Independence and Irredundance in k-Regular Graphs

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ABSTRACT. We show that for each fixed $k \geq 3$, the INDEPENDENT SET problem is NP-complete for the class of k-regular graphs. Several other decision problems, including IRREDUNDANT SET, are also NP-complete for each class of k-regular graphs, for $k \geq 6$.

1 Introduction

A graph is said to be *k-regular* if every vertex has degree k. Well-known examples of regular graphs include Cayley graphs, n-cubes, and toroidal meshes. Regular graphs are important in VLSI design because a chip having a regular structure is generally easier to construct and is more scalable. Most multiprocessors (mesh, butterfly, cube-connected cycles, systolic arrays) have regular designs [7].

Given a graph $G = (V, E)$ with vertex set V, a set $S \subseteq V$ is *independent* if no two vertices in S are adjacent. The decision problem INDEPENDENT SET asks, for a given graph G and integer m, if G has an independent set with at least m vertices. It is well known that this problem is NP-complete ([4], p.194). In fact, the problem's restriction to comparability graphs and bipartite graphs are both NP-complete as well. However for many classes of graphs (e.g. trees, permutation, chordal, split, interval, claw-free, Halin) the problem is in **P** [2, 5].

Relatively few decision problems are known to be NP-complete for k-regular graphs. For example, for 3-regular planar graphs, PARTITION INTO PERFECT MATCHINGS, INDEPENDENT SET, and HAMILTONIAN CIRCUIT are NP-complete. For 4-regular planar graphs, DOMINATING SET, CONNECTED DOMINATING SET, and MAXIMUM LEAF SPANNING TREE are NP-complete [4].

Our main result, demonstrated in Section 3, is that for *every* $k \geq 3$, INDEPENDENT SET is NP-complete for the class of k-regular graphs. *We know of no result like this in the literature.* [1] We suggest that for many problems there exists a constant c such that for all $k \geq c$ the problem restricted to k-regular graphs is NP-complete. In Sections 4 and 5 we show that the decision problems associated with the graph parameters Γ and IR are also NP-complete for k-regular graphs when $k \geq 6$. Note that we cannot expect these results for $k < 3$, since a 1-regular graph consists only of disjoint edges, and a 2-regular graph consists only of disjoint cycles.

2 Main idea

Our reduction will be from the decision problem in propositional logic known as NOT-ALL-EQUAL 3SAT. In this variation of the more familiar 3SAT, we are given a set C of clauses, each of which is a disjunction of exactly three literals over a set of variables U. We are asked whether there is a truth assignment to the variables such that each clause in C has at least one true literal *and* one false literal. This problem is known to be NP-complete ([4], p.259). For the remainder of this paper, let $C = \{C_1, \ldots, C_n\}$ be a fixed instance of NOT-ALL-EQUAL 3SAT.

For simplicity, we will first consider 4-regular graphs, and then later explain how our construction can be modified to k-regular graphs, for any $k \geq 3$. We appeal to the reader's tolerance of our informality, as a precise description of our construction would be mired in a lot of confusing detail.

Given C, we construct a 4-regular graph $G(C)$ like the one shown in Figure 1. For each i, let D_i denote the clause obtained by *negating* each literal in C_i. We will call C_i and D_i *mates*. As seen in Figure 1, $G(C)$ contains a triangle associated with each clause and a triangle for its mate. Thus, there are $2n$ triangles. The vertices of each triangle are labeled

[1] Recently, in [6] Kratochvíl, Proskurowski, and Telle obtained a class of NP-completeness results for the H-COVER problem involving k-regular graphs H for each $k > 2$.

with the literals in the associated clause. It will be helpful to envision the triangles of clauses and their mates appearing next to one another.

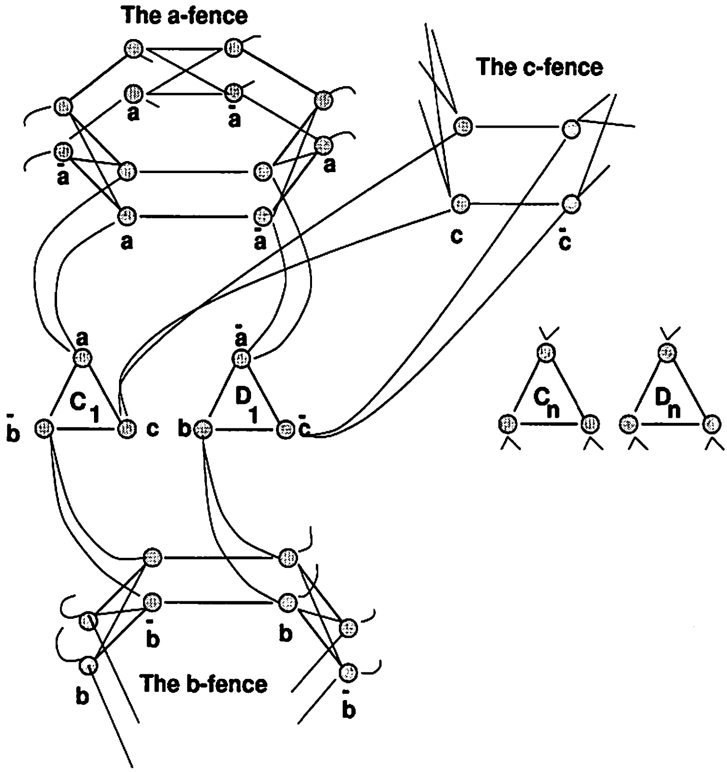

Figure 1. The graph $G(C)$ for $k = 4$

Next, for each variable $x \in U$ we construct a so-called *x-fence*, as follows. For each *occurrence* of either x or $\bar{x}$ in C we associate two parallel edges, called a *gate*. The gates are then placed in a cycle, and bipartite graphs $K_{2,2}$ are inserted between the gates, as shown. The a-fence in Figure 1 has three gates. Without loss of generality we may assume that each $x \in U$ occurs at least twice, so that each x-fence has at least two gates (this avoids a problem of multiple edges). In each x-fence, the vertical "fenceposts" of the gates are labeled alternately with x and $\bar{x}$ as shown. Note that since there are exactly $3n$ occurrences of literals in C, there are exactly $3n$ gates in $G(C)$ and the total number of vertices in $G(C)$ is $(3n)4 + (2n)3 = 18n$.

Finally, edges are placed between triangles and gates. Since each gate in an x-fence corresponds to some occurrence of x or $\bar{x}$, we join the vertices in each fencepost to the corresponding vertices in the clause-triangle and its mate, in the manner shown. It is clear that $G(C)$ is 4-regular. The fenceposts of vertices in an x-fence are marked alternately by x or $\bar{x}$. We say that vertices marked x have *positive parity* and those marked $\bar{x}$ have *negative parity.*

We now claim there is a not-all-equal truth assignment for C if and only if $G(C)$ has an independent set of cardinality at least $8n$. First, suppose C has a not-all-equal assignment. Then each triangle will have a vertex corresponding to a true literal. Select one such vertex in each of the $2n$ triangles. Within each fence, select all the vertices in fenceposts that are associated with false literals. Let S be the set consisting of the $2n$ vertices selected from triangles and the $6n$ vertices selected from the gates. It is easy to see that S is independent.

Conversely, assume $G(C)$ has an independent set S of at least $8n$ vertices. Then since each of the $2n$ triangles can have at most one vertex in S, and each of the $3n$ gates can have at most two vertices in S, S must contain exactly one vertex from each triangle and two vertices from each gate. Note that within each x-fence, either all positive vertices or all negative vertices must be chosen. This defines a truth assignment to the variables. The presence of edges between triangles and x-fences shows that it is a not-all-equal truth assignment.

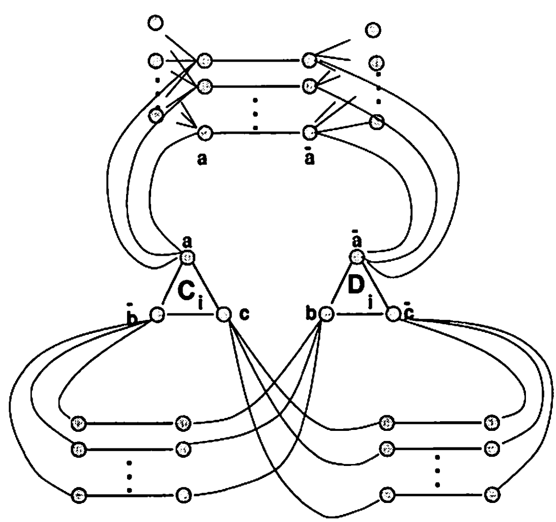

Figure 2. A component of $G_k(C)$

3 Independence for $k \geq 3$

Now let $k \geq 3$ be fixed. We replace each gate of $G(C)$ with $k-2$ parallel edges and each $K_{2,2}$ with $K_{k-2,k-2}$, constructing the graph in a similar way. This graph is denoted $G_k(C)$, a component of which is shown in Figure 2. It is clearly k-regular. Note in the case when $k = 3$ each x-fence is just a cycle. The size of a largest independent set in a graph G is denoted $\beta_0(G)$. It is easy to mimic the argument in the previous section to prove that

Lemma 1 *C has a not-all-equal truth assignment if and only if*

$$\beta_0(G_k(C)) \geq 3(k-2)n + 2n.$$

Since each map $C \to G_k(C)$ is computable in polynomial-time we have

Theorem 1 *For each $k \geq 3$, INDEPENDENT SET is NP-complete for the class of k-regular graphs.*

4 Irredundance for k ≥ 6

A set S is *irredundant* if each $u \in S$ has a *private neighbor*. That is, for each $u \in S$, the set

$$N[u] - N[S-u] \neq \emptyset.$$

Here $N[X]$ denotes the set of vertices either belonging to X or adjacent some member of X. The largest cardinality of an irredundant set in G is denoted $IR(G)$. Independent sets are irredundant, so we always have $\beta_0(G) \leq IR(G)$. It is known that the corresponding decision problem, IRREDUNDANT SET, is NP-complete [3]. For technical reasons (to be explained later), we will assume that $k \geq 6$.

Lemma 2 *C is not-all-equal satisfiable if and only if $IR(G_k(C)) \geq 3(k-2)n + 2n$.*

The "only if" half of this lemma follows from $\beta_0(G_k(C)) \leq IR(G_k(C))$ and Lemma 1. Since the "if" part is somewhat complicated, we postpone it until Section 5. This lemma implies

Theorem 2 *For each $k \geq 6$, IRREDUNDANT SET is NP-complete for the class of k-regular graphs.*

Another well-studied graph parameter is $\Gamma(G)$, the largest cardinality of a minimal dominating set. Its related decision problem is also known to be NP-complete [1]. It is well known that for any graph G,

$$\beta_0(G) \leq \Gamma(G) \leq IR(G). \tag{1}$$

Inequality (1), together with Lemma 1 and Lemma 2, are enough to establish

Lemma 3 *C is not-all-equal satisfiable if and only if $\Gamma(G_k(C)) \geq 3(k-2)n + 2n$.*

Proof: If C has a not-all-equal truth assignment, then by Lemma 1 we have $\beta_0(G_k(C)) \geq 3(k-2)n+2n$, so by (1) we have $\Gamma(G_k(C)) \geq 3(k-2)n+2n$. Conversely, if $\Gamma(G_k(C)) \geq 3(k-2)n+2n$, then by (1) we have $IR(G_k(C)) \geq 3(k-2)n+2n$, and so by Lemma 2, C has a not-all-equal truth assignment.

Theorem 3 *For each $k \geq 6$, the decision problem for Γ is NP-complete for the class of k-regular graphs.*

Note that the conclusion of Theorem 3 will hold for *any* graph parameter sandwiched between β_0 and IR. Another such parameter is called the *upper fractional domination number*, denoted Γ_f, which is studied in [1].

5 Remaining Details

We now outline the proof of Lemma 2. Let S be an irredundant set in $G_k(C)$, of maximum size, having cardinality $\geq 3(k-2)n+2n$. We must show that C has a not-all-equal truth assignment. By Lemma 1, it suffices to show that S is independent.

We will assume that each fence has an odd number of gates. That is, each variable x occurs (as x or $\bar{x}$) an odd number of times. This causes no loss in generality since, if x occurs an even number of times, we may add the two clauses $x \vee v \vee \bar{v}$ and $v \vee x \vee \bar{x}$, for some unused variable v. Both x and v will then occur an odd number of times, and C will have a not-all-equal truth assignment if and only if the modified set of clauses is not-all-equal satisfiable.

In this section, some of our lemmas, in particular Lemma 6, seem to require that $k \geq 4$. And a close look at Lemma 7 seems to require $k \geq 6$. *Therefore we now assume $k \geq 6$ without further mention.*

The following lemma is an interesting observation about cycles having length $\equiv 2 \bmod 4$.

Lemma 4 *Let T be a cycle of length $2(2j+1) \geq 6$, and let U be a maximum irredundant set in T. Then U must be independent.*

Proof: Since T has even length, it has an independent set of size $\frac{|T|}{2}$, and therefore an irredundant set of size $\geq \frac{|T|}{2}$. It suffices to show that any irredundant set of size $\geq \frac{|T|}{2}$ is independent. We induct on j. When $j=1$, T is a 6-cycle and the result is easy to verify. Now assume the hypothesis for all cycles having length $\equiv 2 \bmod 4$, but smaller than the length of T, and let U be an irredundant set containing at least half the vertices of T. By way of contradiction, assume there exist adjacent vertices x and y in U. Consider the path r, u, x, y, v, s. By irredundance, none of r, u, v, s can be in U. Now form the cycle T' by deleting u, x, y, v and joining r and s. The length of T' is $\equiv 2 \bmod 4$, $U' = U \cap T'$ is irredundant in T', and contains at least half the members of T'. By the induction hypothesis, U' is independent. But this implies that the elements in U' alternate in T'. However, neither r nor s is in U'; this is a contradiction.

Lemma 5 *If a gate has three or more vertices in an irredundant set S, then no two of them are adjacent.*

Proof: Suppose there exist $a, b, c \in S$ in a gate such that a and b are adjacent, and b and c have the same parity. It is easy to see that b will not have a private neighbor.

Lemma 6 *Each gate of $G_k(C)$ contains at most $k-2$ vertices in any irredundant set S.*

Proof: If not, two vertices in S would be adjacent, contradicting Lemma 5.

Lemma 7 *Each component of $G_k(C)$, shown in Figure 2, contains at most $2 + 3(k-2)$ vertices in any irredundant set S. Equality is possible only if each triangle has exactly one vertex in S.*

Proof (Sketch): From Lemma 6, each of the three gates contains at most $k-2$ members. Hence it suffices to show that whenever any triangle contains more than one vertex in S, the component has strictly fewer than $2 + 3(k-2)$ vertices in S. An exhaustive case-by-case analysis shows that this is the case. Instead of providing these tedious details we provide the following helpful comments. If a triangle has two vertices in S then each will require a private neighbor in an adjacent gate. This eliminates at least one vertex from each of two gates. Note also that when a gate is adjacent to two vertices in S in (different) triangles C_i and D_i, then the gate can have only two vertices in S, on opposite sides of the gate. One pathological situation occurs when both triangles have three members. Then each gate can have at most two vertices in S, so there are at most 12 vertices in S in the component. Since we require $12 < 3(k-2) + 2$, it is necessary to have $k \geq 6$.

Lemma 8 *Each gate contains exactly $k-2$ vertices in any maximum irredundant set S, each triangle contains exactly one member, and no two vertices in S in a gate are adjacent.*

Proof: The vertices of $G_k(C)$ can be partitioned into exactly n components, like that in Figure 2. By Lemma 7 each component has at most $3(k-2)+2$

vertices in S. Since we are assuming $| S | \geq 3(k-2)n+2n$, we must have exactly $3(k-2)+2$ vertices in each component. By Lemma 6, each gate must each have $(k-2)$ vertices, and by Lemma 5 no two vertices in the same gate are adjacent. By Lemma 7, each triangle each has one vertex in S.

Lemma 9 *The private neighbor y of a vertex x in a maximum irredundant set is never on the opposite side of the same $K_{k-2,k-2}$ component.*

Proof: By Lemma 8, either y or its neighbor y' in the same gate, must be in S.

Lemma 10 *In any maximum irredundant set S in $G_k(C)$, the private neighbor of a vertex in a gate is either itself or its neighbor in the same gate.*

Proof: Since each triangle contains a vertex in S, the private neighbor must be in the same fence. But by Lemma 9, it cannot be outside of the same gate.

Let T be a cycle induced by all vertices in the *same level* of a fence. By Lemma 10, each vertex in $U = S \cap T$ has a private neighbor in T, and so is irredundant in T. We also know U has cardinality half the length of T since (by Lemma 8) exactly one element from each gate is selected. Moreover, the length of T is $\equiv 2 \bmod 4$. By Lemma 4, we must have

Lemma 11 *The vertices in U are independent, and therefore alternate in T.*

This implies that if $v \in U$, the neighbor of v in its gate is *not* a private neighbor. So by Lemma 10, its private neighbor can only be itself. We may apply the same argument to each of the other cycles, and we see that every member of S in a fence has itself as a private neighbor. At this point, we can stop; we have shown S is independent.

We leave open the question regarding Theorems 2 and 3 when $k = 3, 4, 5$. We conjecture that these problems are also NP-complete, but it appears that a new construction will be required to show this.

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