One and Two-Block Configurations in Balanced Ternary Designs

M.A. Francel
Department of Mathematics and Computer Science
The Citadel
Charleston, SC, 29409
francelm@cit1.citadel.edu

D.G. Sarvate
Department of Mathematics
University of Charleston
Charleston, SC 29424
sarvated@ashley.cofc.edu

ABSTRACT. In this paper we count n-block BTD$(V, B, R, 3, 2)$ configurations for $n = 1$ and 2. In particular, we list all configuration types and determine formulae for the number of n-block subsets of a design of each type. A small number of the formulae are shown to be dependent solely on the design parameters The remainder are shown to be dependent on the number of occurrences of two particular two-block configurations as well as the design parameters. Three new non-isomorphic BTD(9; 33;5,3,11; 3; 2) are given that illustrate the independence of certain configurations.

1 Introduction

A *balanced ternary design*, (BTD), with parameters (V, B, R, K, Λ) is a collection of B blocks on V elements such that each element occurs R times in the design; each block contains K elements, where an element may occur 0, 1, or 2 times in a block (i.e. a block is a collection of elements rather than a set of elements); and each pair of distinct elements occurs Λ times in the design. (For a survey of BTD's we refer the reader to Billington [1,2,3].) A BTD is a generalization of a *balanced incomplete block design* (BIBD). In BIBD's no element can occur more than once in a block. Other than this, the definitions for BIBD's and BTD's do not differ.

An *n-line BIBD configuration* is a collection of any n blocks (i.e. lines) of a BIBD. Work has been done on decomposing BIBD$(v, b, r, 3, 1)$'s into two, three, and four-line configurations [6,8,9] and constructing BIBD$(v, b, r, 3, 1)$'s containing no "forbidden configurations" [4,7]. Most recently, Grannell, Griggs and Mendelsohn, [5], have developed formulae for the number of two, three and four-line configurations in BIBD$(v, b, r, 3, 1)$.

The purpose of this paper is to extend the last strand of work by developing formulae for the number of one and two-block BTD$(V, B, R, 3, 2)$ configurations. In Section 2 we present the design examples we will use to show two-block BTD configuration independence and the one-block BTD configurations. In Section 3 we present the two-block BTD configurations and develop formulae for configurations of designs with parameters $(V, B, R, 3, 2)$.

2 Preliminaries and one-block BTD configurations

When we use the term BTD, we assume that the design contains at least one element that appears doubly in some block, and at least one element that appears singly in some block. BTD's are regular in the sense that every element occurs singly in ρ_1 blocks and doubly in ρ_2 blocks where $R = \rho_1 + 2\rho_2$. Thus, our assumption is equivalent to the assumption that both ρ_1 and ρ_2 are nonzero. Under this assumption Λ will always be greater than or equal to two. Also, because of the above described regularity, BTD parameters are usually given as $(V; B; \rho_1, \rho_2, R; K; \Lambda)$ rather than simply (V, B, R, K, Λ).

Before we examine BTD configurations, we present four BTD examples that we will use to illustrate configuration independence. In our examples and throughout the paper, we use bold faced italicized triples of letters/numbers to represent blocks, and sets of bold faced italicized triples to represent block configurations.

All four BTD examples listed below have parameters (9; 33; 5,3,11; 3; 2). Designs D_2, D_3, and D_4 are new. Design D_1, which is included for ease of refernce, was first given by Billington, [B1].

Design D_1 blocks:

112, 114, 116, 133, 159, 177, 188, 223, 224, 225, 267, 267, 288, 299, 335, 336, 348, 348, 377, 399, 445, 447, 449, 466, 556, 557, 558, 668, 669, 778, 799, 899

Design D_2 blocks:

112, 113, 114, 155, 166, 177, 188, 199, 225, 226, 227, 233, 244, 288, 299, 338, 339, 344, 355, 366, 377, 445, 466, 478, 478, 499, 559, 568, 568, 577, 679, 679, 889

Design D_3 blocks:

123, 124, 134, 234, 567; 567, 115, 116, 117, 188, 199, 227, 228, 229, 255, 266, 335, 338, 339, 366, 377, 446, 447, 449, 455, 488, 558, 599, 669, 688, 778, 779, 899

Design D_4 blocks:

123, 124, 135, 145, 166, 234, 345, 117, 118, 119, 255, 277, 226, 228, 229, 336, 337, 338, 399, 446, 447, 449, 488, 556, 557, 588, 599, 668, 669, 677, 788, 779, 899

None of the designs D_1, D_2, D_3, nor D_4 are isomorphic to one another. For two designs to be isomorphic it is necessary for them to both contain the same number of repeated blocks. Only D_1 and D_2 contain the same number of repeated blocks (D_1 and D_2 both contain three blocks each repeated twice. D_3 contains one block repeated twice, and D_4 contains no repeated blocks.) Although D_1 and D_2 contain the same number of repeated blocks they are not isomorphic since in D_2 all three of the repeated blocks taken in pairs intersect, while in D_1 no pair intersects.

We are now ready to examine BTD configurations. Define an *n-block BTD configuration* to be a collection of any n blocks in the BTD. We are interested in determining the number of n-configuration types, and in finding formulae that count the number of times a particular configuration type appears in a design.

Repeated blocks and repeated elements are treated as distinct in BTD block and pair counts. Similarly here, we consider repeated configurations as distinct in configuration counts. For example, assume $b_1 =$ **abc** and $b_2 =$ **abc** are repeated blocks in a BTD that also contains the block $b_3 =$ **def**. Although the two-configurations $\{b1, b3\}$ and $\{b2, b3\}$ are the same set when viewed as $\{abc, def\}$, they will be counted as two configurations. However, for completeness the paper does give formulae for configuration counts where repeats are not counted.

There are two one-block BTD($V, B, R, K = 3, \Lambda$) configurations. They are {**aab**} and {**abc**}. We say the configurations are *constant*, meaning the formulae for the number of each can be given solely in terms of the design parameters. The number of configurations of the form {**aab**} is $V\rho_2$, the number of the form {**abc**} is $B - V\rho_2 = V(\rho_1 - \rho_2)/3$. When the number of configurations of a particular type can not be stated in terms of the design parameters alone, the configuration is said to be *variable*.

Although the number of both one-block configurations are constant, the number of distinct one-block configurations of type {**abc**} is variable. Let t_2 be the number of repeated blocks of type **abc**. The number of distinct configurations of the form {**abc**} is $B - V\rho_2 - t_2$. The value t_2 is independent of the design parameters. We use the design sets given above to illustrate this. Designs D_1, D_2, D_3, D_4 all have the same parameters, (9; 33; 5,3,11; 3; 2). However, D_1 and D_2 contain 3 pairs of repeated blocks of the form **abc**, while D_3 contains one pair, and D_4 contains none. We close the discussion of one-block configurations by noting that in a BTD with $K = 3$ and $\Lambda = 2$, there can be no repeated blocks of type **aab**.

3 Two-block BTD configurations

There are fourteen distinct types of two-block BTD configurations for designs with block size three. The complete listing is shown in Table 3.1. When Λ is small, certain of the configurations can not exist. In Table 3.1, these restrictions are included with the corresponding configuration. Also given in the table are the formulae for counting configurations of a certain type, and the variables used in the formulae.

We examine below the two-block configurations for BTD with parameters $(V, B, R, 3, 2)$. There are eleven possible configurations in this case. Three are constant and eight are variable (i.e. dependent on variables other than the design parameters). Throughout the remainder of the paper we use the configuration notation of Grannell, Griggs and Mendelsohn [5]. C_i will denote a configuration type, and c_i will denote the count for configuration type C_i.

Case 1. $C_6 = \{\mathbf{aab}, \mathbf{aac}\}$, $c_6 = V\rho_2(\rho_2 - 1)/2$.

To construct C_6 configurations, pair each block **aab** with the $\rho_2 - 1$ other blocks containing the same double element. Since the ρ_2 blocks where an element appears doubly are distinct when $\Lambda = 2$, there are $V\rho_2(\rho_2 - 1)$ ways to do this. Each C_6 configuration is produced twice by this construction. Thus, $c_6 = V\rho_2(\rho_2 - 1)/2$.

Case 2. $C_8 = \{\mathbf{aab}, \mathbf{bbc}\}$, $c_8 = V\rho_2^2$.

To construct C_8 configurations, pair each block **aab** with the ρ_2 blocks containing the element **b** doubly. This generates $V\rho_2^2$ pairs **aab**, **bbc**. (The existence of **aab** implies the nonexistence of **bba** when Λ is two, so $c \neq$ **a**.) Each pair produced a distinct C_8 configuration. Thus, $c_8 = V\rho_2^2$.

Case 3. $C_1 = \{\mathbf{abc}, \mathbf{abc}\}$, c_1 is independent of the design parameters.

Let $c_1 = n$.

The number of repeated blocks in a design can not be formulated in terms of the design parameters alone. This is illustrated by the four non-isomorphic BTD(9; 33; 5,3,11; 3; 2) examples of Section 2. Designs D_1 and D_2 each have three repeated blocks, D_3 has one, and D_4 has none.

Case 4. $C_{12} = \{\mathbf{aab}, \mathbf{acd}\}$, $c_{12} = V\rho_2(\rho_1 - \rho_2)$.

To construct C_{12} configurations, pair each block **aab** with the ρ_1 blocks that contain element **a** singly. The blocks being added will have the form

acx where **x** = **c** or **d**. If **x** = **c**, then the pair formed will be a C_8 configuration. If **x** = **d**, then the pair formed will be a C_{12} configuration. Each C_8 and C_{12} is produced once and only once by this construction. Thus, $c_{12} = V\rho_2\rho_1 - c_8 = V\rho_2(\rho_1 - \rho_2)$.

Each block **acd** that is repeated in the design will appear in $6\rho_2$ C_{12} configurations with only $3\rho_2$ of them being unique. Thus, there are $V\rho_2(\rho_1 - \rho_2) - 3\rho_2 n$ distinct C_{12} configurations.

Case 5. $C_2 = \{\mathbf{abc}, \mathbf{abd}\}$, $c_2 = V(\rho_1 - \rho_2)/2 - 3n$.

Each block **abc** contains three pairs of elements, ($\underline{\mathbf{abc}}$, $\underline{\mathbf{a}}\mathbf{b}\underline{\mathbf{c}}$, and $\mathbf{a}\underline{\mathbf{bc}}$). To construct C_2's, for each block **abc** and each pair of elements in the block, match the block with the unique other block in the design containing the same pair. If $\underline{\mathbf{abc}}$ is the block and pair under question, then the block added will be of the form **abx** where **x** = **c** or **d** (**x** $\neq$ **a** or **b** since $\Lambda = 2$). Each of the C_1 configurations $\{\mathbf{abc}, \mathbf{abc}\}$ will be produced a total of six times by the construction. All other pairs produced will have the form **abc**, **abd** and will appear twice each. Thus, $c_2 = [3(B - V\rho_2) - 6c_1]/2 = V(\rho_1 - \rho_1)/2 - 3n$.

None of the blocks that appear in C_2 configurations can be a repeated block in the design. If a block **abc** was repeated in the design and appeared in the C_2 configuration $\{\mathbf{abc}, \mathbf{abd}\}$, the pair **ab** would appear three times in the design. This can't happen since $\Lambda = 2$.

Case 6. $C_{13} = \{\mathbf{aab}, \mathbf{bcd}\}$, c_{13} is independent of the design parameters and n.

Let $c_{13} = m$.

The number of C_{13} configurations $\{\mathbf{aab}, \mathbf{ccb}\}$ in a design can not be formulated in terms of the design parameters and n alone. This is illustrated by design examples given in Section 2. D_1 and D_2 each has parameters (9; 33; 5,3,11; 3, 2) and three repeated block pairs. Yet, D_1 contains 54 C_{13} configurations, while D_2 contains 30 C_{13} configurations.

It can be shown that the number of C_{13} configurations that are duplicates can not be formulated in terms of the design parameters, m, and n alone. Let m' represent the number of distinct C_{13} configurations.

Case 7. $C_9 = \{\mathbf{aab}, \mathbf{bbc}\}$, $c_9 = (V\rho_2(\rho_1 - 1) - m)/2$.

To construct C_9 configurations, pair each block **aab** with the $(\rho_1 - 1)$ blocks, different from **aab**, that contain element **b** singly. These blocks will be of the form **bcx** where **x** = **c** or **d**. Each pair of blocks of the form **aab**, **bcd** will be produced once by the construction except of course, if the block **bcd** appears twice in the design. This will cause two duplicate pairs to be produced. Each C_9 configuration will be produced twice. Thus, $2c_9 = V\rho_2(\rho_1 - 1) - c_{13} = V\rho_2(\rho_1 - 1) - m$.

Case 8. $C_{10} = \{\mathbf{aab}, \mathbf{ccd}\}$, $c_{10} = V\rho_2[(V - 3)\rho_2 - \rho_1 + 1]/2 + m/2$.

To construct C_{10} configurations, pair each block **aab** with the $(V - 2)\rho_2$

blocks ccx where the repeated element is neither a nor b. The element x will be a, b, or d. Each C_9 and C_{10} configuration will be produced twice and each C_8 configuration once by the construction. Thus, $c_{10} = (V\rho_2(V-2)\rho_2 - c_8 - 2c_9)/2 = V\rho_2[(V-3)\rho_2 - \rho_1 + 1]/2 + m/2$.

Case 9. $C_3 = \{\mathbf{abc}, \mathbf{ade}\}$, $c_3 = [V(\rho_1 - \rho_2)(\rho_1 - 3) + 6n - m]/2$.

To construct C_3 configurations, pair each block **abc** with each of the $3(\rho_1 - 1)$ blocks that match **abc** in one of **a**, **b**, or **c** and in which the match appears singly. Using this construction each C_1 is produced six times, each C_2 four times, each C_{13} once, and each C_3 twice. Thus, $2c_3 = 3(B - V\rho_2)(\rho_1 - 1) - 6c_1 - 4c_2 - c_{13} = V(\rho_1 - \rho_2)(\rho_1 - 3) + 6n - m$.

To count the distinct C_3 configurations, produce the pairs as described above. Next remove the $6n(\rho_1 - 1) + (m - m')$ pairs that were produced twice because of repeated blocks. What is left is each C_2 four times, each distinct C_{13} once, and each distinct C_3 twice. Thus, the number of distinct C_3 configurations is $[3(B - V\rho_2)(\rho_1 - 1) - (6n(\rho_1 - 1) + (m - m') - 4c_2 - m']/2 = [V(\rho_1 - \rho_2)(\rho_1 - 3) - 6n(\rho_1 - 3) - m]/2$.

Case 10. $C_4 = \{\mathbf{abc}, \mathbf{def}\}$, $c_4 = V(\rho_1 - \rho_2)[V(\rho_1 - \rho_2) - 9\rho_1 + 15]/18 - n + m/2$.

To construct C_4 configurations, pair each block **abc** with each of the $(B - V\rho_2 - 1)$ other blocks that contain three distinct elements none of which are a, b, or c. In doing this, each C_1, C_2, C_3, and C_4 will appear twice. Thus $c_4 = [(B - V\rho_2)(B - V\rho_2 - 1) - 2c_1 - 2c_2 - 2c_3]/2 = (B - V\rho_2)(B - V\rho_2 - 1)/2 - V(\rho_1 - \rho_2)(\rho_1 - 2)/2 - n + m/2 = V(\rho_1 - \rho_2)[3V(\rho_1 - \rho_2) - 3\rho_1 + 5]/3 - n + m/2$.

To construct distinct C_4 configurations, pair each distinct block **abc** with each of the $(B - V\rho_2 - n - 1)$ other distinct blocks that contain three elements. In doing this, each distinct C_2, C_3, and C_4 will appear twice. Thus, the number of distinct C_4 configurations is $[(B - V\rho_2 - n)(B - V\rho_2 - n - 1)/2 - V(\rho_1 - \rho_2)(\rho_1 - 2)/2 + 3n(\rho_1 - 2) + m/2$.

Case 11. $C_{14} = \{\mathbf{aab}, \mathbf{cde}\}$, $c_{14} = V\rho_2(\rho_1 - \rho_2)(V - 3)/3 - m$.

To construct C_{14} configurations, pair each block **aab** with each of the blocks that contain three distinct elements. In doing this, each C_{12}, C_{13} and C_{14} will be produced once. Thus, $c_{14} = V\rho_2(B - V\rho_2) - c_{12} - c_{13} = V\rho_2(B - V\rho_2 + \rho_2 - \rho_1) - m = V\rho_2(\rho_1 - \rho_2)(V - 3)/3 - m$.

To construct distinct C_{14} configurations, pair each block **aab** with each of the distinct blocks that contain three distinct elements. In doing this, each distinct C_{12}, C_{13} and C_{14} will be produced once. Thus, the number of distinct c_{14} configurations is $V\rho_2(B - V\rho_2 - n) - (V\rho_2(\rho_1 - \rho_2) - 3\rho_2 n) - m' = V\rho_2(B - V\rho_2 - n - \rho_1 + \rho_2) + 3\rho_2 n - m'$.

We conclude by explaining why C_1 and C_{13} were chosen as the independent configurations. BTD blocks are of two types; blocks that contain three distinct elements and blocks that contain only two distinct elements. Viewing the blocks from this perspective, the two-block configurations subdivide naturally into three classes: $C_1 - C_4$, $C_5 - c_{10}$, and $C_{11} - C_{14}$. These subdivisions point to using $\{C_1, C_9, C_{13}\}$ as a basis. However, c_1, c_9, c_{13} are not independent. In particular, $2c_9 + c_{13} = V\rho_2(\rho_1 - 1)$. Since the two blocks of C_9 are "more connected" than the two blocks of C_{13}, it would appear C_{13} should be dropped and $\{C_1, C_9\}$ used for the basis. But recall that our aim was to find formulae for distinct configurations as well as configurations. To do this a third count must be assumed (see Case 6). This count can be easily linked to c_{13}. Because blocks with only two distinct elements cannot be repeated in a BTD where $\Lambda = 2$, the count cannot be linked to c_9. Thus, $\{C_1, C_{13}\}$ was chosen as the basis.

Configuration Type	$\Lambda = 2$ Dependence	Restrictions	Number of Configurations $\Lambda = 2, K = 3$
$C_1 = \{abc, abc\}$	independent		n
$C_2 = \{abc, abd\}$	design parameters c_1		$V(\rho_1 - \rho_2)/2 - 3n$
$C_3 = \{abc, ade\}$	design parameters c_1, c_{13}		$[V(\rho_1 - \rho_2)(\rho_1 - 3)$ $+6n - m]/2$
$C_4 = \{abc, def\}$	design parameters c_1, c_{13}		$V(\rho_1 - \rho_2)[V(\rho_1 - \rho_2)$ $-9\rho_1 + 15]/18 - n + m/2$
$C_5 = \{aab, aab\}$		can not exist if $\Lambda = 2$ or 3	
$C_6 = \{aab, aac\}$	design parameters		$V\rho_2(\rho_2 - 1)/2$
$C_7 = \{aab, bba\}$		can not exist if $\Lambda = 2$ or 3	
$C_8 = \{aab, bbc\}$	design parameters		$V\rho_2^2$
$C_9 = \{aab, ccb\}$	design parameters c_{13}		$(V\rho_2(\rho_1 - 1) - m)/2$
$C_{10} = \{aab, ccd\}$	design parameters c_{13}		$V\rho_2[(V - 3)\rho_2 - \rho_1 + 1]/2$ $+m/2$
$C_{11} = \{aab, abc\}$		can not exist if $\Lambda = 2$	
$C_{12} = \{aab, acd\}$	design parameters		$V\rho_2(\rho_1 - \rho_2)$
$C_{13} = \{aab, bcd\}$	independent		m
$C_{14} = \{aab, cde\}$	design parameters c_{13}		$V\rho_2(\rho_1 - \rho_2)(V - 3)/3 - m$

Table 3.
Two-block BTD configurations (with $K = 3$)

References

[1] Elizabeth J. Billington, Balanced n-ary Designs: A Combinatorial Survey and Some New Results, *Ars Combinatoria* **17A** (1984), 37–72.

[2] Elizabeth J. Billington, Balanced Ternary Designs with Block Size Three, any Λ and R, *Aequationes Mathematicae* **29** (1985), 244–289.

[3] Elizabeth J. Billington, Designs with Repeated elements in blocks: A Survey and Some Recent Results, *Congresses Numerantium* **68** (1989), 123–146.

[4] A E. Brouwer, Steiner triple system without forbidden subconfigurations, *Math. Centrum Amsterdam*, ZW104/77.

[5] M.J. Grannell, T.S. Griggs and E. Mendelsohn, A Small Basis for Four-Line Configurations in Steiner Triple Systems, *Journal of Combinatorial Designs* **3**, No 1 (1995), 51–59.

[6] T.S. Griggs, E. Mendelssohn and A. Rosa, Simultaneous decompositions of Steiner triple systems, *Ars Combinatoria* **37** (1994), 157–173.

[7] T.S. Griggs, J.P. Murphy and J.S. Phelan, Anti-Patch Steiner triple systems, *Journal of Combinatorial Information System Science* **15** (1990), 79–84.

[8] T.S. Griggs, M.J. deResmini and A. Rosa, Decomposing Steiner triple systems into four-line configurations, *Ann. Discrete Math* **52** (1992), 214–226.

[9] P. Horak and A. Rosa, Decomposing Steiner triple systems into small configurations, *Ars Combinatoria* **26** (1988), 91–105.