Two Series of BIB Designs

Roger Logan
Department Of Mathematics
University of Charleston
Charleston, SC 29424

M.K. Singh
Department of Mathematics
Ranchi University
Ranchi-834001, India

K. Sinha
Department of Statistics
Birsa Agricultural University
Ranchi-834006, India

ABSTRACT. In this paper we construct two series of balanced incomplete block (BIB) designs with parameters:

$$v = \binom{2m-3}{2}, \quad r = \frac{(2m-5)!}{(m-3)!}, \quad k = m,$$

$$b = \frac{(2m-3)!}{2m(m-3)!}, \quad \lambda = \frac{(2m-6)!}{(m-3)!}$$

and

$$v = \binom{2m+1}{2}, \quad b = b_1(m+1), \quad r = 2m(\bar{\lambda}_1 - \bar{\lambda}_2), \quad k = m^2,$$

$$\lambda = (m-1)(\bar{\lambda}_1 - 2\bar{\lambda}_2 + \bar{\lambda}_3) + m(\bar{\lambda}_2 - \bar{\lambda}_3),$$

where $k_1, b_1, \bar{\lambda}_i$ are parameters of a special $4-(v,k,\lambda)$ design.

1 Introduction

Constructions of many series of triangular PBIB designs are described in Raghavarao [1], Saha [2], and Sinha [4, 5]. Here, two new series of two-associate-class triangular PBIB designs are constructed; and as special cases, two new series of BIB designs are obtained.

In both constructions we assume the treatments to be the set of all unordered pairs of integers from the set $X = \{1, 2, \ldots, n\}$. In the first construction the blocks are the m-cycles of points from X. In the second construction the blocks are obtained by combining blocks from two special 4-designs. The second construction is motivated by a method presented by Sarvate [3].

2 Construction 1

A two-associate-class triangular scheme can be defined as follows. Let the 2-sets $(ij) = (ji)$; $i, j = 1, 2, \ldots, n$; $i \neq j$ represent the treatments. Two treatments are said to be first associates if there is an integer in common between the corresponding 2-sets, otherwise they are said to be second associates.

A PBIB design based on the above association scheme is called a two-associate-class triangular PBIB design.

Theorem 1. *There exists a series of BIB designs with parameters:*

$$v = \binom{2m-3}{2}, \quad r = \frac{(2m-5)!}{(m-3)!}, \quad k = m,$$
$$b = \frac{(2m-3)!}{2m(m-3)!}, \quad \lambda = \frac{(2m-6)!}{(m-3)!}. \qquad (1)$$

Proof: Let us consider a complete graph K_n with the vertex set $X = \{1, 2, \ldots, n\}$. We take the edges of K_n as the treatments and the m-cycles in it as the blocks of a design T. A block represented by the m-cycles $(a_1, a_2, \ldots, a_m)$ and $(a_i, a_{i+1}, \ldots, a_{i-2}, a_{i-1})$ are assumed to be the same and contains the treatments: (a_1a_2), $(a_2a_3), \ldots, (a_{m-1}a_m)$, (a_ma_1). In K_n there are $\frac{n!}{2m(n-m)!}$ m-cycles in all, and $\frac{(n-2)!}{(n-m)!}$ m-cycles through any pair of intersecting edges. Thus T is a two-associate-class triangular PBIB design with parameters:

$$v = \binom{n}{2}, \quad r = \frac{(n-2)!}{(n-m)!}, \quad k = m, \quad b = \frac{n!}{2m(n-m)!},$$
$$\lambda_1 = \frac{(n-3)!}{(n-m)!}, \quad \lambda_2 = \frac{2(m-3)(n-4)!}{(n-m)!}.$$

Now the condition for balance, $\lambda_1 = \lambda_2$, implies that $n = 2m - 3$. Thus we have a family of BIB designs with parameters (1).

Remark: The blocks of T can also be obtained as follows: Let S_n be the group of permutations on $X = \{1, 2, \dots, n\}$. Let Π^n_m $n > m$) be the set of restrictions of the elements of S_n to the set $\{1, 2, \dots, m\}$ (neglecting repetitions). Let the elements of Π^n_m be represented as ordered m-tuples of elements from X. Let us define an equivalence relation $\sim$ on Π^n_m by $(a_1 a_2 \dots a_m) \sim (b_1 b_2 \dots b_m)$ if $(a_1 a_2 \dots a_m)$ can be obtained by a cyclic permutation of $(b_1 b_2 \dots b_m)$. Then the equivalence classes of $\sim$ on Π^n_m are blocks of T. Clearly the automorphism group of T is S_n.

Illustration. When $m = 4$ in (1) we get a BIB design with parameters $v = 10$, $b = 15$, $r = 6$, $k = 4$, $\lambda = 2$ which is shown below:

4-cycles	Block contents	4-cycles	Block contents
1234	12, 23, 34, 14	1245	12, 24, 45, 15
1243	12, 24, 34, 13	1254	12, 25, 45, 14
1324	13, 23, 24, 14	1425	14, 24, 25, 15
1235	12, 23, 35, 15	1345	13, 34, 45, 15
1253	12, 25, 35, 13	1354	13, 35, 45, 14
1325	13, 23, 25, 15	1435	14, 34, 35, 15
2345	23, 34, 45, 25	2354	23, 35, 45, 24
2435	24, 34, 35, 25.		

3 Construction 2

We first introduce some notation that will be used throughout this section. Let D_1 be a 4-design on v elements and b blocks and k elements per block. Let λ_i, $i = 1, 2, 3, 4$ denote the number of blocks in D_1 which contain a given i-subset of elements. In the following we will denote a 4-design with the parameters $(v,\ b,\ \lambda_1,\ k,\ \lambda_2,\ \lambda_3,\ \lambda_4)$ by $4 - (v,\ k,\ \lambda_4)$.

Theorem 2. *Let D_1 and D_2 be two 4-designs with parameters $4-(n, k_1, \bar{\lambda}_4)$, $4 - (n - k_1, k_2, \hat{\lambda}_4)$, respectively. Then there exists a partially balanced incomplete block design D with parameters*

$$v = \binom{n}{2} \quad b = b_1 b_2, \quad r = 2\hat{\lambda}_1(\bar{\lambda}_1 - \bar{\lambda}_2), \quad k = k_1 k_2$$

$$\lambda_1 = \hat{\lambda}_2(\bar{\lambda}_1 - 2\bar{\lambda}_2 + \bar{\lambda}_3) + \hat{\lambda}_1(\bar{\lambda}_2 - \bar{\lambda}_3), \qquad \lambda_2 = 4\hat{\lambda}_2(\bar{\lambda}_2 - 2\bar{\lambda}_3 + \bar{\lambda}_4).$$

Proof: Let D_1 and D_2 be two 4-designs with parameters as stated in the hypothesis. Let the treatments of D_1 consist of the integers from $X = \{1, 2, \dots, n\}$. We now form a PBIB design D from D_1 and D_2. The treatments of D will consist of the unordered pairs of elements from X. The blocks of D are constructed as follows. Let A be a block of D_1. Once we have a block from D_1, we consider the treatments of D_2 to be the elements of $X \backslash A$. Let B be any block of D_2. The set of unordered pairs

(a, b) where $a \in A$ and $b \in B$ will be a block of D. Thus from each block of D_1 we can form b_2 blocks of D. We now show that D has the desired parameters.

Parameters:

We first determine the replication number. Let (a, b) be an element of D. The pair $\{a, b\}$ occurs in $\bar{\lambda}_2$ blocks of D_1 and $\{a\}$ occurs in $\bar{\lambda}_1$ blocks. It follows that $\{a\}$ occurs in $\bar{\lambda}_1 - \bar{\lambda}_2$ blocks without b. Since $\{b\}$ occurs in $\hat{\lambda}_1$ blocks of D_2, (a, b) appears in $\hat{\lambda}_1(\bar{\lambda}_1 - \bar{\lambda}_2)$ blocks of D. Similarly, (b, a) will appear in $\hat{\lambda}_1(\bar{\lambda}_1 - \bar{\lambda}_2)$ blocks. Hence, $r = 2\hat{\lambda}_1(\bar{\lambda}_1 - \bar{\lambda}_2)$.

By counting the total number of treatments, including repetitions, and using the fact that both D_1 and D_2 are 4-designs, and hence 2-designs, we will show $b = b_1 b_2$, where b_1 and b_2 are the number of blocks in D_1 and D_2, respectively. Since D_1 and D_2 are both 2-designs, we have

$$k = k_1 k_2, \quad \hat{\lambda}_1 = \frac{b_2 k_2}{n - k_1}, \quad n\bar{\lambda}_1 = b_1 k_1, \quad \bar{\lambda}_2 = \frac{\bar{\lambda}_1(k_1 - 1)}{n - 1}.$$

By counting the number of treatments in D we have $vr = bk$. Using the above equalities we have

$$\begin{aligned}
bk_1k_2 &= (n)(n-1)\hat{\lambda}_1(\bar{\lambda}_1 - \bar{\lambda}_2) \\
&= (n)(n-1)\frac{b_2 k_2}{n - k_1}(\bar{\lambda}_1 - \bar{\lambda}_2) \\
&= b_2 k_2 \left[\frac{n(n-1)(\bar{\lambda}_1 - \bar{\lambda}_2)}{n - k_1}\right] \\
&= b_2 k_2 \left[\frac{(n-1)(n\bar{\lambda}_1) - (k_1 - 1)(n\bar{\lambda}_1)}{n - k_1}\right] \\
&= b_2 k_2 \left[\frac{b_1 k_1 (n - k_1)}{n - k_1}\right] \\
&= b_1 b_2 k_1 k_2.
\end{aligned}$$

Hence, $b = b_1 b_2$.

We now count the number of occurences of pairs. Here we have two types

1. $\{(a, b),\ (a, c)\}$; first associates,
2. $\{(a, b),\ (c, d)\}$; second associates.

We will only consider pairs which are first associates. The counting of second associates is done in a similar fashion. Let $\{(a, b),\ (a, c)\}$ be a pair in D. We can not have the sets $\{a, b\}$, $\{a, c\}$, $\{a, b, c\}$ occuring in the same blocks in D_1. Since a occurs in $\bar{\lambda}_1$ blocks and is in $\bar{\lambda}_2$ blocks with b and in $\bar{\lambda}_2$ blocks with c and in $\bar{\lambda}_3$ blocks with both b and c then it follows that a

appears in $\bar{\lambda}_1 - 2\bar{\lambda}_2 + \bar{\lambda}_3$ blocks of D_1 without either b or c. Both b and c must occur in the same blocks of D_2, and this happens $\hat{\lambda}_2$ times. Considering the set $\{b, c\}$ in D_1 and $\{a\}$ in D_2, we have the pair is in $\hat{\lambda}_1(\bar{\lambda}_2 - \bar{\lambda}_3)$ blocks. Hence, the pair appears in $\lambda_1 = \hat{\lambda}_2(\bar{\lambda}_1 - 2\bar{\lambda}_2 + \bar{\lambda}_3) + \hat{\lambda}_1(\bar{\lambda}_2 - \bar{\lambda}_3)$ blocks of D.

Remark: It should be mentioned that the above construction may yield a PBIBD with repeated blocks. In some cases it is possible to remove the extra blocks to obtain a PBIBD with $\frac{1}{2}b_1b_2$.

Corollary 1. *When $n = 2s + 1$ and $k_1 = k_2 = s$ in Theorem 2, there is a BIBD with parameters*

$$v = \binom{2s+1}{2}, \quad b = \frac{1}{2}b_1(s+1), \quad r = s(\bar{\lambda}_1 - \bar{\lambda}_2) = \frac{(s+1)}{2}\bar{\lambda}_1,$$

$$k = s^2, \quad \lambda = \frac{s^2 - 1}{2(2s-1)}\bar{\lambda}_1,$$

(when we remove duplicate blocks, b, r and λ are all halved). Here, the parameters b_1, $\bar{\lambda}_1$ come from the $4 - (n, k_1, \bar{\lambda}_4)$ design of Theorem 2.

Proof: In Theorem 2 if we set $\lambda_1 = \lambda_2$ we obtain $n = 2s + 1$. In this case if D_1 is a $4 - (2s + 1, s, \lambda_4)$ design then D_2 is a $4 - (s + 1, s, s - 3)$ design. It follows that $b_2 = s + 1$. Hence, D_2 contains all the s-subsets of a s+1 set. It now follows that each block in D is repeated. If we remove all the duplicates we obtain $b = \frac{1}{2}b_1(s + 1)$.

Corollary 2. *If in Theorem 2 we let $n = m^2 + 2$, $k_1 = \frac{1}{2}(m^2 + m + 2)$, and $n - k_1 = k_2 = \frac{1}{2}(m^2 - m + 2)$, $b_2 = 1 = \hat{\lambda}_1 = \hat{\lambda}_2 = \hat{\lambda}_3 = \hat{\lambda}_4$, we get a BIBD with parameters*

$$v = \binom{m^2+2}{2}, \quad b = b_1, \quad r = \frac{2(m^2 + m + 2)}{m^2 + 1}\bar{\lambda}_1,$$

$$k = \frac{1}{4}(m^4 + 3m^2 + 4), \quad \lambda = \frac{m^2 + m + 2}{2(m^2 + 1)}\bar{\lambda}_1.$$

Aknowledgement: We would like to thank the referee for his suggestions concerning the proof of theorem 1. We shall take up the construction of triangular designs by the action of symmetric groups on special subgraphs of strongly regular triangular graphs in the future.

References

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