. This new parameter was recently introduced by Sampathkumar and Pushpa Latha in [3], in which they prove the following.

Theorem 1 *(a) For any graph G of order $p, \gamma(G) \leq \gamma_w(G) \leq p - \delta(G)$.*
(b) $\gamma_w(G) + \gamma_w(\overline{G}) \leq p + 1$.
(c) If T is a tree of order $p \geq 3$ with e endvertices and b vertices adjacent to an endvertex, then $e \leq \gamma_w(T) \leq p - b$. Furthermore, if $T \not\cong K_{1,p-1}$, then $\gamma_w(T) \leq p - e$, while $\gamma_w(T) + \gamma_w(\overline{T}) \leq 2p - 3$.

The *independent weak domination number* of G, denoted by $i_w(G)$, is the minimum cardinality of a weak-dominating set which is also independent. Allan and Laskar (see [1]) have shown that if a graph G does not contain $K_{1,3}$ as an induced subgraph, then $i(G) = \gamma(G)$. Similarly, Sampathkumar and Pushpa Latha (see [3]) show that if a graph G does not contain $K_{1,3}$ as an induced subgraph, then $i_w(G) = \gamma_w(G)$. Lastly, Sampathkumar and Pushpa Latha (see [3]) conjecture that $i(T) \leq \gamma_w(T)$ for a tree T.

We show that this conjecture is true, prove that $\gamma_w(T) \leq \beta(T)$ for a tree T, exhibit an infinite class of trees for which the differences $\gamma_w - i$ and $\beta - \gamma_w$ can be made arbitrarily large, and show that the decision problem corresponding to the computation of $\gamma_w(G)$ is NP-complete, even for bipartite graphs. Lastly, we provide a linear algorithm to compute $\gamma_w(T)$ for a tree T.

2 Main results

In this section we prove that $i(T) \leq \gamma_w(T)$ for a tree T, thus settling a conjecture of Sampathkumar and Pushpa Latha ([3]). We then prove that $\gamma_w(T) \leq \beta(T)$ for a tree T and then exhibit an infinite class of trees for which the differences $\gamma_w - i$ and $\beta - \gamma_w$ can be made arbitrarily large.

We start by proving a conjecture of Sampathkumar and Pushpa Latha ([3]).

Theorem 2 *Let $T = (V, E)$ be a tree. Then $i(T) \leq \gamma_w(T)$.*

Proof. Suppose the result is not true and let T be the smallest order tree such that $i(T) > \gamma_w(T)$. Note that $p = p(T) \geq 3$. Root T at any nonleaf vertex r and let v be an endvertex of T for which $d(v, r)$ is a maximum. We first consider the case when $d(v, r) \leq 2$. Then T is either isomorphic to a $K_{1,n}$ or a $K_{1,n}$ with some edges divided once. Let s be the number of vertices at distance 1 from r and let t be the number of vertices at distance 2 from r. If $s > 0$, then $i(T) = 1 + t \leq s + t = \gamma_w(T)$. If $s = 0$, then $i(T) = t < 1 + t = \gamma_w(T)$. Both cases lead to a contradiction and we may therefore assume that $d(v, r) \geq 3$. Let S be a weak dominating set of T of cardinality $\gamma_w(T)$. Before proceeding further, we prove the following claim.

Claim 1 *If v is a leaf of T, u is the parent of v and w is the parent of u, then $v \in S$ and $u \notin S$.*

Proof. Since $deg(v) = 1$ and $deg(u) \geq 2$ we must have that $v \in S$. Suppose, to the contrary, that $u \in S$. Let $T' = T - v$. Then, if $x \neq v$ is weakly dominated by u in T, i.e. $xu \in E$ and $deg_T(x) \geq deg_T(u)$, then $xu \in E(T')$ and $deg_{T'}(x) = deg_T(x) \geq deg_T(u) > deg_T(u) - 1 = deg_{T'}(u)$, so that x is also weakly dominated by u in T'. Hence, $S' = S - \{v\}$ is a weak dominating set of T', so that $\gamma_w(T') \leq |S'|$. Now, since $p(T') < p(T)$, we have that $i(T') \leq \gamma_w(T') \leq |S'| = |S| - 1 = \gamma_w(T) - 1$. Let I be a minimum dominating set of T'. If $u \in I$, then I is also an independent dominating set of T, so that $i(T) \leq |I| = i(T') < \gamma_w(T)$. On the other hand, if $u \notin I$, then $I \cup \{v\}$ is an independent dominating set of T, so that $i(T) \leq |I| + 1 = i(T') + 1 \leq \gamma_w(T)$. Both possibilities lead to a contradition, so that $u \notin S$. □

Since $d(v, r) \geq 3$, let u be the parent of v and let w be the parent of u. Suppose $v' \in N[u] - \{u, v, w\}$. If $deg_T(v') \geq 2$, then there must be a leaf x of T such that $d(r, x) > d(r, v)$, which contradicts our choice of v. We conclude that v' must be a leaf of T. Also, by Claim 1, $v, v' \in S$, while $u \notin S$. Let $S' = S - \{v\}$. Then S' is a weak dominating set of $T' = T - v$, so that $\gamma_w(T') \leq |S'| = |S| - 1$. Now, since $p(T') < p(T)$, we must have that $i(T') \leq \gamma_w(T') \leq |S| - 1 = \gamma_w(T) - 1$. Let I be a minimum independent dominating set of T'. If $u \in I$, then I is also an independent dominating set of T, so that $i(T) \leq |I| = i(T') < \gamma_w(T)$. If $u \notin I$, then $I \cup \{v\}$ is an independent dominating set of T, so that $i(T) \leq |I| + 1 = i(T') + 1 \leq \gamma_w(T)$. These contradictions show that v has no siblings.

Note that $deg_T(w) \geq 2$. Before proceeding further, we prove another claim.

Claim 2 *If $u' (\neq u)$ is a child of w, then u' is not a leaf of T.*

Proof. Suppose, to the contrary, that u' is a leaf of T. By Claim 1, we have that $v, u' \in S$, while $u, w \notin S$. The set $S' = S - \{v\}$ is a weak dominating set of $T' = T - \{u, v\}$. Then $i(T') \leq \gamma_w(T') \leq |S| - 1 = \gamma_w(T) - 1$. Let I be a minimum independent dominating set of T'. Then $I \cup \{v\}$ is an independent dominating set of T, so that $i(T) \leq |I|+1 = i(T')+1 \leq \gamma_w(T)$, which is a contradiction. □

By Claim 2 and the choice of v, it follows that T_w, the maximal subtree rooted at w, is isomorphic to a $K_{1,n}$ where each edge is divided once. Since $d(v,r) \geq 3$, w has a parent, say ℓ. Let $v_1, \ldots, v_n$ be the leaves of T and let $u_1, \ldots, u_n$ be their parents. By Claim 1, $\{v_1, \ldots, v_n\} \subseteq S$, while $u_i \notin S$ for $i = 1, \ldots, n$. Suppose that $w \in S$. The set $S' = S - (\cup_{i=1}^{n}\{v_i\} \cup \{w\}) \cup \{\ell\}$ is a weak dominating set of $T' = T - (\cup_{i=1}^{n}\{u_i, v_i\} \cup \{w\})$. As before, $i(T') \leq \gamma_w(T') \leq |S|-(n+1)+1 = |S|-n = \gamma_w(T)-n$. Let I be a minimum dominating set of T'. Then $I \cup \cup_{i=1}^{n}\{u_i\}$ is an independent dominating set of G, so that $i(T) \leq |I|+n \leq \gamma_w(T)$, which is a contradiction. Hence $w \notin S$. Since $u_i \notin S$ for $i = 1, \ldots, n$ and $w \notin S$, we must have that $\ell \in S$. Then $S' = S - (\cup_{i=1}^{n}\{v_i\})$ is a weak dominating set of $T' = T - (\cup_{i=1}^{n}\{u_i, v_i\} \cup \{w\})$. As before, $i(T') \leq \gamma_w(T') \leq |S| = \gamma_w(T) - n$. Let I be a minimum dominating set of T'. Then $I \cup \cup_{i=1}^{n}\{u_i\}$ is an independent dominating set of G, so that $i(T) \leq |I| + n \leq \gamma_w(T)$, which is a contradiction. This final contradiction shows that our assumption, namely that there is a tree T such that $i(T) > \gamma_w(T)$, is false and the theorem is proved. This result is best possible, since $i(P_4) = \gamma_w(P_4) = 2$. ■

Theorem 3 *Let T be a tree. Then $\gamma_w(T) \leq \beta(T)$.*

Proof. It is easily verified that $\gamma_w(T) \leq \beta(T)$ for all trees of order at most 3. Suppose $\gamma_w(T) \leq \beta(T)$ for all trees of order p and let T be a tree of order at least $p+1 \geq 4$. Let r be a nonleaf of T and root T at r. Let S be a maximum independent set of T and let v be a leaf of T such that $d(v,r)$ is a maximum. Furthermore, let u be the parent of v and let T_u be the maximal subtree of T rooted at u. If u has no parent, then T is isomorphic to $K_{1,n}$ for some integer $n \geq 1$. Since every weak dominating set of T contains all the leaves of T, $\gamma_w(T) = n = \beta(T)$. Suppose, therefore, that u has a parent, say w. Let $X = \{v_1, \ldots, v_n\}$ be the leaves of T_u. Without loss of generality, we may assume that $X \subseteq S$, so that $u \notin S$. It now follows that $S' = S - X$ is an independent set of $T' = T - X - u$, which, clearly, is also maximum in T'. Hence, by the induction hypothesis, we have $\gamma_w(T') \leq \beta(T') \leq |S'|$. Let W be a minimum weak dominating set of T'. If $w \notin W$, then $W \cup X$ is a weak dominating dominating set of T, so that $\gamma_w(T) \leq |W| + |X| \leq |S'| + n = |S| = \beta(T)$ and we are done. We henceforth assume that $w \in W$. Our choice of v implies that every child of w in T' is either in W or adjacent to a leaf of T', which is in W. So, if the subtree T_u is reinserted into the tree T' to obtain the original T, the only vertex that may not be weakly dominated by $W \cup X$ is the parent of w, say w'. If this is the case, then $deg_{T'}(w) = deg_{T'}(w')$. Then

$W' = W - \{w\} \cup \{w'\}$ is a weak dominating set of T', since w' now weakly dominates itself and w. It now follows that $W' \cup X$ is a weak dominating set of T, so that $\gamma_w(T) \le |W'| + |X| = |W| + |X| \le |S'| + n = |S| = \beta(T)$ and we are done. The result is best possible, since $\gamma_w(K_{1,n}) = \beta(K_{1,n})$. ■

That there exists an infinite class of trees in which the differences $\gamma_w - i$ and $\beta - \gamma_w$ can be made arbitrarily large, may be seen as follows. Let $e \ge 1$ and $d \ge 2$ be integers. Let S_1 ($S_2 = \{a_1, \ldots, a_{d-1}\}$, respectively) be a set of e ($d-1$, respectively) independent vertices. Join each vertex in S_1 (S_2, respectively) to a new vertex c_1 (c_2, respectively). Now join c_1 with c_2. Let $H_1, \ldots, H_{d-1}$ be disjoint copies of a $K_{1,d}$ with each edge subdivided once and identify the central vertex of H_i with a_i for $i = 1, \ldots, d-1$. The set S_3 will be used to denote the vertices obtained in the subdivision process, while S_4 will be used to denote the leaves of the subdivided stars. Denote the resulting tree by $T(e,d)$. The tree $T(2,3)$ is shown in Figure 1.

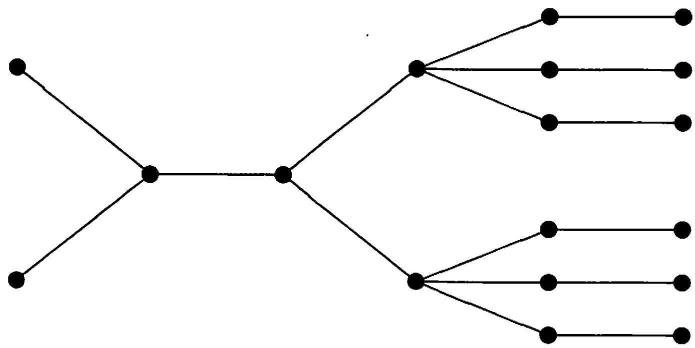

Figure 1: The tree $T(2,3)$

Theorem 4 *Let $e \ge 1$ and $d \ge 2$ be integers and $T = T(e,d)$. Then $\gamma_w(T) - i(T) = e$ and $\beta(T) - \gamma_w(T) = d - 2$.*

Proof. Let I be an independent dominating set of T. Then $|I \cap (S_3 \cup S_4)| \ge (d-1)d$ and $|I \cap (S_1 \cup \{c_1\})| \ge 1$, so that $|I| \ge 1 + (d-1)d$. The set $S_3 \cup \{c_1\}$ is an independent dominating set of T of cardinality $1 + (d-1)d$, so that $i(T) = 1 + (d-1)d$.

Let D be a weak dominating set of T. Then, since every weak dominating set must contain all the leaves of T, $S_1 \cup S_4 \subseteq D$. Also, in order to weakly dominate c_2, we must have that $D \cap \{c_1, c_2\} \ne \emptyset$, so that $|D| \ge$

$|S_1| + |S_4| + |D \cap \{c_1, c_2\}| \geq e + (d-1)d + 1$. The set $S_1 \cup S_4 \cup \{c_2\}$ is a weak dominating set of T of cardinality $e + (d-1)d + 1$, so that $\gamma_w(T) = e + (d-1)d + 1$.

Let I be a maximum independent set of T. Then $S_1 \cup S_4 \subseteq I$, so that $c_1 \notin I$ and $S_3 \cap I = \emptyset$. It now follows that $S_2 \subseteq I$, whence $c_2 \notin I$. Hence, $I = S_1 \cup S_2 \cup S_4$, so that $\beta(T) = e + (d-1) + (d-1)d$.

This shows that $\gamma_w(T) - i(T) = e$ and $\beta(T) - \gamma_w(T) = d - 2$. ■

3 Complexity results

Consider the decision problem

WEAK DOMINATING SET (WDS)

INSTANCE: A graph $G = (V, E)$ and a positive integer $k \leq |V|$.

QUESTION: Is there a weak dominating set of cardinality at most k?

In this section we will show that **WDS** is NP-complete, even when restricted to bipartite graphs, by describing a polynomial transformation from the following well-known NP-complete problem:

EXACT COVER BY 3-SETS (X3C)

INSTANCE: A finite set X with $|X| = 3q$ and a collection $\mathcal{C}$ of 3-element subsets of X. Each element $x \in X$ appears in at least two subsets.

QUESTION: Does $\mathcal{C}$ contain an exact cover for X, that is, a subcollection $\mathcal{C}' \subseteq \mathcal{C}$ such that every element of X occurs in exactly one member of $\mathcal{C}'$. Note that if $\mathcal{C}'$ exists, then its cardinality is precisely q.

Theorem 5 **WDS** *is NP-complete, even for bipartite graphs.*

Proof. It is clear that **WDS** is in NP.

To show that **WDS** is a NP-complete problem, we will establish a polynomial transformation from **X3C**. Let $X = \{x_1, \ldots, x_{3q}\}$ and $\mathcal{C} = \{C_1, \ldots, C_n\}$ be an arbitrary instance of **X3C**. We will construct a bipartite graph G and a positive integer k such that this instance of **X3C** will have an exact three cover if and only if G has a weak dominating set of cardinality at most k.

The graph G is constructed as follows. Let F be the graph obtained from a $K_{1,2}$ with each edge divided exactly once. The central vertex is denoted by v, the vertices obtained by the subdivision process by d and d' and the leaves of F adjacent to d and d' by e and e'. Let $F_1, \ldots, F_{3q}$ be $3q$ disjoint copies of F. Corresponding to each variable x_i we associate the graph F_i. Let $\{v_i, d_i, d'_i, e_i, e'_i\}$ be the names of the vertices in F_i that are names v, d, d', e, e', respectively, in F. Corresponding to each set C_j we associate the graph $H_j \cong K_2$ with $V(H_j) = \{c_j, c'_j\}$. The construction of G is completed by joining v_i and c_j if and only if the variable $x_i \in C_j$. Finally, set $k = m + 7q$. Note that G is bipartite and that the construction of G is accomplished in polynomial time.

Let $D = \cup_{i=1}^{3q}\{d_i, d'_i\}$, $E = \cup_{i=1}^{3q}\{e_i, e'_i\}$, $V = \{v_1, \ldots, v_{3q}\}$, $C' = \{c'_1, \ldots c'_m\}$ and $C = \{c_1, \ldots, c_m\}$.

Suppose $\mathcal{C}$ has an exact 3-cover, say $\mathcal{C}'$. Then it is easily verified that $S = E \cup C' \cup \{c_j | C_j \in \mathcal{C}'\}$ is a weak dominating set of cardinality $2(3q) + m + q = m + 7q$. (Note that $deg(v_i) \geq 4 = deg(c_j)$ for all i and all j.)

Suppose, conversely, that S is a minimum weak dominating set of cardinality at most $m + 7q$. Note that $E \cup C' \subseteq S$, since S must contain every endvertex of G. Let $S' = S - (E \cup C')$. Then $|S'| \leq m + 7q - (6q + m) = q$. We now prove that $S' \subseteq C$. Suppose $|S' \cap D| = x$ and $|S' \cap V| = y$. Then $|S' \cap C| \leq q - (x + y)$, so that $|N[S' \cap C] \cap V| \leq 3(q - (x + y)) = 3q - 3x - 3y$. Note that $|N[S' \cap D] \cap V| = |S' \cap D| = x$. It then follows that $|V - N[S' \cap D] \cap V - S' \cap V - N[S' \cap C] \cap V| \geq 3q - x - y - (3q - 3x - 3y) = 2x + 2y$. If $x > 0$ or $y > 0$, then $x_i \notin N[S]$ for some $i \in \{1, \ldots, 3q\}$, which contradicts the fact that S is a weak dominating set of G. This implies that $S' \subseteq C$.

Let $\mathcal{C}' = \{C_j | c_j \in S\}$. Then, since S is a weak dominating set of G, $\mathcal{C}'$ must be a cover for X. Since $\mathcal{C}'$ is a cover of X such that $|\mathcal{C}'| \leq q$, it follows that $C_i \cap C_j = \emptyset$ for distinct C_i and C_j in $\mathcal{C}'$. Also, if x_k is covered by distinct elements C_i and C_j, then, by the construction of G, $x_k \in C_i \cap C_j$, which is a contradiction. Hence $\mathcal{C}'$ is an exact three cover for X. ■

4 A linear algorithm for computing $\gamma_w(T)$ for a tree T

In this section, we present a linear algorithm for computing the value of $\gamma_w(T)$ for any tree T. We construct a dynamic style algorithm using the methodology of Wimer (see [4]).

We make use of the well-known fact that the class of (rooted) trees can be constructed recursively from copies of the single vertex K_1, using only one rule of composition, which combines two trees (T_1, r_1) and (T_2, r_2) by adding an edge between r_1 and r_2 and calling r_1 the root of the resulting larger tree. We denote this as follows: $(T, r_1) = (T_1, r_1) \circ (T_2, r_2)$.

In particular, if S is a weak dominating set of T, then S splits into two subsets S_1 and S_2 according to this decomposition. We express this as follows: $(T, S) = (T_1, S_1) \circ (T_2, S_2)$.

We will find it convenient to know, a priori, what the degree of each vertex in the tree T is. This is accomplished by the following (linear) algorithm. Suppose we have as input the parent array $parent[1 \ldots p]$ for the input tree. The output will be the array $deg[1 \ldots p]$ where $deg[i]$ will be the degree of the ith vertex.

```
procedure degree;

begin


for i:=1 to p do
    deg[i]:=0;

for i:=p downto 2 do
begin
    deg[i]:=deg[i]+1;
    deg[parent[i]]:=deg[parent[i]]+1;
end;

end; {degree}
```

In order to construct an algorithm to compute $\gamma_w(T)$ for any tree T, we characterise the classes of possible tree-subset pairs (T, S) which can occur. For this problem there are four classes:

$[1] = \{(T_1, S_1) | r_1 \in S_1, S_1$ is a dominating set of T_1 and $\forall v \in V(T_1)\ \exists s \in S_1 \cap N[v]$ such that $deg_T(v) \geq deg_T(s)\}$.
$[2] = \{(T_1, S_1) | r_1 \notin S_1, S_1$ is a dominating set of T_1 and $\forall v \in V(T_1)\ \exists s \in S_1 \cap N[v]$ such that $deg_T(v) \geq deg_T(s)\}$.
$[3] = \{(T_1, S_1) | r_1 \notin S_1, S_1$ is a dominating set of T_1 and $\forall v \in V(T_1) - \{r_1\}\ \exists s \in S_1 \cap N[v]$ such that $deg_T(v) \geq deg_T(s)\}$.
$[4] = \{(T_1, S_1) | r_1 \notin S_1, S_1$ is not a dominating set of T_1 and $\forall v \in V(T_1) - \{r_1\}\ \exists s \in S_1 \cap N[v]$ such that $deg_T(v) \geq deg_T(s)\}$.

	[1]	[2]	[3]	[4]	Conditions
[1]	[1]	[1]	X	X	$deg_T(r_1) > deg_T(r_2)$
	[1]	[1]	[1]	[1]	$deg_T(r_1) = deg_T(r_2)$
	[1]	[1]	[1]	[1]	$deg_T(r_1) < deg_T(r_2)$
[2]	[2]	[2]	X	X	$deg_T(r_1) > deg_T(r_2)$
	[2]	[2]	X	X	$deg_T(r_1) = deg_T(r_2)$
	[2]	[2]	X	X	$deg_T(r_1) < deg_T(r_2)$
[3]	[2]	[3]	X	X	$deg_T(r_1) > deg_T(r_2)$
	[2]	[3]	X	X	$deg_T(r_1) = deg_T(r_2)$
	[3]	[3]	X	X	$deg_T(r_1) < deg_T(r_2)$
[4]	[2]	[4]	X	X	$deg_T(r_1) > deg_T(r_2)$
	[2]	[4]	X	X	$deg_T(r_1) = deg_T(r_2)$
	[3]	[4]	X	X	$deg_T(r_1) < deg_T(r_2)$

Figure 2:

Next we must consider the effect of composing a tree T_1 having a set S_1 which is of class $[i]$ with a tree T_2 having a set which is of class $[j]$ for every possible combination of classes $1 \leq i, j \leq 4$. That is, we must describe the appropriate class of the combined set $S_1 \cup S_2$ in the composed tree $T = T_1 \circ T_2$. This is described in Figure 2 . An 'X' in the table signifies that this composition cannot happen, that is, no set S can ever decompose into two subsets S_1 and S_2 of the classes indicated.

From Figure 2, we can now write out a system of recurrence relations, as follows.

if $deg(r_1) > deg(r_2)$
then begin
    $[1] = [1] \circ [1] \cup [1] \circ [2]$
    $[2] = [2] \circ [1] \cup [2] \circ [2] \cup [3] \circ [1] \cup [4] \circ [1]$
    $[3] = [3] \circ [2]$
    $[4] = [4] \circ [2]$
  end
else if $deg(r_1) = deg(r_2)$
  then begin
    $[1] = [1] \circ [1] \cup [1] \circ [2] \cup [1] \circ [3] \cup [1] \circ [4]$
    $[2] = [2] \circ [1] \cup [2] \circ [2] \cup [3] \circ [1] \cup [4] \circ [1]$
    $[3] = [3] \circ [2]$
  end

```
else begin
    [1] = [1] ∘ [1] ∪ [1] ∘ [2] ∪ [1] ∘ [3] ∪ [1] ∘ [4]
    [2] = [2] ∘ [1] ∪ [2] ∘ [2]
    [3] = [3] ∘ [1] ∪ [3] ∘ [2] ∪ [4] ∘ [1]
    [4] = [4] ∘ [2]
  end
```

To illustrate this, a tree-subset pair of class [3] can be read as follows: a tree-subset pair (T, S) which is of class [3] can be obtained only by composing a pair (T_1, S_1) of class [3] with a pair (T_2, S_2) of class [2] if $deg(r_1) \geq deg(r_2)$ and by composing a pair (T_1, S_1) of class [3] with a pair (T_2, S_2) of class [1] or by composing a pair (T_1, S_1) of class [3] with a pair (T_2, S_2) of class [2] or composing a pair (T_1, S_1) of class [4] with a pair (T_2, S_2) of class [1] if $deg(r_1) < deg(r_2)$.

To prove the correctness of this dynamic programming algorithm for computing $\gamma_w(T)$ for any tree T, we would have to prove a theorem asserting that each of these recurrences are correct. Space limitations prevent us from doing this here, but it is easy to do. It is even easier to verify the correctness of Figure 2, which can be done by inspection. The final step in specifying a γ_w-algorithm is to define the initial vector. In this case, for trees, the only basis graph is the tree with single vertex K_1. We need to know the minimum cardinality of a set S in a class of type [1] to [4] in the graph K_1, if any exists. It is easy to see that the initial vector is $[1, -, -, 0]$ where '-' means undefined.

We now have all the ingredients for a γ_w-algorithm, where the input is the parent array $parent[1 \ldots p]$ for the input tree and where the output is the 4-tuple corresponding to the root (i.e. vertex 1) of T which is computed repeatedly by applying the recurrence system to each vertex in the parent array, with the initial vector $[1, -, -, 0]$ being associated with every vertex in the parent array as the computation begins.

The basic structure for the algorithm is a simple iteration.

```
procedure γw;

for i:=1 to p do
    initialise vector [1 ...4] to [1,-,-,0];

call degree;

for j:=p downto 2 do
begin
```

```
   k:=parent[j];
   combine(vector,k,j);
 end;

γw(T) := min {vector[1,1], vector[1,2]};

end; {γw}
```

The combine procedure is derived directly from the recurrence system:

```
procedure combine (vector, k, j);

if deg[k] > deg[j]
then begin
       vector[k,1]:= min {vector[k,1]+vector[j,1],
                          vector[k,1]+vector[j,2]};
       vector[k,2]:= min {vector[k,2]+vector[j,1],
                          vector[k,2]+vector[j,2]},
                          vector[k,3]+vector[j,1],
                          vector[k,4]+vector[j,1];
       vector[k,3]:= vector[k,3]+vector[j,2];
       vector[k,4]:= vector[k,4]+vector[j,2];
     end
else if deg[k] = deg[j]
     then begin
             ⋮
          end
     else begin
             ⋮
          end;

end; { combine }
```

It is clear that procedure γ_w has linear execution time.

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