Graph Decompositions into Generalized Cubes

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Abstract

For a positive integer d, the usual d-dimensional cube Q_d is defined to be the graph $(K_2)^d$, the Cartesian product of d copies of K_2. We define the generalized cube $Q_{d,k}$ to be the graph $(K_k)^d$ for positive integers d and k. We investigate the decompositions of the complete graph K_{k^d} and the complete k-partite graph $K_{k\times k^{d-1}}$ into generalized cubes when k is the power of a prime and d is any positive integer, and some generalizations. We also use these results to show that Q_5 divides K_{96}.

1 Generalized Cubes

By a *decomposition* of a graph G, we mean a collection of subgraphs $G_1, \ldots, G_n$ whose edge sets partition the edge set of G. In this case we may write $G = G_1 + \ldots + G_n$. If each subgraph G_i is isomorphic to a fixed graph G_0 we say that G_0 *divides* G, and write $G = nG_0$.

Let d and k be positive integers. We define the *generalized cube* $Q_{d,k}$ (hereafter called a *cube*) to be the graph $(K_k)^d$, the Cartesian product of d copies of K_k. Alternately, if F if a set with k elements, then $Q_{d,k}$ is the graph with vertex set $V = F^d$ and edge set E the set of all $\{x, y\}$ with x and y in V such that x and y differ in exactly one coordinate. If $k = 2$ we get the usual d-dimensional cube, denoted Q_d. It is easily seen that $|V| = k^d$ and $|E| = d(k-1)k^d/2$.

If F is an additive group we define w and W on V by $w(x_1, x_2, \ldots, x_d) = |\{i : x_i \neq 0\}|$ and $W(x_1, x_2, \ldots, x_d) = x_1 + x_2 + \ldots + x_d$, and call $w(x)$ and $W(x)$ the *weight* and *parity* of x, respectively. Then $\{x, y\}$ is an edge of $Q_{d,k}$ iff $w(x - y) = 1$, which implies $W(x) \neq W(y)$. Thus $Q_{d,k}$ is k-partite, with V partitioned into sets

$$V_\alpha = \{x \in V : W(x) = \alpha\}, \alpha \in F, \tag{1}$$

each with k^{d-1} elements. We note that this partition may depend on the group F. For example if $F = Z_4$, then $Q_{2,4}$ contains a cycle of length 8 with vertices in $V_0 \bigcup V_1$, but if $F = Z_2^2$, then in $Q_{2,4}$ no cycle of length 8 has vertices with only two parities.

Let $K_{m \times n}$ denote the complete m-partite graph with exactly n vertices in each part and let $K_{m_1,\ldots,m_t}$ denote the complete t-partite graph whose i^{th} color set has m_i elements. Let ${}^mG$ denote the multigraph formed by replacing each edge of a graph G with m parallel edges.

In this study, we examine the decompositions of complete multigraphs and of complete multipartite multigraphs into cubes. We note here that since K_k divides $(K_k)^d$ (i.e. $Q_{d,k}$) for all positive integers k and d, these decompositions are somewhat related to both affine geometries (in the complete graph case) and to group divisible designs (in the complete multipartite graph case). (See Hall [4].)

We remind the reader that an *affine plane of order k* is a decomposition of K_{k^2} into K_k's. Thus, an affine plane of order k can be obtained from a decomposition of K_{k^2} into $Q_{2,k}$'s.

Despite the relation of our decompositions to affine geometries and to group divisible designs, we are not aware of any similar results or method in the literature.

2 Cube Decomposition of Complete Graphs

We will prove theorems about decompositions of both $K_{k \times k^{d-1}}$ and K_{k^d} into cubes when k is a prime power, but need two lemmas first.

If F is a ring with unit 1 we let e_i denote the element of V with ith coordinate 1 and all other coordinates 0.

Lemma 1 *Let F be a field with k elements, and regard V as a vector space over F. Suppose B is a linearly independent subset of V with m elements. Define $C = \{\alpha b : \alpha \in F\backslash\{0\}, b \in B\}$, and let $G(B)$ be the graph with edge set $E(B) = \{\{x, x+c\} : x \in V, c \in C\}$. Then $G(B)$ is isomorphic to k^{d-m} vertex-disjoint copies of $Q_{m,k}$.*

Proof: Let $B = \{b_1, b_2, \ldots, b_m\}$. Extend B to a basis $B' = \{b_1, b_2, \ldots, b_d\}$ for V. Define $f : V \to V$ by $f(x_1, x_2, \ldots, x_d) = \Sigma x_i b_i$. Since B' is a basis, this is a permutation of V and so when extended to the edges of K_{k^d} gives a graph isomorphism. Now the edges $\{\{x, x+\alpha e_i\} : x \in V, \alpha \in F\backslash\{0\}, 1 \le i \le m\}$ are isomorphic to k^{d-m} isomorphic copies of $Q_{m,k}$, one for each choice of the last $d-m$ coordinates of x. Note that f takes such an edge into $\{\Sigma x_i b_i, \Sigma x_i b_i + \alpha b_i\} \in E(B)$. Since $|C| = (k-1)m$ and $|E(B)| = k^d(k-1)m/2$, which is k^{d-m} times the number of edges of $Q_{m,k}$, the theorem statement follows. □

Lemma 2 *Let d and k be integers greater than 1. If m is an integer such that $0 < m < d$, then*

$$\frac{k^{m-1}}{m} \leq \lfloor \frac{k^{d-1}}{d} \rfloor,$$

and

$$\frac{k^m - 1}{(k-1)m} \leq \lfloor \frac{k^d - 1}{(k-1)d} \rfloor.$$

Proof: To prove the first inequality it suffices to show that $k^{m-1}/m \leq \lfloor k^m/(m+1) \rfloor$ for each integer $m > 0$. This is straightforward for $m \leq 3$, so assume $m \geq 4$. Note that $mk/(m+1) \geq 1.5$ and $k^{m-1}/m \geq 2$. Then

$$\frac{k^{m-1}}{m} \leq (\frac{mk}{m+1})\frac{k^{m-1}}{m} - 1 = \frac{k^m}{m+1} - 1 \leq \lfloor \frac{k^m}{m+1} \rfloor.$$

Likewise to prove the second inequality it suffices to show that $(k^m - 1)/(k-1)m \leq \lfloor (k^{m+1}-1)/(k-1)(m+1) \rfloor$ for $m > 0$. This is straightforward for $m \leq 3$, so assume $m \geq 4$. Note that $(k^{m+1} - 1)m/(k^m - 1)(m+1) \geq k^{m+1}m/k^m(m+1) \geq 1.5$ and $(k^m - 1)/(k-1)m \geq k^m/km \geq 2$. Then

$$\frac{k^m - 1}{(k-1)m} \leq \frac{(k^{m+1} - 1)m}{(k^m - 1)(m+1)} \frac{k^m - 1}{(k-1)m} - 1 =$$

$$\frac{k^{m+1} - 1}{(k-1)(m+1)} - 1 \leq \lfloor \frac{k^{m+1} - 1}{(k-1)(m+1)} \rfloor.$$

Thus, the proof is complete. □

The following theorem is a consequence of the Edmonds matroid partition theorem [2]; its first appearance is [6].

Theorem 1 *Let n be a positive integer, and let S be a subset of a vector space such that for any finite subset T of S we have $|T| \leq n \cdot \operatorname{rank} T$. Then S can be partitioned into n disjoint linearly independent subsets.*

In the theorems that follow we make the convention that $Q_{r,k}$ is empty for $r = 0$. The $k = 2$ case of the following theorem appears in [3].

Theorem 2 *Suppose that k is a power of a prime. Write $k^{d-1} = qd + r, 0 \leq r < d$. Then $K_{k \times k^{d-1}}$ can be decomposed into q copies of $Q_{d,k}$ and k^{d-r} vertex-disjoint copies of $Q_{r,k}$.*

Proof: We take the vertices of $K_{k \times k^{d-1}}$ to be $V = F^d$, where F is a field with k elements, and V is partitioned into the sets V_α of (1), so that the edges of $K_{k \times k^{d-1}}$ are all $\{x, y\}$ with $W(x) \neq W(y)$. Let $R =$

$\{e_1, e_2, \ldots, e_r\}$. We claim that $V_1 \backslash R$ can be partitioned into q bases for V, where V is regarded as a vector space over F. By Theorem 1 it suffices to show that if T is any subset of $V_1 \backslash R$, then

$$|T| \leq q \cdot \operatorname{rank} T \tag{2}$$

Let rank $T = m$. Then (2) is clear if $m = 0$, and, since $|V_1 \backslash R| = qd$, if $m = d$. Note that if Y is any m-dimensional subspace of V, then $|Y \bigcap V_1| \leq k^{m-1}$. Thus it suffices to show that $k^{m-1} \leq \lfloor k^{d-1}/d \rfloor m$ for $0 < m < d$, and this is the first inequality of Lemma 2.

Now we invoke Lemma 1. By it each basis $B \subseteq V_1$ for V generates a subgraph $G(B)$ of $K_{k \times k^{d-1}}$isomorphic to $Q_{d,k}$, while if $r > 0$, then $G(R)$ is isomorphic to k^{d-r} edge-disjoint copies of $Q_{r,k}$. We must show that all these subgraphs of $K_{k \times k^{d-1}}$ are edge-disjoint.

Suppose that B_1 and B_2 are disjoint linearly independent subsets of V_1, but $E(B_1)$ and $E(B_2)$ are not disjoint. Say that $\{x, x + \alpha b_1\} = \{y, y + \beta b_2\}$, with $\alpha, \beta \in F \backslash \{0\}, b_1 \in B_1, b_2 \in B_2$. If $x = y$, then $\alpha b_1 = \beta b_2$. But $\alpha = W(\alpha b_1) = W(\beta b_2) = \beta$, so $b_1 = b_2$, which is a contradiction. Otherwise $x = y + \beta b_2$ and $y = x + \alpha b_1$. Then $\alpha b_1 = -\beta b_2$, leading to the same contradiction.

Since the number of edges of $K_{k \times k^{d-1}}$ is

$$(k-1)k^{2d-1}/2 = q \cdot d(k-1)k^d/2 + k^{d-r} \cdot r(k-1)k^r/2,$$

the proof is complete. □

Theorem 3 *Suppose that k is a power of a prime. Write $(k^d - 1)/(k-1) = qd + r, 0 \leq r < d$. Then K_{k^d} can be decomposed into q copies of $Q_{d,k}$ and k^{d-r} vertex-disjoint copies of $Q_{r,k}$.*

Proof: As in the last proof we take F to be a field with k elements and consider V to be a vector space over F. Let W be the set of all elements of $V_0 \backslash \{0\}$ which have first nonzero coordinate 1, and let $R = \{e_1, e_2, \ldots, e_r\}$. Note that $|W| = (|V_0| - 1)/(k-1) = (k^{d-1} - 1)/(k-1)$. We will show that $Z = (V_1 \bigcup W) \backslash R$ can be partitioned into q bases for V.

By Theorem 1, it suffices to show that if $T \subseteq Z$, then $|T| \leq q \cdot$rank T. Let $m =$ rank T. The desired inequality holds for $m = 0$ and $m = d$, the latter case following from $|Z| = qd$. Now if Y is a subspace of V of rank m, then $Y \bigcap V_1 \leq k^{m-1}$ and $Y \bigcap W \leq (k^{m-1} - 1)/(k-1)$, since if $x \in Y \bigcap W$ and $\alpha \neq 1$, then $\alpha x \notin Y \bigcap W$. Thus $|T| \leq k^{m-1} + (k^{m-1} - 1)/(k-1) = (k^m - 1)/(m-1)$, and so it suffices to show $(k^m - 1)/(k-1) \leq qm$ for $0 < m < d$. But this is the second inequality of Lemma 2.

Now by Lemma 1 each basis B generates a subgraph $G(B)$ of K_{k^d} isomorphic to $Q_{d,k}$, while if $r > 0$ then $G(R)$ consists of k^{d-r} edge-disjoint copies of $Q_{r,k}$. We claim that these subgraphs are all edge-disjoint. It suffices to show that if B_1 and B_2 are disjoint linearly independent subsets of $V_1 \bigcup W$, then $G(B_1)$ and $G(B_2)$ are edge-disjoint. Say that $\{x, x+\alpha b_1\} = \{y, y+\beta b_2\}$, with $\alpha, \beta \in F\backslash\{0\}, b_1 \in B_1, b_2 \in B_2$. If $x = y$, then $\alpha b_1 = \beta b_2$. Then $\alpha W(b_1) = W(\alpha b_1) = W(\beta b_2) = \beta W(b_2)$, and so b_1 and b_2 are both in V_1 or both in W. In the first case, the proof is as for the previous theorem. In the second case, $\alpha = \beta$ by the definition of the set W. But then $b_1 = b_2$, which is a contradiction. Likewise if $x = y + \beta b_2$ and $y = x + \alpha b_1$, then $\alpha b_1 = -\beta b_2$. Then $\alpha = -\beta$ and $b_1 = b_2$ in the same way.

Since the number of edges of K_{k^d} is

$$k^d(k^d - 1)/2 = q \cdot d(k-1)k^d/2 + k^{d-r} \cdot r(k-1)k^r/2,$$

the proof is complete. □

By counting edges we see that a necessary condition for $Q_{d,k}$ to divide K_{k^d} is that $d(k-1)$ divide $k^d - 1$, and Theorem 3 says that this is sufficient when k is a prime power. However, this condition is not sufficient for general k. For example, $Q_{2,21}$ cannot divide K_{441} since K_{21} divides $Q_{2,21}$ and it is well known that K_{21} does not divide K_{441} (see [4], Theorem 12.3.2.)

Our next result determines exactly when $d(k-1)$ divides $k^d - 1$ when d (but not necessarily k) is a prime power.

Theorem 4 *Let p be prime and $d = p^t$, with $t > 0$ and $k > 1$. Then $d(k-1)|k^d - 1$ if and only if $k \equiv 1 \pmod p$.*

Proof: First assume $d(k-1)|k^d - 1$. Since $p|k^d - 1$, p does not divide k. Then $k^p \equiv k \pmod p$ by Fermat's theorem. Successively raising to the power p gives $k^{p^2} \equiv k^p \pmod p$, $k^{p^3} \equiv k^{p^2} \pmod p$, etc. Thus

$$k \equiv k^p \equiv k^{p^2} \equiv \ldots \equiv k^d \pmod p.$$

But $k^d \equiv 1 \pmod p$ by assumption.

Now suppose $k \equiv 1 \pmod p$. If $K \equiv 1 \pmod{p^S}$ for some $S > 0$, then

$$K^p = (1 + p^S u)^p = 1 + p \cdot p^S u + (p(p-1)/2)(p^S u)^2 + \ldots \equiv 1 \pmod{p^{S+1}}.$$

Applying this result t times gives $K^{p^t} \equiv 1 \pmod{p^{S+t}}$. In particular, if $k - 1 = p^s v$, where p does not divide v, then $k^{p^t} \equiv 1 \pmod{p^{s+t}}$. We see that $p^{s+t}|k^d - 1$. Also $v|k-1$ and $k - 1|k^d - 1$, so $v|k^d - 1$. Thus $vp^{s+t} = (k-1)d|k^d - 1$. □

There are 272 cases for d and k less than 100 when $d = p^t$ and k are both prime powers and $k \equiv 1 \pmod p$, implying that $Q_{d,k}$ divides K_{k^d}.

Theorems 2 and 3 can be extended to regular complete multigraphs.

Theorem 5 *Let d, k, and m be positive integers with k a power of a prime. Write $mk^{d-1} = q_1 d + r_1, 0 \leq r_1 < d$, and $m(k^d - 1)/(k-1) = q_2 d + r_2, 0 \leq r_2 < d$, and denote ${}^m K_{k \times k^{d-1}}$ and ${}^m K_{k^d}$ by G_1 and G_2, respectively. Then G_i can be decomposed into q_i copies of $Q_{d,k}$ and k^{d-r_i} copies of $Q_{r_i,k}$ for $i = 1, 2$.*

Proof: We will only sketch the proof for $i = 1$. Let $k^{d-1} = qd + r, 0 \leq r < d$. We consider ${}^m K_{k \times k^{d-1}}$ as m copies of $K_{k \times k^{d-1}}$ and apply the proof of Theorem 2 to each one, replacing the set R in that proof by sets

$$R_1 = \{e_1, \ldots, e_r\}, R_2 = \{e_{r+1}, \ldots, e_{2r}\}, \ldots, R_m = \{e_{(m-1)r+1}, \ldots, e_{mr}\},$$

where the subscripts of the e_i are taken modulo d. This gives mq copies of $Q_{d,k}$.

Now write $mr = dq_3 + r_3, 0 \leq r_3 < d$. Then the sequence $e_1, e_2, \ldots, e_{mr}$ can be split into q_3 bases $\{e_1, \ldots, e_d\}$ and a set R with r_3 elements, generating q_3 additional copies of $Q_{d,k}$ and k^{d-r_3} copies of $Q_{r_3,k}$. But $mk^{d-1} = (mq + q_3)d + r_3$, so by the uniqueness of quotient and remainder in the division algorithm $q_1 = mq + q_3$ and $r_1 = r_3$. □

3 An Application of Hamming Codes

In the remainder of this paper we will consider only ordinary cubes Q_d, so that $k = 2$ and F is the field Z_2.

If $d = 2^t, t > 0$, then Theorem 2 says that $K_{2^{d-1},2^{d-1}}$ can be decomposed into $2^{d-1}/d = 2^{d-t-1}$ copies of Q_d. By means of Hamming codes we will exhibit decompositions of more general complete bipartite graphs into copies of Q_d.

We remind the reader of some basic Hamming code facts; for details see [5]. Given an integer $t > 1$ set $n = 2^t - 1$ and $m = n - t$. The corresponding Hamming code is a subgroup C of Z_2^n with 2^m elements such that every nonzero element of C has weight > 2. This implies that if x and y are distinct elements of C, then $w(x - y) > 2$.

Now set $d = n + 1$, and let H be the subset of Z_2^d formed by appending a d^{th} coordinate (0 or 1) at the end of each element of C so as to make its weight even. Then H is a subgroup of V_0 (see (1)), and $w(x - y) > 2$ for distinct elements x and y of H. In fact this holds if x and y are distinct elements of any fixed coset of H in V_0. Note that there are $2^n/2^m = d$ such cosets. If, as usual, we consider the vertices of Q_d to be Z_2^d, then if distinct vertices x and y of V_0 are adjacent in Q_d to the same vertex in V_1 we have $w(x - y) = 2$, and so x and y must be in different cosets of H. Let the cosets of H in V_0 be $H_1, H_2, \ldots, H_d$.

Theorem 6 *Let $t > 1$ be an integer, $d = 2^t$, and $m = d - t - 1$. It is possible to list the vertices of Q_d of even weight as $v_1, v_2, \ldots, v_{2^{d-1}}$ so that if ϕ is the graph isomorphism on $K_{2^{d-1},2^{d-1}}$ that is the identity on V_1 and sends $v_i \in V_0$ into v_{i+d}, the subscripts taken modulo 2^{d-1}, then $Q_d, \phi(Q_d), \phi^2(Q_d), \ldots, \phi^{2^m-1}(Q_d)$ forms a decomposition of $K_{2^{d-1},2^{d-1}}$ into copies of Q_d.*

Proof: Let $n = d - 1$, and order the elements of V_0 as $v_1, v_2, \ldots, v_{2^n}$, where $v_1 \in H_1, v_2 \in H_2, \ldots, v_d \in H_d, v_{d+1} \in H_1$, and in general $v_i \in H_r$ whenever $i \equiv r \pmod d$. Let $V_1 = \{w_1, w_2, \ldots, w_{2^n}\}$. Note that $K_{2^n,2^n}$ has 2^{2n} edges, Q_d has $d2^n$ edges, and $2^{2n}/d2^n = 2^m$. Thus it suffices to show that the graphs $\phi^i(Q_d), 0 \le i < 2^m$, are edge disjoint.

Suppose the edge $\{v, w\} \in \phi^i(Q_d) \bigcap \phi^j(Q_d)$, where $0 \le i < j < 2^m$. Then there exist edges $\{v_a, w\}, \{v_b, w\} \in Q_d$ such that $\phi^i(v_a) = \phi^j(v_b)$, so that $a + id \equiv b + jd \pmod{2^n}$. But $d|2^n$, so this implies that $a \equiv b \pmod d$. Then v_a and v_b are in the same coset of H. But both are adjacent to w in Q_d, so we must have $a = b$. Then $id \equiv jd \pmod{2^n}$, or $i \equiv j \pmod{2^{n-t}}$. Since $n - t = m$, this is a contradiction. □

In the language of [10] the last theorem shows that $K_{2^{d-1},2^{d-1}}$ has an r, s-cyclic decomposition into copies of Q_d, where $r = d$ and $s = 0$, and Theorem 1 of that paper can be applied to extend this to a cyclic decomposition of $K_{2^{d-1}+qd,2^{d-1}}$ for any positive integer q. There is also a direct proof, which we omit since it is nearly identical to that of Theorem 6.

Theorem 7 *Let $q > 0$ and $t > 1$ be integers, $d = 2^t$, and $m = d-t-1$. Let $K_{2^{d-1}+qd,2^{d-1}}$ have the vertex partition V_0', V_1. Then it is possible to list the elements of V_0' as $v_1, v_2, \ldots, v_{2^{d-1}+qd}$ so that if ψ is the graph isomorphism that is the identity on V_1 and sends $v_i \in V_0'$ into $v_i + d$, the subscripts taken modulo $2^{d-1}+qd$, then $Q_d, \psi(Q_d), \ldots, \psi^{2^m+q-1}(Q_d)$ forms a decomposition of $K_{2^{d-1}+qd,2^{d-1}}$ into copies of Q_d.*

4 A Cube Decomposition of K_{96}

Some necessary conditions for the existence of a d-cube decomposition of K_n are:

(4.1) if $n > 1$ then $n \ge d$,

(4.2) $d|(n-1)$ (since Q_d is d-regular and K_n is $(n-1)$-regular), and

(4.3) $d2^d|n(n-1)$ (since $|E(Q_d)| = d2^{d-1}$ and $|E(K_n)| = n(n-1)/2$.)

For a fixed d, these necessary conditions require that n lies in certain congruence classes modulo d.

In 1981, Anton Kotzig [7] proved the following results concerning Q_d-decompositions of K_n:

Theorem 8 *If d is even and there is a Q_d-decomposition of K_n, then $n \equiv 1 \pmod{d2^d}$.*

Theorem 9 *If d is odd and there is a Q_d-decomposition of K_n, then either*
(a) $n \equiv 1 \pmod{d2^d}$ or
(b) $n \equiv 0 \pmod{2^d}$ and $n \equiv 1 \pmod{d}$.

Theorem 10 *There is a Q_d-decomposition of K_n if $n \equiv 1 \pmod{d2^d}$.*

The previous three theorems by Kotzig established the sufficiency of conditions (4.1)-(4.3) for the cases when d is even and when d is odd and n is odd. The case d odd and n even remains open for $d \geq 5$. If d is an odd prime number, then the open case reduces to solving the following problem [8].

Problem 1 *If d is an odd prime and $n \equiv (d+1)2^{d-1} \pmod{d2^d}$, then Q_d divides K_n.*

We note that case $d = 3$ and $n \equiv 16 \pmod{24}$ was first settled in [9]. Moreover, 3-cube decompositions of both ${}^{\lambda}K_n$ and ${}^{\lambda}K_{m,n}$ are investigated in [1].

Our previous results can be applied to show the following.

Theorem 11 *We have $K_{96} = 57Q_5$.*

Proof: Using Theorem 2 with $k = 2$ and $d = 4$ and 5 gives

$$K_{8,8} = 2Q_4 \tag{3}$$

and

$$K_{16,16} = 3Q_5 + 16Q_1, \tag{4}$$

and Theorem 3 with $d = 5$ gives

$$K_{32} = 6Q_5 + 16Q_1. \tag{5}$$

Note that since the copies of Q_1 are vertex-disjoint in (4) and (5), $16Q_1$ amounts to a 1-factor of the original graph. We will denote a 1-factor with n edges by F_n.

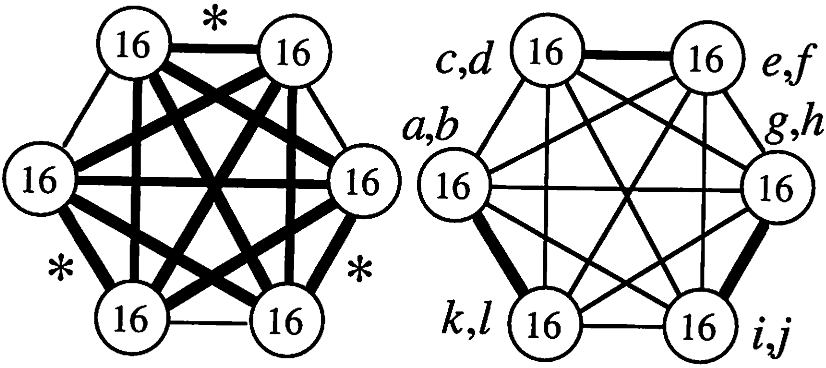

Figure 1

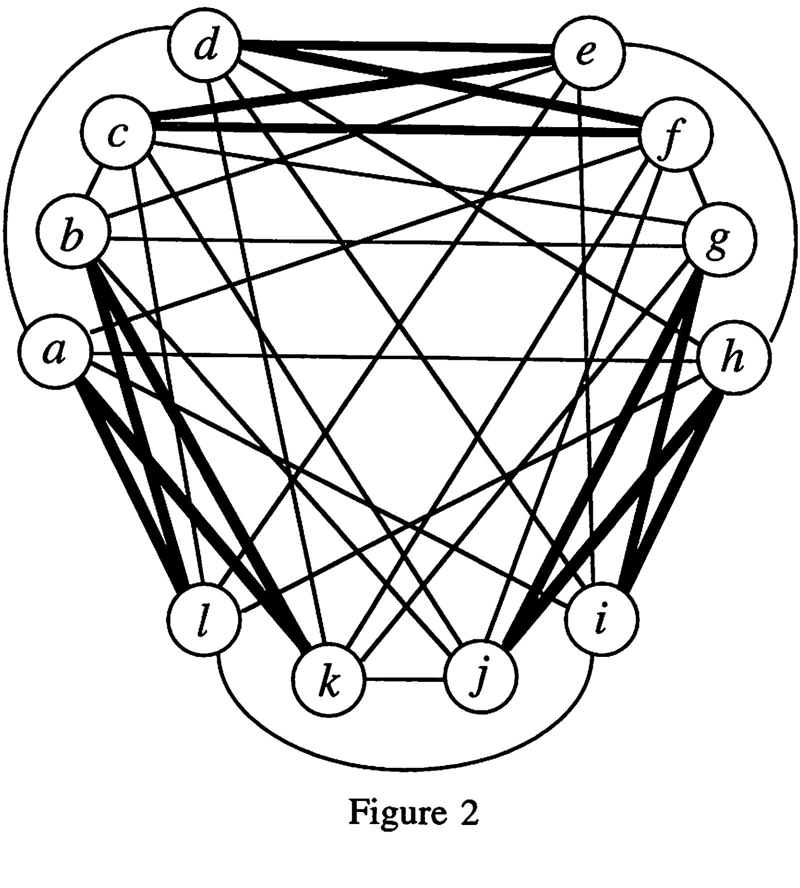

Figure 2

Now $K_{96} = 3K_{32} + K_{32,32,32}$, so (5) allows us to remove from K_{96} 18 copies of Q_5, leaving $3F_{16}+K_{32,32,32} = 3F_{16}+12K_{16,16}$. See the first graph in Figure 1, where the thin lines represent F_{16} and the thick lines $K_{16,16}$. Let the 16 vertices in the leftmost group be $a_1, a_2, \ldots, a_8, b_1, \ldots, b_8$, then in succeeding groups (clockwise) $c_1, \ldots, c_8, d_1, \ldots, d_8$, etc., alphabetically. Suppose these are numbered so that we have the edges $\{a_1, d_1\}, \ldots, \{a_8, d_8\}$, $\{b_1, c_1\}, \ldots, \{b_8, c_8\}$ in the remaining F_{16}. Likewise let the other two copies of F_{16} have edges $\{e_t, h_t\}$, $\{f_t, g_t\}$, $\{i_t, l_t\}$, and $\{j_t, k_t\}$.

We use (4) on each of the 9 copies of $K_{16,16}$ not marked with a *, removing 27 more copies of Q_5 and leaving the second graph in Figure 1. In fact we can take out any 1-factor F_{16} of $K_{16,16}$ and what is left can be decomposed into three copies of Q_5. Thus we can get the graph of Figure 2, where each circle represents 8 vertices, the thin lines copies of F_8 in which adjacent vertices have the same subscript, and the thick lines copies of $K_{8,8}$.

Now we will use (3) and the fact that $Q_5 = Q_4 \times K_2$ to decompose the graph remaining into copies of Q_5. There exists a copy of Q_4 inside the $K_{8,8}$ consisting of edges $\{c_s, e_t\}$ that is symmetric in the sense that $\{c_s, e_t\}$ is an edge if and only if $\{e_s, c_t\}$ is. Denote this by (c, e) and its complement in $K_{8,8}$ by $[c, e]$; by (3) $[c, e]$ is also a symmetric copy of Q_4. Likewise let eh denote the F_8 with edges $\{e_t, h_t\}$. We extend the notation to other letters in the obvious way, so that, for example, $\{c_s, e_t\}$ is in (c, e) iff $\{h_s, j_t\}$ is in (h, j). Then $(c, e) + eh + (hj) + jc$ is isomorphic to Q_5, and, for that matter, so is $[c, e] + el + [lb] + bc$.

We can find 12 such examples in the graph of Figure 2, namely

$$\begin{array}{ccc} (ce) + eh + (hj) + jc, & (gi) + il + (lb) + bg, & (ka) + ad + (df) + fk, \\ (cf) + fj + (jg) + gc, & (hi) + ia + (al) + lh, & (kb) + be + (ed) + dk, \\ [ce] + el + [lb] + bc, & [hj] + jk + [ka] + ah, & [df] + fg + [gi] + id, \\ [cf] + fa + [al] + lc, & [hi] + ie + [ed] + dh, & [kb] + bj + [jg] + gk. \end{array}$$

It can be checked that each F_8 appears once in this list and each pair corresponding to a $K_{8,8}$ twice, once with parentheses and once with brackets. Thus we have 12 more copies of Q_5, for a grand total of $18 + 27 + 12 = 57$. □

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Bicyclic Directed Triple Systems

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Abstract. A directed triple system of order v, denoted $DTS(v)$, is said to be *bicyclic* if it admits an automorphism whose disjoint cyclic decomposition consists of two cycles. In this paper, we give necessary and sufficient conditions for the existence of bicyclic $DTS(v)$s.

1. Introduction

A *directed triple system* of order v, denoted $DTS(v)$, is a v-element set X of points, together with a set β of ordered triples of elements of X, called *blocks*, such that any ordered pair of points of X occur in exactly one block of β. We denote by $[x, y, z]$ the block containing the ordered pairs (x, y), (y, z) and (x, z). A $DTS(v)$ exists if and only if $v \equiv 0$ or $1 \pmod 3$ [6].

An *automorphism* of a $DTS(v)$ is a permutation of X which fixes β. The *orbit* of a block under an automorphism π is the image of the block under the powers of π. A set of blocks B is said to be a *set of base blocks for a $DTS(v)$ under the permutation* π if the orbits of the blocks of B produce the $DTS(v)$ and exactly one block of B occurs in each orbit. A $DTS(v)$ admitting an automorphism consisting of a single cycle is said to be *cyclic*. A cyclic $DTS(v)$ exists if and only if $v \equiv 1$, 4, or 7 (mod 12) [4]. A $DTS(v)$ admitting an automorphism consisting of a fixed point and a cycle of length $v-1$ is said to be *rotational* (or *1-rotational*) and exists if and only if $v \equiv 0 \pmod 3$ [2].

These types of automorphism questions have also been addressed for other triple systems. Colbourn [3] proved that if π is an automorphism of a two-fold triple system of order v then, under the appropriate necessary conditions, the two-fold triple system can be directed to form a $DTS(v)$ which also admits π as an automorphism. A Steiner triple system of order v, denoted $STS(v)$, is said to be *bicyclic* if it admits an automorphism consisting of two disjoint cycles. A bicyclic $STS(v)$ admitting an automorphism consisting of a cycle of length N and a cycle of length M (where $N < M$) exists if and only if $N \equiv 1$ or $3 \pmod 6$, $N \neq 9$, $N \mid M$, and $v = N + M \equiv 1$ or $3 \pmod 6$ [1, 5]. The purpose of this paper is to present necessary and sufficient conditions for the existence of a bicyclic $DTS(v)$.

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