A NOTE ON THE ROAD-COLORING CONJECTURE

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ABSTRACT. Some results relating to the road-coloring conjecture of Alder, Goodwyn, and Weiss, which give rise to an $O(n^2)$ algorithm to determine whether or not a given edge-coloring of a graph is a road-coloring, are noted. Probabilistic analysis is then used to show that, if the outdegree of every edge in an n-vertex digraph is $\delta = \omega(\log n)$, a road-coloring for the graph exists. An equivalent re-statement of the conjecture is then given in terms of the cross-product of two graphs.

DEFINITIONS

Let $\mathcal{G}$ be an n-vertex digraph. $V(\mathcal{G})$ will denote the vertex-set of $\mathcal{G}$, and $E(\mathcal{G})$ will denote the edge-set of $\mathcal{G}$. $\mathcal{G}$ is *strongly connected* if for every pair of vertices v and w in $V(\mathcal{G})$, there is a directed path from v to w. The *outdegree* of vertex $v \in V(\mathcal{G})$, $d^+(v)$, is the number of edges originating at v. $\mathcal{G}$ is *aperiodic* if the set of lengths of simple directed cycles in $\mathcal{G}$ has gcd 1. (See [Br] or [BR] for a discussion of aperiodic digraphs.)

HISTORY

The road-coloring conjecture of Alder, Goodwyn, and Weiss [AGW] (hereinafter referred to as RCC) is a graph-theoretic characterization of a problem from ergodic theory. Let $\mathcal{G}$ be a strongly connected digraph such that $d^+(v) = 2$ for every $v \in V(\mathcal{G})$. (In this paper, loops and multiple edges are not permitted. Strictly speaking, this is not essential to the problem, but it is required if we are to follow the path laid out by O'Brien [O] in theorem 3 below.) Let $\chi : E(\mathcal{G}) \to \{R, B\}$ be an edge coloring of $\mathcal{G}$ such that for each $v \in V(\mathcal{G})$, v has one red (R) edge and one blue (B) edge going out from it. χ is called a *road-coloring* of $\mathcal{G}$. A string $I \in \{R, B\}^*$ will be called a *set of instructions*. Given χ and $v \in V(\mathcal{G})$, let $I(v)$ designate the vertex $w \in V(\mathcal{G})$ which is arrived at if one begins at v and follows the path labeled by I.

We now give the following definition.

Definition. Let $v \in V(\mathcal{G})$, and let χ be a road coloring of $\mathcal{G}$. χ is a *resolving road-coloring for* v if and only if there exists an $I \in \{R, B\}^*$ such that for all $w \in V(\mathcal{G})$, $I(w) = v$.

(In terms of cities and roads, the problem may be stated as follows: Suppose every city has two one-way roads coming out of it. Further, we are interested in answering the question, "Do you know the way to San Jose?". Our road system has a resolving road coloring if there is a way to paint the roads red and blue, one of each color out of each city, so that a single list of colors correctly answers the question regardless of the city in which it is asked.)

More generally, we allow $d^+(v)$ to be any fixed number, δ, and we color the edges from a set of δ colors, κ, so that no two edges out of the same vertex have the same color. The graph then has a resolving road coloring for vertex v if there exists an $I \in \kappa^*$ such that for all $w \in V(\mathcal{G})$, $I(w) = v$. (Theorem 2 below shows that we only need consider the case for $\delta = 2$, but the more general case will be needed when we consider the conjecture probabilistically later.)

Alder, Goodwyn, and Weiss [AGW] showed the following.

Theorem 1 (Alder, et al.). *Let $\mathcal{G}$ be a strongly connected digraph such that $d^+(v) = \delta$ for all $v \in V(\mathcal{G})$. Then*

$$\mathcal{G} \textit{ has a resolving road coloring} \Longrightarrow \mathcal{G} \textit{ is aperiodic}$$

They further conjectured that the above theorem can be strengthened to "if and only if". This conjecture has become known as the *road-coloring conjecture* (RCC).

O'Brien [O] offered a partial solution by showing:

Theorem 2 (O'Brien). *RCC is true for $\delta = 2 \Rightarrow$ RCC is true $\forall \delta \geq 2$*

and

Theorem 3 (O'Brien). *Let $\mathcal{G}$ be a strongly connected digraph such that $d^+(v) = \delta$ ($\delta \geq 2$) for all $v \in V(\mathcal{G})$ and such that $\mathcal{G}$ contains a prime-length cycle. Then*

$$\mathcal{G} \textit{ has a resolving road-coloring} \Longleftrightarrow \mathcal{G} \textit{ is aperiodic}$$

Friedman [F] offers further results about RCC. Observations on particular digraphs and resolving road-colorings for them may be found in [Br].

In this paper we offer an $O(n^2)$ algorithm for determining whether or not a particular road-coloring is a resolving road-coloring, and we show that for $\delta = \omega(\log n)$ RCC is true for almost every strongly aperiodic digraph of outdegree δ.

The Road-Coloring Conjecture in the Terminology of Formal Language Theory

RCC is easily restated as a problem about finite automata. (See, e.g., [HU] for the essential principles of finite automata.) From here on, $\mathcal{M}$ will refer to a strongly connected aperiodic digraph such that $d^+(v) = 2$ for all $v \in V(\mathcal{M})$. Further, we will assume that $\mathcal{M}$ is an n-vertex graph with the vertices labeled $\{1, 2, \ldots, n\}$. Our goal is an $O(n^2)$ algorithm for determining whether or not $\mathcal{M}$ has a resolving road coloring. In the interest of space, we offer the following lemmas without proof.

Lemma 1. *$\mathcal{M}$ has a resolving road-coloring for a particular vertex i*

$$\Updownarrow$$

$\mathcal{M}$ has a resolving road-coloring for every vertex $j \in V(\mathcal{M})$

Lemma 2. *χ is a resolving road-coloring for $\mathcal{M}$*

$$\Updownarrow$$

For all $i \in V(\mathcal{M})$ there exists a corresponding set of instructions, I, such that $I(i) = I(n) = n$

We now switch to the terminology of finite automata. Let χ be a road coloring of $\mathcal{M}$. To turn $\mathcal{M}$ into a finite automata we need only name an initial state and a set of final states. (χ describes the transition function.) Let $\mathcal{M}_i$ be the automaton with initial state i and final state n. (There are no other final states.) Let L_i denote the language accepted by $\mathcal{M}_i$. The above lemmas imply:

Theorem 4. *χ is a resolving road-coloring for $\mathcal{M}$*

$$\Updownarrow$$

$$\forall i \in \{1, 2, \ldots, n-1\} \; L_i \cap L_n \neq \emptyset$$

The following is a well known construction for creating a finite automaton, $\mathcal{M}^2_\chi$, such that $L(\mathcal{M}^2_\chi) = L_i \cap L_n$. ($L(\mathcal{M})$ here denotes the language accepted by automaton $\mathcal{M}$. We call our automaton $\mathcal{M}^2_\chi$ since $\mathcal{M}^2_\chi$ is a sub-graph of the usual cross-product $\mathcal{M} \times \mathcal{M}$.) The states of $\mathcal{M}^2_\chi$ are the elements of $V(\mathcal{M}) \times V(\mathcal{M})$. There is an edge labeled "R" ("B") from (j, k) to (l, m) if and only if in $\mathcal{M}$ there is an edge labeled "R" ("B") from j to l and from k to m.

The final state of $\mathcal{M}^2_\chi$ is (n, n) and the initial state is (i, n). $L(\mathcal{M}^2_\chi) \neq \emptyset$ if and only if there is a directed path from (i, n) to (n, n). Let a *root-directed arborescence* be a directed tree in which all paths are directed to the root. The above automata construction implies the next theorem.

Theorem 5. χ *is a resolving road-coloring for* $\mathcal{M}$

$$\Updownarrow$$

$\mathcal{M}_\chi^2$ *contains a root-directed arborescence rooted at* (n,n) *which includes nodes* (i,n) *for all* $1 \le i \le n-1$

The above ideas give rise to the following algorithm.

(1) Create $\mathcal{M}_\chi^2$ from $\mathcal{M}$ and χ.

(2) Perform a depth-first search from vertex (n,n) in $\mathcal{M}_\chi^2$ (going against the direction of the edges), and return "true" if every vertex of the form (i,n) for $1 \le i \le n-1$ is encountered; otherwise, return "false".

It is easy to see that the above algorithm has running time $O(n^2)$, where n is the number of vertices in $\mathcal{M}$.

PROBABILISTIC ANALYSIS

In this section we examine the road-coloring conjecture from the point of view of random graph theory. (See [Bo] or [P] for the essential ideas of random graph theory.) Let $\mathcal{G}$ be an n-vertex digraph such that $\mathcal{G}$ is strongly connected, aperiodic, and such that $d^+(v) = \delta$ for all $v \in V(\mathcal{G})$, where $\delta = \omega(\log n)$. We will show that almost every such $\mathcal{G}$ has a resolving road coloring.

Considering only digraphs such that every vertex has outdegree δ, let

$$\Omega_{\delta,n} = \{\, \mathcal{G} \mid |V(\mathcal{G})| = n \text{ and } \mathcal{G} \text{ is loopless and without multiple edges} \,\}$$

and

$$\widehat{\Omega}_{\delta,n} = \{\, \mathcal{G} \mid |V(\mathcal{G})| = n \,\}$$

(i.e., loops and multiple edges are allowed in $\widehat{\Omega}_{\delta,n}$ but not in $\Omega_{\delta,n}$.)

For $\mathcal{G}$ (in either $\widehat{\Omega}_{\delta,n}$ or $\Omega_{\delta,n}$), let X_k be the number of directed k-cycles in $\mathcal{G}$. Let $\widehat{\Pr}(E)$ be the probability of E in $\widehat{\Omega}_{\delta,n}$, and let $\Pr(E)$ be the probability of E in $\Omega_{\delta,n}$. Recalling that cycles cannot repeat vertices, it is easy to see that

$$\Pr(X_k = m) \ge \widehat{\Pr}(X_k = m) \qquad \text{for } k \ge 2 \text{ and } m \ge 2$$

and

$$\Pr(X_k = 0) \le \widehat{\Pr}(X_k = 0) \qquad \text{for } k = 2$$

since allowing loops and multiple edges decreases the chance that a k-cycle exists for $k \geq 2$. Thus if almost every graph in $\widehat{\Omega}_{\delta,n}$ has a k-cycle, then almost every graph in $\Omega_{\delta,n}$ does, too.

Now consider $\widehat{\Omega}_{1,n}$. (See pages 364 - 376 in [Bo] for a discussion of this sample space.) Further consider the $\binom{n}{k}$ k-subsets of $V(\mathcal{G})$. Let X_i be the Bernoulli random variable which equals 1 if and only if the ith k-subset forms a directed k-cycle. Let $Y = \sum_{i=1}^{\binom{n}{k}} X_i$. In [Bo] it is shown that $\Pr(Y = m)$ tends to a Poisson distribution with parameter $\lambda = \frac{1}{k}$. In particular, this means

$$E(Y) \to \frac{1}{k} \quad \text{and} \quad Var(Y) \to \frac{1}{k}.$$

Now consider $\mathcal{G}_1, \mathcal{G}_2, \ldots \mathcal{G}_\delta$, all elements of $\widehat{\Omega}_{1,\delta}$ superimposed on one another to form a graph $\mathcal{G} \in \widehat{\Omega}_{\delta,n}$. Let $\widehat{Y} = \sum_{i=1}^{\delta} Y_i$, where Y_i is the Y associated with $\mathcal{G}_i$. Then $\widehat{Y}$ is less than or equal to the number of k-cycles in $\mathcal{G}$. Further,

$$E(\widehat{Y}) = \delta E(Y_1) \to \frac{\delta}{k} \quad \text{and} \quad Var(\widehat{Y}) = \delta Var(Y_1) \to \frac{\delta}{k}$$

Using Chebyshev's inequality we get

$$\Pr(\widehat{Y} = 0) \leq \frac{Var(\widehat{Y})}{E(\widehat{Y})^2} \to \frac{k}{\delta}$$

Thus if $\delta \to \infty$, then $\Pr(\widehat{Y} = 0) \to 0$, and almost every graph in $\widehat{\Omega}_{\delta,n}$ has a directed k-cycle (hence, so does almost every graph in $\Omega_{\delta,n}$).

McDiarmid [McD] has shown that if $\delta = \omega(\log n)$ then almost every graph in $\Omega_{\delta,n}$ is Hamiltonian (hence strongly connected). (The strong connectivity of every $\mathcal{G} \in \Omega_{\delta,n}$ for $\delta = \omega(\log n)$ can also be shown using the usual methods of probabilistic graph theory.) Thus if we take $k = 2$ and $k = 3$ it follows that almost every $\mathcal{G}$ in $\Omega_{\delta,n}$ is aperiodic, strongly connected, and contains a prime-length cycle. O'Brien's result (Theorem 3 in this paper) then implies

Theorem 6. *Almost every $\mathcal{G} \in \Omega_{\delta,n}$ for $\delta = \omega \log(n)$ has a resolving road coloring.*

We conjecture that the above theorem is true for $\delta \geq 2$, but to prove such a result would, it seems, require working in the sample space of *strongly connected*, outdegree-δ digraphs. We do not see how to do that at this time.

As further food for thought we point out the following tantalizing result of McAndrew [McA].

Theorem 7 (McAndrew). *For $\mathcal{G}$ a digraph,*

$$\mathcal{G} \times \mathcal{G} \text{ is strongly connected } \Longleftrightarrow \mathcal{G} \text{ is strongly connected and aperiodic.}$$

Now, for $\mathcal{M}$ a strongly connected digraph with outdegree 2 and road-coloring χ, let χ^2 be a road-coloring of $\mathcal{M} \times \mathcal{M}$ (which has outdegree 4) be defined by $\chi^2((i,j),(k,l)) = \chi(i,j)\chi(k,l)$. Then (by the discussions above)

Theorem 8. *χ is not a resolving road-coloring for $\mathcal{M}$ if and only if there exists a partition of $V(\mathcal{M} \times \mathcal{M})$ into subsets W_1 and W_2 such that $(n,n) \in W_1$ and $(i,n) \in W_2$ for some $1 \leq i \leq (n-1)$, and all edges going from W_2 to W_1 are colored RB or BR under χ^2.*

(Note that *some* edges will go from W_2 to W_1 follows from Theorem 6.)

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