String Decompositions of Graphs

Atsushi KANEKO
Dept. of Electronic Engineering
Kogakuin University
Tokyo 163-91, Japan

Mamoru WATANABE
Dept. of Computer Science and Mathematics
Kurashiki University of Science and the Arts
Kurashiki-shi 712, Japan

ABSTRACT. For a graph G,if F is a nonempty subset of the edge set $E(G)$, then the subgraph of G whose vertex set is the set of end of edges in F is denoted by $<F>_G$. Let $E(G) = \cup_{i \in I} E_i$ be a partition of $E(G)$, let $D_i = <E_i>_G$ for each i, and let $\phi = (D_i \mid i \in I)$, then ϕ is called a partition of G and E_i (or D_i) is an element of ϕ. Given a partition $\phi = (D_i \mid i \in I)$ of G, ϕ is an admissible partition of G if for any vertex $v \in V_2(G)$ there is an unique element D_i which contains vertex v as an inner point. For two distinct vertices $u$and v, u-v walk of G is a finite, alternating sequence $u = u_0, e_1, u_1, e_2, \ldots, v_{n-1}, e_n, u_n = v$ of vertices and edges, beginning with vertex u and ending with vertex v, such that $e_i = u_{i-1} u_i$ for $i = 1, 2, \ldots, n$. A u-v string is a u-v walk such that no vertex is repeated except possibly u and v, i.e. u and v are allowed to appear at most two times. Given an admissible partition ϕ, ϕ is a string decomposition or *SD* of G if every element of ϕ is a string.

In this paper, we prove that 2-connected graph G has an *SD* if and only if G is not a cycle. We also give a characterization of the graphs with cut vertices such that each graph has an *SD*.

ARS COMBINATORIA 49(1998), pp. 225-236

1 Introduction The graph decomposition problem has been studied by many mathematicians. In [1] many results on this problem concerning various topics are summarized. This problem is also appeared in [2, Chapter II.5]. In this paper we consider a new kind of graph decomposition problem and give a complete solution of it. We will propose a concept of string decomposition of a graph. When we regard a graph as a figure be constructed by a lot of wires which form pieces of it, how we cut a wire to slightly many pieces (will be called strings) which are bent possibly at most two as a technical simplicity and to be reconstructed it under some suitable conditions. We are expect string decomposition to have a wide range of technological applications.

All graphs considered in this paper are only finite, undirected, simple graphs without loops. Let G be a graph and let u and v be (not necessarily distinct) vertices of G.

For two distinct vertices $u$and v, ***u-v walk*** of G is a finite, alternating sequence $u = u_0, e_1, u_1, e_2, \ldots, v_{n-1}, e_n, u_n = v$ of vertices and edges, beginning with vertex u and ending with vertex v, such that $e_i = u_{i-1} u_i$ for $i = 1, 2, \ldots, n$. A ***u-v string*** is a u-v walk such that no vertex is repeated except possibly u and v, i.e. u and v are allowed to appear at most two times. For two distinct vertices u and v, ***u-v path*** is u-v string such that no vertex is repeated, and is denoted by $\boldsymbol{P}$ $(\boldsymbol{u}, \boldsymbol{v})$. A u-v walk such that no vertex is repeated, $u = v$ and $n \geq 3$, is called a ***cycle***.

For every u-v string H, the maximum degree in H is less than or equal to three. For every u-v string S, the set E (S) of the edges of S can be decompose to at most two cycles and at most one path (see Fig 1). We note that a cycle does not any string.

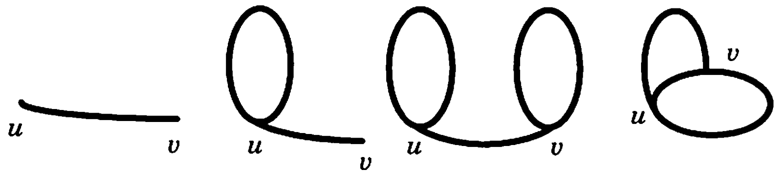

Figure 1.

A graph with order $n \geq 3$ is said to be ***2-connected*** if the minimum number of vertices whose removal results in a disconnected is two or in the trivial graph. For a graph G, the vertex set of G is denoted by $V(G)$, while the edge set is denoted by $E(G)$. If U is a nonempty subset of $V(G)$, then the subgraph of G whose vertex set is U and whose edge set is the set of those edges of G that have both ends in U is called an ***induced subgraph*** of G induced by U and is denoted by $<U>_G$. If F is a nonempty subset of $E(G)$, then the subgraph of G whose vertex set is the set of end of edges in F and is denoted by $<F>_G$. Let $E(G) = \cup_{i \in I} E_i$ be a partition of $E(G)$, let $D_i = <E_i>_G$ for each i, and let $\phi = (D_i \mid i \in I)$, then ϕ is called a ***partition*** of G and E_i(or D_i) is an ***element of*** ϕ.
For an integer i, we denote $V_i(D) = \{ v \in V(G) \mid deg_G(v) \geq i \}$.
If D is a cycle or a string and $v \in V_2(G)$, then v is called an ***inner point*** of D.
A complete graph of n vertices is denoted by K_n. A ***subdivision*** of a graph G is a graph which can be obtained from G by a sequence of edge subdivision.
We denote by M_3 a graph that can be obtained from K_3 by replacing each edge (a,b) of K_3 by two parallel edges(see Fig. 2).

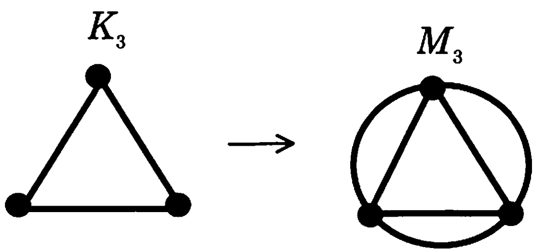

Figure 2.

Given a partition $\phi = (D_i \mid i \in I)$ of G, ϕ is an ***admissible*** partition of G if for any vertex $v \in V_2(G)$ there is an unique element D_i which contains vertex v as an inner point. Given an admissible partition ϕ, ϕ is a ***path decomposition***, or ***PD***, of G if every element of ϕ is path of order at least two. Given an admissible partition ϕ, ϕ is a ***string decomposition*** or ***SD*** of G if every element of ϕ is a string. Moreover a partition ϕ is a ***cycle decomposition*** or ***CD***, of G if every element of ϕ is a path or a cycle. Therefore, if ϕ is a PD, then ϕ is a CD.
For a string D, the number of cycles contained in D is denoted by $\boldsymbol{m}(\boldsymbol{D})$.

An ***elementary partition*** of a string D is a partition of D in which D is partitioned into paths and cycles as follows:

Case 1. $m(D) = 1$. The edge set $E(D)$ is partitioned into a cycles and a path(see Fig. 3(a)).

Case 2. $m(D) = 2$. If D is 2-connected, $E(D)$ is partitioned into a cycle a path (see Fig. 3(c)). If D has cut vertices, $E(D)$ is partitioned into two cycles and a path (see Fig. 3(b)).

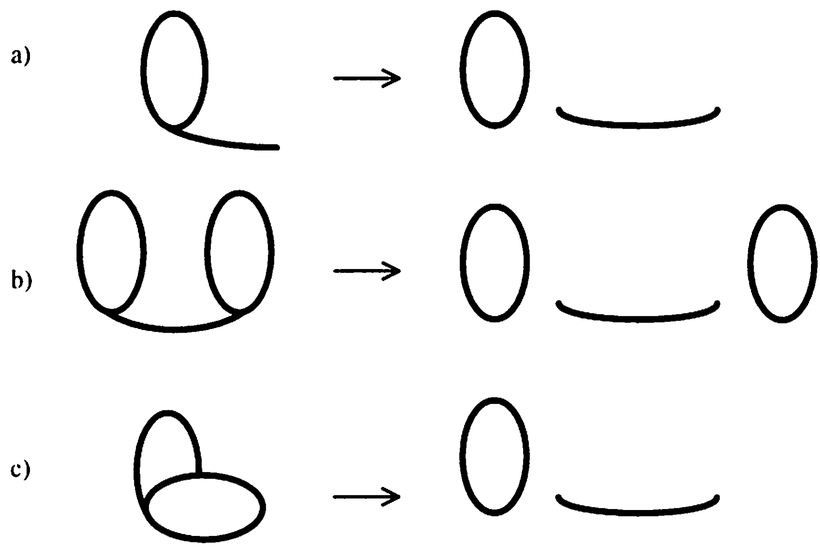

Figure 3.

Given a ***SD*** ϕ of G, a ***subdivision*** of ϕ is a partition ϕ' obtained from ϕ by making an elementary partition for each element of ϕ. By the definition of a string, ϕ'is obviously a *CD* of G, since for any $u \notin V(G)$ the element which contains vertex u as an inner point is uniquely determined.

Let A and B be subgraphs of G such that $V(A) \cap V(B) = \varnothing$. An ***A-B path P*** is a path connecting between a vertex of B and a vertex of B which has no points in common with $V(A) \cup V(B)$ except its end points.

Let H be a subgraph of G or a subset of $V(G)$. An path P is ***H-bridge*** if P is a path connecting between distinct vertices w_1 and w_2 of H such that $E(H) \cap E(P) = \varnothing$ and $V(H) \cap V(P) = \{w_1, w_2\}$.

A collection $\{P_1, P_2, \ldots, P_n\}$ of u-v paths is a ***internally disjoint*** if $V(P_i) \cap V(P_j) = \{u, v\}$ for all i and j, $1 \le i, j \le n$, $i \ne j$. In generally, a collection $\{P_1, P_2, \ldots, P_n\}$ of H-bridges is ***internally disjoint*** if $V(P_i) \cap V(P_j) \subseteq V(H)$ (or H) for all i and j, $1 \le i, j \le n$, $i \ne j$.

Let G be a 2-connected graph and suppose $S \subseteq V(G)$. a subgraph F of G is ***good*** with respect to S if either $E(F) = \emptyset$ and $S = V(F)$, or $E(F) \ne \emptyset$ and there is a partition ϕ of F satisfying three conditions:

(i) Each element of G is a path;

(ii) For any vertex v of S, there is no elements which contains v as an inner point;

(iii) For any $v \in V(F) - S$, there is uniquely element of ϕ which contains v as an inner point.

If F is a good subgraph of G, ϕ is called a ***good decomposition*** of F with respect to S.If we regard any subset S as a subgraph of G with no edge, that is $<S>_{G-E(H)}$, where $H = <S>_G$, by the definition, S is a good subgraph with respect to itself.

2 Preliminary Lemmas

Lemma 1. *Let G be a 2-connected and let H be a subgraph with $|V(H)| \ge 2$. Then $G - E(H)$ is a good subgraph with respect to $V(H)$.*

Proof: From Remark 3, by regarding H as a good subgraph with respect to $V(H)$, we may consider a maximal subgraph F such that $V(H) \subseteq V(F)$, where the maximality is considered with the number of edges. If $(G - E(H)) - E(F) \ne \emptyset$, then there exists a F*-bridge* P, since G is 2-connected and $|V(F)| \ge |V(H)| \ge 2$. Moreover, it is obvious that $F \cup P$ is a good subgraph of G with respect to $V(H)$, which contradicts the maximality of $|E(F)|$.

Let G be a 2-connected graph and let $u \in V(G)$. A partition $\phi = \{E_1, E_2, \ldots, E_k\}$ of G is a ***pseudocycle decomposition*** (***PCD***) of G with respect to v if ϕ satisfies the following conditions:

(i) For each i, $1 \le i \le k$, $D_i = <E_i>_G$ is a path or a cycle;

(ii) For any $v \in V(G) - \{u\}$, there exists unique i, $1 \le i \le k$ such that $v \in V_2(D_i)$.

In the above definition, if each D_i is a path, ϕ is called in particular a ***pseudopath decomposition*** (***PPD***) of G with respect to u.

Lemma 2. *Let G be a 2-connected graph and let $u \in V(G)$. Let $G - u$ be a tree such that G has a PCD $\phi = \{E_1, E_2, ..., E_k\}$ of G with respect to u, where $D_i = \langle E_i \rangle_G$. Then there exists an integer i, $1 \le i \le k$ such that D_i is a cycle and any other D_j ($j \ne i$) is a path(therefore, ϕ is a PCD and not PPD).*

Proof: If G has a cycle C, then $u \in V(C)$ since $G - u$ is a tree. Then we prove by induction on $|V(G)|$ that G has a cycle. If $|V(G)| = 3$, then $G = K_3$. Therefore G has an unique PCD consisting of only cycle *i.e.* G itself. Suppose that any graph H with $|V(G)| < |V(G)|$ satisfying the conditions in Lemma 2 holds Lemma 2.

Assume $|V(G)| \ge 4$ and put $T = G - u$. Then T is a tree. Let x is be a endpoint of T and let $e_1 = xy \in E(T)$. Since G is 2-connected, $e_u = xu \in E(G)$. Assume, to the contrary, that there exists a PPD $\phi = \{E_1, E_2, ..., E_k\}$ of G with respect to u, that is, each $P_i := \langle E_i \rangle_G$ is a path.

Let x be a interior vertex of a path P_i. Let G' is the subgraph obtained from G by contracting edge xu to one vertex u'. Let x be a interior vertex of a path P_1. Let G' is the subgraph obtained from G by contracting edge xu to one vertex u'. Trivially, G' is 2-connected and $G' - u'$ is a tree. We consider two cases.

If $yu \notin E(G)$, then $\{E_1 - e_1, E_2, ..., E_k\}$ is a PPD of G', which reads $\phi = \{E_1, E_2, ..., E_k\}$ is a PPD of G, a contradiction. Assume that $e_2 := yu \in E(G)$ and $e_2 \notin E(P_2)$. By the definition there exists at most one path which contains y as interior vertex, so either P_1 or P_2 contains y as an endpoint. Then either $\{E_1, E_2 - e_2, ..., E_k\}$ is a PPD of G', which is a contradiction. Therefore, there exists at least one cycle $D_i := \langle E_i \rangle_G$, $1 \le i \le k$. Let C be a any cycle of G. Trivially $|V(C)| \ge 3$. Since G is 2-connected, by Lemma 1, $G - E(C)$ has a good decomposition ϕ with respect to $V(C)$. The set $\phi \cup \{E(C)\}$ is trivially a CD.

Lemma 3. *If a 2-connected graph G contains one of the following subgraphs (1) ~ (3) as a subgraph, G has a PD.*

(*1*) *a subdivision of K_4.*

(2) *a subdivision of* M_3.

(3) *two cycles* C_1 *and* C_2 *such that* $V(C_1) \cap V(C_2) = \varnothing$.

Proof. Let G be a 2-connected graph and let H be a subgraph of G of order at least two. If G has a PD, then by Lemma 1, then $G - E(H)$ has a good decomposition $\phi \cup \varphi$ is a PD of G, where φ is a PD of H. Thus to prove Lemma 3, it is enough to show that if G contains one of the subgraphs (1)~(3), G contains a subgraph H having a PD.

we consider two cases.

Case 1. G contains a subdivision H of K_4.

Let w_1, w_2, w_3 and w_4 be four vertices of G. Let $H := \cup_{1 \le i < j \le 4} P(w_i, w_j)$, where $\{P(w_i, w_j)\}$ is the set of disjoint $\{w_1, w_2, w_3, w_4\}$ – { bridges }.

Put $P_1 := P(w_1, w_2) \cup P(w_2, w_3) \cup P(w_3, w_4)$ and $P_2 := P(w_2, w_4) \cup P(w_1, w_4) \cup P(w_1, w_3)$. Then H has a PD containing P_1 and P_2.

Case 2. G contains a subdivision H of M_3.

Let w_1, w_2, and w_3 be tree vertices of G. Put $H := P_1(w_1, w_2) \cup P_2(w_1, w_2) \cup P_1(w_2, w_3) \cup P_2(w_2, w_3) \cup P_1(w_1, w_3) \cup P_2(w_1, w_3)$, where $\{P_i(w_j, w_k)\}$ is the set of disjoint { w_1, w_2, w_3 }-bridges. Put $R_1 := P_1(w_1, w_2) \cup P_1(w_2, w_3)$, $R_2 := P_2(w_2, w_3) \cup P_1(w_1, w_3)$and $R_3 := P_2(w_1, w_2) \cup P_2(w_1, w_3)$. Then H has a PD containing R_1, R_2 and R_3.

Case 3. G has cycles C_1 and C_2 such that $|V(C_1) \cap V(C_2)| \le 1$.

Since G is 2-connected, there exist two paths connecting C_1 and C_2, $P_1 := P(u_1, u_2)$ and $P_2 := P(v_1, v_2)$ such that $u_1 \ne v_1$, $\{u_1, v_2\} \subseteq V(C_1)$, $u_2 \ne v_2$, $\{u_2, v_2\} \subseteq V(C_2)$, and $V(P_1) \cap V(P_2) = \varnothing$.

Let Q_1 and R_1 be the disjoint $u_1 - v_1$ paths of C_1 and let Q_2 and R_2 be the disjoint $u_2 - v_2$ paths of C_2. Putting $H = C_1 \cup C_1 \cup P_1 \cup P_2$, $T_1 = Q_1 \cup P_1 \cup Q_2$, and $T_2 = R_1 \cup P_2 \cup R_2$, H has a PD containing T_1 and T_2. Thus we finish the discussion of all cases and the proof of Lemma 3.

Lemma 4. *If a 2-connected graph G contains none of subgraph in (1)~(3) in Lemma 3, then there exists a vertex u such that* $G - u$ *is a tree.*

Proof. Since G is 2-connected G has a cycle C. Let B_C be the set of paths (possibly not internally disjoint) which are C-bridges. We consider two cases :

Case 1. $B_C = \varnothing$. In this case clearly $G = C$, so G contains none of the subgraphs in (1)~ (3) in Lemma 3. Then there exists a vertex u of G such that $G - u$ is a tree.

Case 2. $B_C \neq \varnothing$. Put $W_C := \cup_{P(u,v)\in B} \{u,v\}$, where $B = B_C$. Note that $|W_C|$ =2 .

Subcase 2.1. $|W_C|$ =2 for every cycle C. Since each component of $G - W_C$ is a path, G has none of the subgraphs in (1)~ (3) in Lemma 3, so for every $u \in W_C$ $G - u$ is a tree.

Subcase 2.2. $|W_C| \geq 3$ for some cycle C. Suppose $\{ P_1 := P(u,v), P_2 := P(x,w)\} \subseteq B_C$. Then $|\{u,v,w,x\}| \geq 3$. If P_1 and P_2 are not internally disjoint C-bridge, then $H := C \cup P_1 \cup P_2$ contains a subdivision of K_4, so we may suppose that P_1 and P_2 are internally C-bridge. If $|\{u,v,w,x\}| \geq 4$, then H contains two cycles which are disjoint. Then we suppose that $|\{u,v,w,x\}| = 3$ and $u = x$.
Let Q_1 be a u-v path which does not contain w.
Similarly, Let Q_2 be a v-w path which does not contain u and let Q_3 be a u-w path which does not contain v. If $G - u$ has a cycle C', then $V(C') \cap V(P_1 \cup Q_1 - u) \neq \varnothing$ and $V(C') \cap V(P_2 \cup Q_2 - u) \neq \varnothing$,since G has not two disjoint cycles. Putting $H' = C \cup P_1 \cup P_2 - u$, G contains at least one H'-bridge R. However, if H' has an endvertex of R, then $H \cup R$ contains some of subgraphs in (1)~ (3), which is a contradiction. Then $G - u$ has no cycle.

3 Main Results

Theorem 5. *Let G be a 2-connected graph. G has a PDD if and only if there exists a vertex v such that $G - v$ has no cycle.*

Proof. Necessity: Suppose that there exists a vertex u such that $G - v$ has no cycles. If G has a *PD*, then this partition is also a *PDD* with respect to u, which contradicts Lemma 2.
Sufficiency: Suppose G has no *PD*. Then, by Lemma 3 and Lemma 4, there exists a vertex u such that $G - v$ is a tree.

Corollary. *Every 3-connected graph has a PD.*

Theorem 6. *Let G be a 2-connected graph. G has a SD if and only if G is not a cycle.*

Proof. Necessity is trivial by the definition of *SD*. Suppose that G has no *SD*. Since a *PD* is an *SD*, by Theorem 5, there exists a vertex v such that $G - v$ is a tree. By Lemma 2 G has a *CD* $:D_1, D_2, \ldots, D_k$ such that exactly one of these, say D_1, is a cycle. If G is not a cycle, in this partition there exists a path such that either of endvertices of it is a vertex of D_1. Then $D_1 \cup D_2, D_3, \ldots, D_k$ is an *SD* of G.

A ***block*** is a graph G is a maximal subgraph which has no cut vertex. Then a block of a graph is the subgraph induced by a cut edge or a maximal 2-connected subgraph of G.
Let $C(G)$ be the set of cut vertices of G. A block B of G is an ***end-block*** if $|V(G) \cap C(G)| \leq 1$. Let $\boldsymbol{X(G)} := \{v \in C(G) ; \exists B \in End(G), v \in V(B)\}$, where $\boldsymbol{End(G)}$ is the set of end-blocks of G. Let T_G be a connected subgraph that contains $X(G)$ and is a maximal with respect to this property. T_G is a tree whose end vertices are in $X(G)$. Let $A_G = \cup_{B \in NE(G)} B$.

Lemma 6. $C(G) \subseteq V(T_G)$.

Proof. To the contrary, if there exists a vertex $v \in C(G) - V(T_G)$, then there exists a component D of $G - v$ such that $V(D) \cap V(T_G) = \varnothing$, since T_G is connected and $v \notin V(T_G)$. Since $X(G) \subseteq V(T_G)$, $v \notin V(X)$ and $X(G) \cap V(D) = \varnothing$. Then there exists an end-block
$B \in End(G)$ such that $V(B) \subseteq V(D)$. From the definition of $X(G)$, $V(B) \cap X(G) = \varnothing$, which contradicts $V(D) \cap V(X) = \varnothing$.

Lemma 7. *Let B a block that is not an end-block of G. Then* $|V(T_G) \cap V(B)| \geq 2$.

Proof. Since B is not an end-block, $|C(G) \cap V(B)| \geq 2$. By Lemma 6, $C(G) \subseteq V(T_G)$. This completes the proof of Lemma 7.

Lemma 8. *If G is a tree, G has a PD.*

Proof. The proof is given easily by induction on $|V(G)|$, and is omitted.

Lemma 9. *If a graph G has a cut vertex, then A_G has a good decomposition with respect to X (G).*

Proof. Let $B \in NE(G)$. If B is a cut edge of G, by Lemma 7, then $B = e \in E(T_G)$ (*1).

Put $NE'(G)$, by Lemma 7, then $|V(T_G) \cap V(B)| \geq 2$. Putting $H := <V(T_G) \cap V(B)>_B$, by Lemma 1, $B - E(H)$ is a good sub graph with respect to $V(H)$. Putting $NE'(G) = \{B_1, B_2, ..., B_m\}$, for any i, B_i is a good decomposition ϕ_i with respect to $V(B_i) \cap V(T_G)$. On the other hand, by Lemma 8, a tree T_G has a PD ϕ_{m+1}. By $B = e \in E(T_G)$, we have $A(G) = T_G \cup (\cup_i B_i)$, where i runs from 1 to m. Therefore $\{\phi_1, \phi_2, ..., \phi_{m+1}\}$ is a good decomposition of A_G with respect to X(G).

Lemma 10. *Let G be a 2-connected graph and $u \in V(G)$. If $G - u$ has a cycle, then G has a good decomposition with respect to u.*

Proof. Let C be a cycle of $G - u$. Since G is 2-connected, there exists a C-bridge $P(x,y)$ that contains u. Denote two internally disjoint paths connecting x and y by $P_1(x,y)$ and $P_2(x,y)$. Consider also $P_3(x,u)$ and $P_4(y,u)$, which are subpaths of $P(x,y)$. Putting $H := P(x,y) \cup C$, both $E(P_2(x,y)) \cup E(P_4(y,u))$ and $E(P_1(x,y)) \cup E(P_3(x,u))$ are a *PDD* of H with respect to u. Since $|V(H)| \geq 2$ and G is 2-connected, by Lemma 1, $G - E(H)$ is a good subgraph with respect to $V(H)$. By (1) and (2) of Lemma3, G has a good decomposition with respect to u.

Let $B \in End(G)$ and $V(B) \cap C(G) = \{v\}$. An end-block B of G withcut vertex v is ***good*** if $B - v$ is either one point set or $B - v$ has a cycle, and otherwise B is **bad**, B is said if $B - v$ is a tree with order at least two.

Theorem 11. *Let G be a graph with cut vertices. Then G has a PD if and only if G has no bad end- block.*

Proof. Necessity: Suppose that G has a bad end-block and has a PD ϕ. For $u \in V(B) \cap C(G)$, the restriction $\phi|_B$ of ϕ to B is a PDD of B with respect to u, which contradicts Lemma 2. Sufficiency: Suppose every end-block B is good. Let $u \in V(B) \cap X(G)$.
If B is 2-connected, by Lemma 10, B has a good decomposition with respect to u. From now on, when B is a edge, for convenience sake we regard B itself as an element of a good decomposition with respect to u.

Case 1. $|X(G)| \geq 2$. By Lemma 9, E_{RG} has a good decomposition ϕ_0 with respect to $X(G)$. For each element u of $X(G)$, we correspond to an element, say B_u, of $End(G)$ containing u(*[1]). Let ϕ_1 be a good decomposition of B_u with respect to u by the same argument in the proof of Lemma10. Choose any element of ϕ_0 containing u and denote it by $P_0(u)$. Also chose any element of ϕ_1 containing u and denote by $P_1(u)$(*[3]). Then $\phi_0(u) \cup \phi_1(u)$ is a path of G containing u as an interior vertex. This operation applies to all elements of $X(G)$. If for each $B \in End(G)$ add ϕ_0 to ϕ_B, which is a good decomposition with respect to $V(B) \cap C(G)$, then we have a PD of G.

Case 2. $|X(G)| \geq 1$. Put $X(G) = \{u\}$. Every block of G is end-block containing u and there exists at least two such blocks. Let B_0 and B_1 be any two end-blocks of G, and for each i, $1 \leq i \leq 1$, let ϕ_i be a good decomposition of B_i *with* respect to u. On the discussion in the case 1, considering a decomposition ϕ_0 of B_i instead of B_u and ϕ_0 of B_0 instead of A_G, we can show the existence a PD of G by a similar discussion as in the case 1.

Theorem 12. *Let G be a graph having cut vertices. Then G has SD if and only if no two bad end-blocks have a common vertex.*

Proof. Necessity: Suppose B_1 and B_2 be bad end-blocks having u as a common vertex and both have an SD, ϕ. Let ϕ' be a subdivision of ϕ. Then ϕ' is a CD of G. Considering restrictions of ϕ' to B_1 and B_2, B_1 and B_2 give PCD's ϕ_1 and ϕ_2 with respect to u respectively.
By Lemma 2, there exists a cycle $C_1 \in \phi_1$ and a cycle $C_2 \in \phi_2$ such that $E(C_1) \subseteq E(B_1)$, $E(C_2) \subseteq E(B_2)$. Then $C_1 \in \phi'$ and $C_2 \in \phi'$, so it means that there exists at least two bad end-blocks of ϕ' having u as an interior vertex, which contradicts that ϕ' is a CD of G.

Hence if G has an SD, then G has no bad end-blocks having exactly one common vertex.

Sufficiency: For each $u \in X(G)$ suppose that there exists at least one bad-block containing u. The proof is similar to the proof of sufficiency of Theorem 11. However, in ($*^1$) in the proof of Theorem11, for each $u \in X(G)$ when we correspond to an element of $End(G)$ containing u, if there exists a bad end-block B containing u, we choose B as B_u. Then we choose a CD in Lemma 2 as decomposition ϕ_1 of B_u in ($*^2$). Since ϕ_1 has exactly one cycle C we may consider C as a element of ϕ_1 containing u in ($*^3$). Then $C \cup P_0(u)$ is a string of G and contains u as an interior vertex. Under the above condition the existence of an SD of G can be proved by the similar discussion in the proof of the sufficiency of Theorem 11.

References

[1] J. Bosák, "Decompositions of Graphs", Kluwer Academic Publishers, Dordrecht/Boston/London, 1990.

[2] B. Bollobás, "Extremal Graph Theory", Academic Press, London/New York/San Francisco, 1978.