Cyclic Niche Graphs and Grids

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Abstract. A graph $G = (V, E)$ is a *loop niche* graph if there is a digraph $D = (V, A)$ such that $xy \in E$ iff there exists $z \in V$ such that either xz and $yz \in A$ or zx and $zy \in A$. If D has no loops, G is a *cyclic niche* graph, and if D is acyclic, G is a *niche* graph. We give a characterization of triangle-free cyclic niche graphs, and apply this to classify grids.

1. Introduction

Let $D = (V, A)$ be an acyclic digraph. Then the *niche graph* corresponding to D is the graph $G = (V, E)$ where there is an edge between two distinct vertices x and y of V if and only if for some z in V either there are arcs xz and yz in D or there are arcs zx and zy in D. A graph G is a *niche graph* if there exists an acyclic digraph D such that G is the niche graph corresponding to D, and D is then called a *food web* for G. Niche graphs are a variant of competition graphs; see [4] for a survey of this area. As has been done with competition graphs, the definition of niche graph has been relaxed to allow food webs that are not acyclic [1,6]. If a graph has a food web that may have loops and/or cycles, the graph is a *loop niche graph*, and if it has a food web that may have cycles but not loops, it is a *cyclic niche graph*. Clearly all niche graphs are cyclic niche graphs, and all cyclic niche graphs are loop niche graphs. However graphs can be loop niche without being cyclic niche (for example $K_{1,3}$) and cyclic niche without being niche (for example C_4). Figure 1 gives the food webs for these two examples.

Classifying niche graphs in general appears to be difficult. In the remainder of this paper we will restrict our attention to triangle-free graphs. In this case some classification results have been obtained.

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Theorem 1 [1,6]. *No triangle-free graph of maximum degree at least 5 is a loop niche graph.*

Theorem 2 [5]. *Every tree of maximum degree at most* 3 *except* K_2 *and* $K_{1,3}$ *is a niche graph.*

Theorem 3 [6]. *Let C be a caterpillar; i.e. a tree in which all vertices of degree more than* 1 *lie on a path called the spine. Then C is loop niche if and only if C has maximum degree at most 4 and the spine of C does not consist solely of an even number of vertices of degree 4.*

We will extend the technique that was used in proving Theorem 3 to cyclic niche graphs, and use this to obtain some further classification results.

2. Interlacings and Interlaceable Coverings

Let G be a triangle-free graph. We start by considering how to create food webs for certain of the subgraphs of G that consist of only a pair of paths or a pair of cycles.

Given two (not necessarily disjoint) paths P and P' in G of lengths differing by at most 1, we first define an **interlacing** from P to P'. Choose a consecutive ordering of the vertices along P, say $v_0v_1...v_n$, and along P', say $w_0w_1...w_m$ (where $m = n-1$, n or $n+1$). Then the interlacing I is one of the following sets of $m+n+1$ arcs:
If $m = n+1$ then $I = \{v_0w_0, v_0w_1, v_1w_1, v_1w_2, ..., v_nw_n, v_nw_{n+1}\}$.
If $m = n-1$ then $I = \{v_0w_0, v_1w_0, v_1w_1, v_2w_1, ..., v_{n-1}w_{n-1}, v_nw_{n-1}\}$.
If $m = n$ then $I = \{v_0w_0, v_0w_1, v_1w_1, v_1w_2, ..., v_{n-1}w_n, v_nw_n\}$ or $\{v_0w_0, v_1w_0, v_1w_1, v_2w_1, ..., v_nw_{n-1}, v_nw_n\}$.

If one of the paths has length 0, there is a unique interlacing from P to P'. Otherwise, since there are two possible orderings of the vertices for a path, there are two possible interlacings from P to P' when their lengths differ by exactly 1, and four possible interlacings when their lengths are equal. For example, the paths abc and def have four possible interlacings: $\{ad, ae, be, bf, cf\}$, $\{ad, bd, be, ce, cf\}$, $\{af, ae, be, bd, cd\}$ and $\{af, bf, be, ce, cd\}$.

Similarly, given two cycles C and C' in G, both of length n (where $n \geq 4$) we define an **interlacing** from C to C'. Choose a consecutive ordering of the vertices around C, say $v_1 \ldots v_n$, and around C', say $w_1 \ldots w_n$. Then the corresponding interlacing is the set $\{v_1w_1, v_1w_2, v_2w_2, v_2w_3, \ldots, v_nw_n, v_nw_1\}$ consisting of $2n$ arcs. Since there are two possible orientations for a cycle, and n possible starting points in ordering it, C and C' have $2n$ possible interlacings. Figure 2 illustrates one of the eight possible interlacings of two 4-cycles.

From the definitions of loop niche and interlacing we get immediately the following lemma.

Lemma 1. *Let H and H' be two subgraphs of a graph G that are either a pair of paths of length differing by at most* 1 *or a pair of cycles of equal length, and let I be an interlacing from H to H'. Then $H \cup H'$ is a loop niche graph with food web $D = (V(H \cup H'), I)$.*

For a triangle-free graph G, we define an *interlaceable covering* to be two subgraphs H and K of G, each of maximum degree at most 2, such that the (not necessarily disjoint) union $H \cup K$ contains every edge of G and there exists a one-to-one correspondence between the components of H and the components of K that maps cycles to cycles of the same length and paths to paths of length differing by at most 1. The following theorem (from [6], in which an interlaceable covering was called a Pd/Py decomposition) is a useful tool in detecting loop niche graphs. We sketch the proof since we will be adapting it to cyclic niche graphs.

Theorem 4 [6]. *Let G be a triangle-free graph. Then G is a loop niche graph if and only if G has an interlaceable covering.*
Sketch of proof: If G has an interlaceable covering with subgraphs H and K, then for each pair of corresponding components choose an interlacing from the component in H to the component in K. By Lemma 1, the union of all the interlacings provides the arc-set for a food web, so G is a loop niche graph. Conversely, suppose that $G = (V, E)$ is a loop niche graph with food web $D = (V, A)$. Then since G is triangle-free each vertex in D has in-degree and out-degree at most 2. Define a graph H by $V(H) = \{v \in V | v$ has out- degree at least 1 in $D\}$ and

$E(H) = \{xy|$ there exists $z \in V$ such that $xz, yz \in A\}$. Similarly, define a graph K by $V(K) = \{v \in V | v$ has in-degree at least 1 in $D\}$ and $E(K) = \{xy|$ there exists $z \in V$ such that $zx, zy \in A\}$. Since G is the loop niche graph of D, H and K are subgraphs of G and $H \cup K$ covers the edges of G. Since G is triangle-free, each component of H and K has maximum degree at most 2. For each component C of H, let C' be the subgraph of G induced by the heads of all arcs of D whose tails are in $V(C)$. Then C' is a component of K; pair C and C'. This gives the required interlaceable covering. □

Figure 3 gives an example of a loop niche graph G with food web D and interlaceable covering P. In this example, the paths gh and abc of P are paired, and the arcs in the food web D that create the edges gh, ab and bc of these two paths in G are the arcs of the interlacing $\{ga, gb, hb, hc\}$. Similarly, the cycles $cefd$ and $ehif$ are paired, with interlacing $\{ce, ch, eh, ei, fi, ff, df, de\}$. Finally, the paths ij and d are paired, with interlacing $\{id, jd\}$.

Determining whether an acyclic food web can result from a given interlaceable covering is difficult, since a cycle can result from combinations of arcs from several different pairs. However, determining whether a loopless food web can result from a given interlaceable covering is much easier, since it is only necessary to check whether each pair of components has an interlacing that does not contain a loop. The graph G in the example above is in fact a cyclic niche graph, since the cycles $cefd$ and $ehif$ have the alternate interlacing $\{cf, ci, ei, eh, fh, fe, de, df\}$ that does not contain a loop. Thus the next result, which follows from the proof of Theorem 4, can be useful in determining cyclic niche graphs.

Theorem 5. *Let G be a triangle-free graph. Then G is cyclic niche if and only G has an interlaceable covering such that each corresponding pair of components in the covering has an interlacing that does not contain a loop.*

As an example of the application of Theorems 4 and 5, we give an infinite class of loop niche graphs that are not cyclic niche. Let $G_1 = K_{1,4}$, and for $n > 1$ let G_n be determined from

G_{n-1} by adding to each vertex v of degree 1 in G_{n-1} three new vertices of degree 1 adjacent to v (so v now has degree 4 in G_n). G_3 is illustrated in Figure 4.

Theorem 6. *The graph G_n is loop niche but not cyclic niche for all $n \geq 1$.*

Proof. We first define the subgraphs H and K of an interlaceable covering (see Figure 4). We use P_k to refer to a subgraph which is a path of length $k-1$. For each vertex v of degree 4 having exactly three neighbours of degree 1, choose two of these neighbours and remove the P_3 connecting them (leaving v as a vertex now of degree 2). Altogether, this removes $4 \times 3^{n-2}$ P_3's from G_n. From the remains of G_n, for each vertex of degree 4 at distance 2 from exactly three vertices of degree 1, choose two of these pendant vertices and remove the P_5 connecting them. This removes $4 \times 3^{n-3}$ P_5's from G_n. Now repeat this process, removing $4 \times 3^{n-i-1}$ P_{2i+1}'s from G_n for $i = 1, 2, ..., n-1$. What remains is a spider with a central vertex of degree 4 and four legs each of length n; split this at the vertex of degree 4 giving two P_{2n+1}'s. Put one of these P_{2n+1}'s into H and the other into K. Then put into K all of the P_{2n-1}'s that were removed from a vertex of the P_{2n+1} now in H, and put into H all of the P_{2n-1}'s that were removed from a vertex of the P_{2n+1} now in K. Repeat this process outward, finally putting into K all of the P_3's that were removed from a vertex of a path now in H and vice versa. The subgraphs H and K now each contain one P_{2n+1} and $2 \times 3^{n-i-1}$ P_{2i+1}'s for $i = 1, 2, ..., n-1$. Thus we can pair each P_k in H with a distinct P_k in K, and so H and K form an interlaceable covering for G_n. By Theorem 4, G_n is a loop niche graph. Now let H and K be the two subgraphs of any interlaceable covering for G_n. Since all components of H and K have maximum degree at most 2, and since $H \cup K$ covers the edges of G_n, it follows that for every vertex of degree 4 in G_n, two of the incident edges must be in H and the other two in K. Thus all components must be paths starting and ending at vertices of degree 1 in G_n, and so every interlaceable covering for G must, as the one above does, consist of $4 \times 3^{n-i-1}$ P_{2i+1}'s, for $i = 1, 2, ..., k-1$, and two P_{2n+1}'s, each of which has the central vertex of G_n as its centre vertex. The two P_{2n+1}'s must be paired with each other, since all other path components have length at most $2n-2$. But all four of the possible interlacings between two P_{2n+1}'s contain an arc from the centre vertex of one P_{2n+1} to the centre vertex of the other, which in G_n gives a loop. Thus, by Theorem 5, G_n is not a cyclic niche graph.

3. Application to Grids

We now apply the results of the last section to classify the $m \times n$ grids (i.e. the Cartesian products $P_m \times P_n$).

The $1 \times n$ grids are the paths P_n; these are all known to be niche graphs [3] except for P_2, which is loop niche only.

The 2×2 grid is the cycle C_4. By Figure 1, C_4 is a cyclic niche graph (and by [3] it is not a niche graph). The 2×3 grid is a niche graph, since it can be decomposed into two C_4's that pair with each other without cycles (Figure 5). Similarly the 2×4 grid is a niche graph, since it can be decomposed into two P_6's that pair with each other without cycles (Figure 5). Note that in the food webs for both the 2×3 grid and the 2×4 grid, the two leftmost vertices have in-degree 0 and the two rightmost vertices have out-degree 0. Thus if, for example, two of the food webs for the 2×3 grid are placed side by side and the two rightmost vertices of the first identified with the two leftmost vertices of the second, no cycles are created and the result is a food web for the 2×5 grid (Figure 5). This construction can be generalized to any $2 \times n$ grid.

Theorem 7. *Every $2 \times n$ grid is a niche graph, for $n \geq 3$.*

Proof. If n is odd, take $(n-1)/2$ copies of the food web for the 2×3 grid, place them in a row, and identify the two vertices on the right of each web with the two vertices on the left of the next web. The result is an acyclic food web for the $2 \times n$ grid. If n is even, take one copy of the food web for the 2×4 grid and $(n-4)/2$ copies of the food web for the 2×3 grid, place them in a row, and identify the two vertices on the right of each web with the two vertices on the left of the next web. The result is an acyclic food web for the $2 \times n$ grid. □

Once the grids become large enough to have a vertex of degree 4, they do not appear to be niche graphs. We tested the 3×3 grid by exhaustive computer search, and determined that it is not a niche graph. It is not practical to do this for much larger grids, but we have found no way of showing them to be niche graphs (nor have we found any reason why they should not be). However by defining appropriate interlaceable coverings we can show that these grids are all cyclic niche graphs.

Theorem 8. *Every grid other than the* 1×2 *grid is a cyclic niche graph.*

Proof. Let G be an $m \times n$ grid. We have already considered all cases where $m \leq 2$ (and, by symmetry, where $n \leq 2$), so suppose that $m, n \geq 3$. Label the rows of the grid from top to bottom with $1, 2, ..., m$, and label the columns of the grid from left to right with $1, 2, ..., n$. Then we will describe for each case how to dissect the grid into the subgraphs H and K of an interlaceable covering.

Case (i) m and n are both odd (Figure 6). Consider the vertex v in row i and column j. If $i + j$ is even, split v so that the edges (if any) above and to the left remain together and the edges (if any) below and to the right remain together. If $i + j$ is odd, split v so that the edges (if any) above and to the right remain together and the edges (if any) below and to the left remain together. Once this has been done for every vertex, the result is $(m-1)(n-1)/2$ C_4's and $n + m - 2$ P_2's. Place all the C_4's whose top edge is in an odd-numbered row into H, and the remainder into K, then pair each C_4 in H with the C_4 in K immediately below and to the right. Place the P_2's on the top row and on the leftmost column into K, and the P_2's on the bottom row and on the rightmost column into H, then pair each P_2 in the top row with the corresponding P_2 one column to the left in the bottom row, and each P_2 in the leftmost column with the corresponding P_2 one row up in the rightmost column. This gives an interlaceable covering. Since each C_4/C_4 pair has only one vertex in common, and since none of the P_2/P_2 pairs have any vertices in common, each pair can be interlaced without creating a loop. Thus G is a cyclic niche graph.

Case (ii) m is even and n is odd (Figure 7). Consider any vertex v in row i. If i is even, split v so that the edges (if any) above and to the left remain together and the edges (if any) below and to the right remain together. If i is odd then split v so that the edges (if any) above and to the right remain together and the edges (if any) below and to the left remain together. Once this has been done for every vertex, the result is $n-1$ P_{2m}'s and $m-1$ P_2's. Put the P_{2m}'s alternately (from left to right) into H and K, pairing each P_{2m} going into H with the following P_{2m} going into K. Put the P_2's in the leftmost column into K and those in the rightmost column into H, pairing each P_2 in the rightmost column with the corresponding P_2 one row down in the leftmost column. Place the vertex at the top right corner as a P_1 into H and pair it with the P_2 at the top of the leftmost column. This gives an interlaceable covering. The only paired components with any common vertices are the paired P_{2m}'s, which have m vertices in common: vertex $4k+2$ (numbered from the top) of the H component is vertex $4k+1$ (also numbered from the top) of the K component and vertex $4k+3$ of the H component is vertex $4k+4$ of the K component, for $k = 0, 1, ..., (m-2)/2$. In particular, neither of the two centre vertices of the H component are in the K component, all common vertices before the centre of the H component occur in the first half of the K component, and all common vertices after the centre of the H component occur in the second half of the K component. So, if the H component ordered from top to bottom is interlaced to the K component ordered from bottom to top, the interlacing has no loops. Thus G is a cyclic niche graph.

Case (iii) m and n are both even, and without loss of generality $m \leq n$ (Figure 8). For each vertex v, split v so that the edges (if any) above and to the left remain together and the edges (if any) below and to the right remain together. The result is $m+n-2$ paths, in the order (from top left corner to bottom right) $P_3, P_5, P_7, ..., P_{2m-1}, P_{2m}, ..., P_{2m}, P_{2m-1}, ...P_7, P_5, P_3$ (with $n-m$ P_{2m}'s). Put these alternately into H and K, and pair the first with the last, the second with the second last, and so on finally pairing the two middle paths with each other. This gives an interlaceable covering. The only paired components with any vertices in common are the two middle paths, which are both P_{2m-1}'s if $m = n$, and both P_{2m}'s if $m < n$.

Suppose first $m = n$, and consider the two middle P_{2m-1}'s, one an H component and one a K component. These share vertices 1, 3, 5, ..., $2m - 1$ (counting from the top left). So the centre (mth) vertices of these H and K components are not common vertices, all common vertices before the centre of the K component occur before the centre of the H component, and all common vertices after the centre of the K component occur after the centre of the H component. Thus if the H component ordered from the top left is interlaced to the K component ordered from the bottom right, the interlacing has no loops (though it does have 2-cycles). Thus G is a cyclic niche graph.

Now suppose $m < n$, and consider the two middle P_{2m}'s, one an H component and one a K component. Then (still counting from the top left) vertex $2k-1$ of the H component is vertex 2k of the K component, for k = 1,2,...,m. Then the set consisting of the arcs from vertex 1 of the H component to vertex 1 of the K component, and from vertex i of the H component to vertices $i-1$ and i of the K component, i = 2,3,...,2m, is an interlacing with no loops. Thus G is a cyclic niche graph. □

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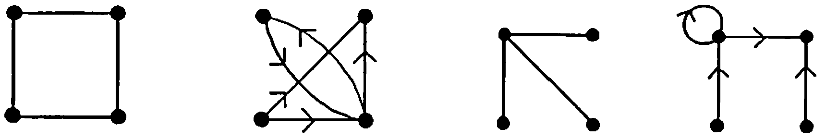

Figure 1. C_4 is cyclic niche, and $K_{1,3}$ is loop niche

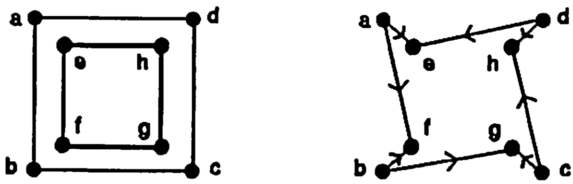

Figure 2. One of the interlacings of two 4-cycles

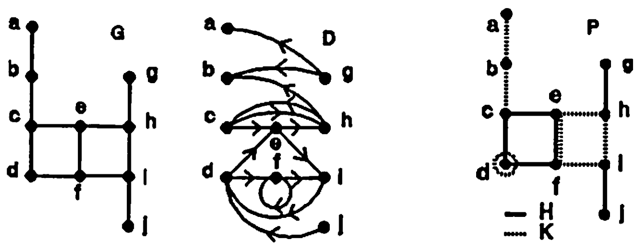

Figure 3. An interlaceable covering P of a graph G, with a corresponding food web D

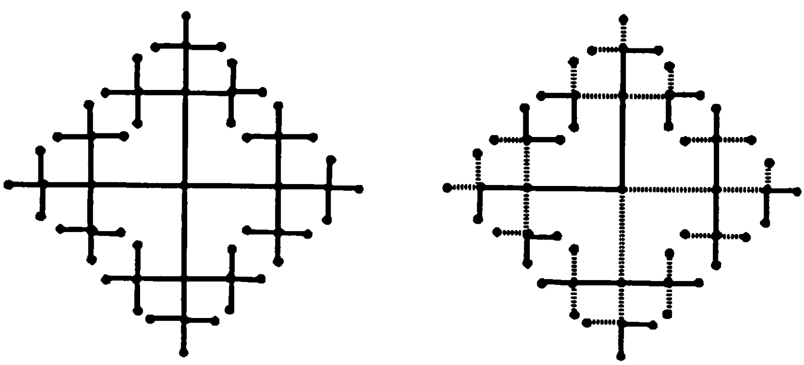

Figure 4. The graph G_3 with an interlaceable covering

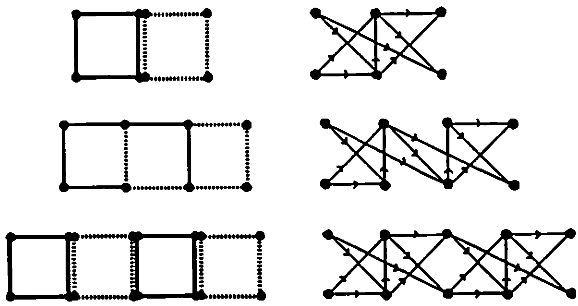

Figure 5. Interlaceable coverings and acyclic food webs for the 2 × 3, 2 × 4 and 2 × 5 grids

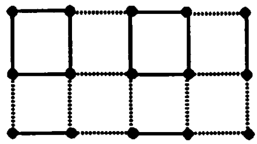

Figure 6. An interlaceable covering for an odd by odd grid

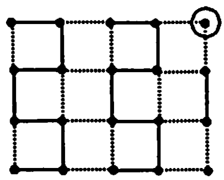

Figure 7. An interlaceable covering for an even by odd grid

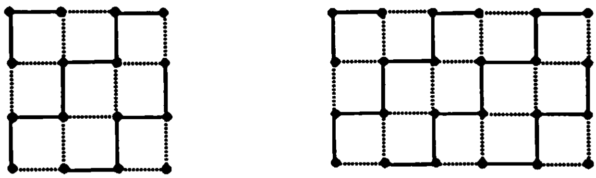

Figure 8. Interlaceable coverings for even by even grids