Note on the Union-Closed Sets Conjecture

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ABSTRACT. We establish an improved bound for the Union-Closed Sets Conjecture.

1 Introduction

A union-closed family of sets is defined as a non-empty finite collection of distinct finite sets closed under union. The following conjecture is referred to as the Union-Closed Sets Conjecture [5,7].

Conjecture. *Let $\mathcal{F} = \{A_1, A_2, \dots, A_n\}$ be a family of union-closed sets. Then there exists an element which belongs to at least $\lceil n/2 \rceil$ of the A_i's, where*

$$\lceil n/2 \rceil = \begin{cases} n/2, & \text{if } n \text{ is even,} \\ (n+1)/2, & \text{if } n \text{ is odd.} \end{cases} \tag{1.1}$$

From the definition, the union A of all sets of $\mathcal{F}$ is in $\mathcal{F}$, and $\mathcal{F} \subseteq \mathcal{P}(A)$, the power set of A. We call $\mathcal{F}$ a union-closed family over A. Let $m = |A|$ and $n = |\mathcal{F}|$. Then $\mathcal{F}$ can be denoted as $\mathcal{F} = \{A_1, A_2, \dots, A_n\}$ with $A_n = A = \{1, 2, \dots, m\}$ and $|A_1| \leq |A_2| \leq \dots |A_n|$.

The origin of the conjecture traced back, according to [4], to P. Frankl [6, p 525]. It was also recorded in [8, p. 161 and p. 186] as an open problem. D.G. Sarvate and J.C. Renaud initiated the research of the Conjecture ([2,3]) and confirmed it for $n \leq 18$, or for $|A_1| \leq 2$. In [1], B. Poonen discussed a number of equivalent conjectures and proved the Conjecture for $m \leq 7$ or $n \leq 28$.

Let $\mathcal{F}_i = \{S \in \mathcal{F} | i \in S\}$. Then $\mathcal{F}_i = \mathcal{F}_j$ defines an equivalence relation $i \sim j$ on A. We call the equivalence class $B \subseteq A$ a block. Now, the Conjecture may be formulated equivalently as: There exists an element $i \in A$ such that $|\mathcal{F}_i| \geq \lceil n/2 \rceil$.

In this note, we shall confirm the conjecture for $m \leq 8$ or $n \leq 32$ or $n \geq 2^m - 12(3/2)^{\lfloor m/3 \rfloor} - 1/2\binom{m}{3} - \binom{m}{2} - (5/4)m + 44.5$.

2 Preliminaries

The following two lemmas may be found in [1].

Lemma 1.1. *For each n, it suffices to consider families $\mathcal{F}$ such that $\emptyset \in \mathcal{F}$.*

Lemma 1.2. *For each n, it suffices to consider families for which all the blocks are singletons.*

In what follows, we consider only union-closed families containing the empty set and having singleton blocks.

Lemma 1.3. *Let $\mathcal{F}$ be a union-closed family defined above. If*

$$\sum_{j=1}^{m}(j - m/2)n_j > 0,$$

where n_j is the number of sets of $\mathcal{F}$ of cardinality j. Then Conjecture holds for $\mathcal{F}$.

Proof: Assume to the contrary that $|\mathcal{F}_i| \leq \lceil n/2 \rceil - 1$, for each $i \in A$, where

$$\lceil n/2 \rceil = \begin{cases} n/2, & \text{if } n \text{ is even,} \\ (n+1)/2, & \text{if } n \text{ is odd.} \end{cases}$$

Then

$$\sum_{j=1}^{m} jn_j = \sum_{i=1}^{n} |A_i| = \sum_{i \in A} |\mathcal{F}_i| \leq m(\lceil n/2 \rceil - 1).$$

Since $n_0 = n_m = 1$ and $\sum_{j=1}^{m-1} n_j = n - 2$.

If n is even, then $\sum_{j=1}^{m} jn_j \leq m(n/2 - 1) = m/2(\sum_{j=1}^{m} n_j - 1)$, and $\sum_{j=1}^{m}(j - m/2)n_j \leq -m/2$, a contradiction.

If n is odd, then $\sum_{j=1}^{m} jn_j \leq m((n+1)/2 - 1) = m/2\sum_{j=1}^{m} n_j$, hence $\sum_{j=1}^{m}(j - m/2)n_j \leq 0$, a contradiction. □

For any integer k with $3 \leq k < n$. Define L_k to be the least integer t such that if a union-closed family $\mathcal{F}$ contains all k-subsets of a t-set B, then there exists an element of B which appears in at least $\lceil |\mathcal{F}|/2 \rceil$ members of $\mathcal{F}$. The well-definedness of L_k may be deduced from the following proposition.

Proposition 1.4. *For any $k \geq 3$, $L_k \leq 2k - 2$.*

For the proof of Proposition 1.4, we need some lemmas.

Lemma 1.5. *Let A, B, C be three sets. Then $A \cup B = A \cup C$ if and only if $B \Delta C \subseteq A$, where 'Δ' denotes the symmetric difference.*

Lemma 1.6. *Let $\mathcal{F}$ be a union-closed family over A, and let B be a fixed nonempty subset of A. For any $C \subset B$, Let $\mathcal{F}_c(B)$ denote the family of sets T of $\mathcal{F}$ with $T \cap B = C$. Suppose $D \subset C \subset B$ and $C \in \mathcal{F}$. Then $|\mathcal{F}_D(B)| \leq |\mathcal{F}_C(B)|$.*

Proof: Let $\mathcal{F}_D(B) = \{T_1, \ldots, T_r\}$. Then $D \subseteq T_i$, $i = 1, 2, \ldots, r$. We first show that $T_1 \cup C, \ldots, T_r \cup C$ are pairwise distinct.

Assume to the contrary, that $T_i \cup C = T_j \cup C$ for some $1 \leq i < j \leq r$. Since $C = D \cup (C - D), (T_i \cup D) \cup (C - D) = (T_j \cup D) \cup (C - D)$. Thus $T_i \cup (C - D) = T_j \cup (C - D)$. It follows from Lemma 1.5 that $T_i \Delta T_j = (T_i - T_j) \cup (T_j - T_i) \subset C - D \subset B - D$. Therefore

$$\begin{aligned} T_i \Delta T_j &\subseteq (B - D) \cap (T_i \cup T_j) \\ &= ((B - D) \cap T_i) \cup ((B - D) \cap T_j) \\ &= \emptyset. \end{aligned}$$

This yields $T_i = T_j$, a contradiction.

Now, the result follows since $T_i \cup C \in \mathcal{F}$ and $(T_i \cup C) \cap B = (T_i \cap B) \cap (C \cap B) = D \cup C = C$. □

We also need a lemma due to de Brujin.

Lemma 1.7. [9, Th.3.1.1] *The subsets of an n-element set can be expressed as a disjoint union of symmetric chains such as $A_1 \subset A_2 \subset \ldots A_h$, where $|A_{i+1}| = |A_i| + 1$ and $|A_1| + |A_h| = n$.*

Proof of Proposition 1.4: Without loss of generality, let $\mathcal{F}$ be a union-closed family over $A = \{1, 2, \ldots, m\}$ containing all k-subsets of $B = \{1, 2, \ldots, 2k-2\}$, where $k \geq 3$, and $|\mathcal{F}| = n$. Clearly, any subset $C \subset B$ with $|C| \geq k$ is in $\mathcal{F}$, and $\sum_{T \subseteq B} N_T(B) = |\mathcal{F}| = n$.

Assume to the contrary, that for each $i \in B = \{1, 2, \ldots, 2k-2\}$, $|\mathcal{F}_i| \leq \lceil n/2 \rceil - 1$. Since for $T \neq S \subseteq B$, $\mathcal{F}_T(B) \cap \mathcal{F}_S(B) = \emptyset$, then

$$\sum_{\substack{i \in T \\ T \subseteq B}} |\mathcal{F}_T(B)| = |\mathcal{F}_i| \leq \lceil n/2 \rceil - 1.$$

Thus

$$\sum_{i \in B} \sum_{i \in T \subseteq B} |\mathcal{F}_T(B)| \leq (\lceil n/2 \rceil - 1)(2k - 2). \tag{1.2}$$

On the other hand, we have

$$\begin{aligned}
&\sum_{i\in B}\sum_{i\in T\subseteq B}|\mathcal{F}_T(B)| \\
&=\sum_{\emptyset\neq T\subseteq B}|T|\cdot|\mathcal{F}_T(B)| \\
&=(k-1)\sum_{|T|=k-1}|\mathcal{F}_T(B)|+(2k-2)|\mathcal{F}_T(B)| \\
&+\sum_{j=1}^{k-2}\left(j\sum_{|T|=j}|\mathcal{F}_T(B)|+(2k-2-j)\sum_{|T|=2k-2-j}|\mathcal{F}_T(B)|\right).
\end{aligned}$$

Put $t=\binom{2k-2}{j}=\binom{2k-2}{2k-2-j}$, and let $P_1,P_2,\ldots,P_t$ be all j-subsets of B, and $Q_1,Q_2,\ldots,Q_t$ be all $(2k-2-j)$-subsets of B, where $1\le j\le k-2$.

By Lemma 1.7, since all members of 2^B can be arranged in $\binom{2k-2}{k-1}$ symmetric chains, we may assume (by rearranging the subscripts if necessary) that

$$P_1\subseteq Q_1,\ldots,P_t\subseteq Q_t.$$

Since all Q_i's are in $\mathcal{F}$, by Lemma 1.2, $|\mathcal{F}_{P_i}(B)|\le|\mathcal{F}_{Q_i}(B)|$, for $i=1,2,\ldots,t$. So we have, for $1\le j\le k-2$,

$$\begin{aligned}
&j\sum_{|T|=j}|\mathcal{F}_T(B)|+(2k-j-2)\sum_{|T|=2k-2-j}|\mathcal{F}_T(B)| \\
&=\sum_{i=1}^{t}(j|\mathcal{F}_{P_i}(B)|+(2k-j-2)|\mathcal{F}_{Q_i}(B)|) \\
&\ge\sum_{i=1}^{t}(j|\mathcal{F}_{P_i}(B)|+(k-j-1)|\mathcal{F}_{P_i}(B)|+(k-1)|\mathcal{F}_{Q_i}(B)|) \\
&=\sum_{i=1}^{t}(k-1)(|\mathcal{F}_{P_i}(B)|+|\mathcal{F}_{Q_i}(B)|) \\
&=(k-1)\left(\sum_{|T|=j}|\mathcal{F}_T(B)|+\sum_{|T|=2k-j-2}|\mathcal{F}_T(B)|\right).
\end{aligned}$$

Therefore, we have

$$\sum_{i\in B}\sum_{i\in T\subseteq B}|\mathcal{F}_T(B)|$$

$$\geq (k-1)\sum_{j=1}^{2k-3}\sum_{|T|=j}|\mathcal{F}_T(B)|+(2k-2)|\mathcal{F}_B(B)|$$

$$\geq (k-1)\sum_{j=1}^{2k-3}\sum_{|T|=j}|\mathcal{F}_T(B)|+(k-1)N_B(B)+(k-1)|\mathcal{F}_\emptyset(B)|$$

$$=(k-1)(\sum_{j=0}^{2k-2}\sum_{|T|=j}|\mathcal{F}_T(B)|)$$

$$=(k-1)\sum_{T\subseteq B}|\mathcal{F}_T(B)|$$

$$=(k-1)n.$$

This contradicts (1.2). □

For any $l \geq L_k$ ($k \geq 3$), define $f_k(l)$ to be the least integer t such that for any family $\mathcal{F}'$ of t k-subsets of an l-set B and any union-closed family $\mathcal{F} \supseteq \mathcal{F}'$, there exists an element i of B which appears in at least $\lceil|\mathcal{F}|/2\rceil$ members of $\mathcal{F}$.

Lemma 1.8. *For any $k \geq 3$ and any $l \geq L_k$, we have*

$$f_k(l+1) \leq \frac{l+1}{l+1-k}(f_k(l)-1)+1.$$

Proof: Let $\mathcal{F}'$ be a family consisting of $f_k(l+1)-1$ k-subsets of $\{1,2,\ldots,l+1\}$ with the property that for every l-subset B, at most $(f_k(l)-1)$ k-subsets are in $\mathcal{F}'$. We consider the sum $N=\sum_{i=1}^{l+1}N_i$, where N_i denotes the number of elements in $\mathcal{F}'$ which does not contain i. Clearly, $N_i \leq f_k(l)-1$. So we have $N \leq (l+1)(f_k(l)-1)$. It is easy to see that every element of $\mathcal{F}'$ has been counted in exactly $l+1-k$ times in the sum N, hence $N=(l+1-k)(f_k(k+1)-1)$. So

$$(l+1-k)(f_k(k+1)-1) \leq (l+l)(f_k(l)-1),$$

and the result follows. □

Lemma 1.9. $f_3(4) \leq 3$, $f_3(5) \leq 6$, $f_3(6) \leq 11$, $f_3(7) \leq 18$, $f_3(8) \leq 28$.

Proof: By a result of [1] (Corollary 4), we have $f_3(4) \leq 3$, and the others follow from Lemma 1.8. □

3 Main results

Let $\mathcal{F}$ be a union-closed family over A. For $S \subseteq A$, let $\overline{\mathcal{F}}_S$ be the subfamily of $\mathcal{F}$ of sets disjoint from S. Then $\overline{\mathcal{F}}_S$ is also a union-closed family or $\{\emptyset\}$. Let M_S be the largest set of $\overline{\mathcal{F}}_S$. If $\alpha, \beta \in A$ and $M_{\{\alpha\}} = M_{\{\beta\}}$, then α, β are in the same block, so $\alpha = \beta$.

Theorem 3.1. *Conjecture holds for $m = 8$.*

Proof: Let $\mathcal{F}$ be a union-closed family over $A = \{1,2,3,4,5,6,7,8\}$. Assume to the contrary that Conjecture fails for $\mathcal{F}$, then Lemma 1.3 implies that

$$\sum_{j=1}^{7}(j-4)n_j \leq -4,$$

by a result of [2] (Th.2), we have $n_1 = n_2 = 0$, so

$$-n_3 + n_5 + 2n_6 + 3n_7 \leq -4. \tag{3.1}$$

By Lemma 1.9, $n_3 \leq 27$. Thus

$$n_5 + 2n_6 + 3n_7 \leq 23. \tag{3.2}$$

We assert that $|M_{\{i\}}| \geq 6$ for every $i = 1, 2, \ldots, 8$. Otherwise, $|M_{\{i\}}| \leq 5$ for some $1 \leq i \leq 8$ Since $f_3(5) \leq 6$, we have $|M_{\{i\}}| \leq f_3(5) - 1 + \binom{5}{4} + 2 \leq 12$. Therefore $|\mathcal{F}| \leq 2|M_{\{i\}}| - 1 \leq 23$, by a result of [1] (Th.3), Conjecture holds for $\mathcal{F}$, a contradiction. This proves the assertation.

Since all $M_{\{i\}}$ are distinct, so we have

$$2n_6 + 3n_7 \geq 16, \tag{3.3}$$

it follows from (3.2) that $n_5 \leq 7$.

Since $f_3(5) \leq 6$, $f_3(4) \leq 3$, by enumerating the sum $\sum_P$ {the number of 3-subset of $\mathcal{F}$ containing in P}, where P runs over all 5-subset of A, we have

$$5n_5 + 2(\binom{8}{5} - n_5) \geq \binom{5}{2} n_3.$$

So $n_3 \leq 0.3n_5 + 11.2$. Since $n_5 \leq 7$, we have $n_3 \leq 2.1 + 11.2 = 13.3$. So $n_3 \leq 13$ and by (3.1) we have $n_5 + 2n_6 + 3n_7 \leq 9$. This contradicts (3.3). □

For a union-closed family $\mathcal{F}$ over A with $|\mathcal{F}| = n$. Since $\overline{\mathcal{F}}_A = \{\emptyset\}$, let $K = \{\alpha_1, \alpha_1, \ldots, \alpha_k\}$ be the smallest subset of A such that $\mathcal{F}_K = \{\emptyset\}$. From the discussion in [1, pp. 261–262], we may assume that each $\mathcal{F}_{\{\alpha_i\}}$ is minimal, i.e., if $\beta \in A$ and $\mathcal{F}_{\{\beta\}} \subseteq \mathcal{F}_{\{\alpha\}}$. Then $\beta = \alpha$. We may assume $K = \{1, 2, \ldots, k\}$ without loss of generality. For $0 \leq j \leq k$, let S_j be the number of sets of $\mathcal{F}$ which contain exactly j elements of K. Clearly, $S_0 = 1$.

Theorem 3.2. *If $\sum_{j=0}^{k}(j-k/2)S_j > -k/2$, then Conjecture holds for $\mathcal{F}$.*

Proof: Assume to the contrary that for each i, with $1 \le i \le k$, $|\mathcal{F}_i| \le \lceil n/2 \rceil - 1$. Then $\sum_{i=1}^{k} |\mathcal{F}_i| \le (\lceil n/2 \rceil - 1)k \le (n-1)k/2$. Note that

$$0 \cdot S_0 + 1 \cdot S_1 + \dots kS_k = \sum_{i=1}^{k} |\mathcal{F}_i|$$

and $S_0 + S_1 + \cdots + S_k = n$.

We have

$$\sum_{i=1}^{k}(j-k/2)S_j \le -k/2$$

a contradiction. This completes the proof. □

Lemma 3.3. [1] *For $0 \le j \le k$, $S_j \ge \binom{k}{j}$, $S_k \ge m-k+1$ and $S_0 = 1$.*

Theorem 3.4. *Conjecture holds for $n \le 32$.*

Proof: It suffices to consider the case for $m \ge 9$ and $n \le 31$. The case for $k \le 2$ is trivial.

If $k=3$, by Lemma 3.3, $S_3 \ge m-3+1 \ge 7$ and $S_1 = n-(S_0+S_2+S_3) \le 31-(1+\binom{3}{2}+7) = 20$. So

$$\begin{aligned}\sum_{j=0}^{k}(j-k/2)S_j &= (-3/2)S_0 + (-1/2)S_1 + (1/2)S_2 + (3/2)S_3 \\ &\ge (-3/2) + (-1/2)20 + (1/2)\binom{3}{2} + (3/2)7 \\ &= 1/2 > -3/2.\end{aligned}$$

And we are done by Theorem 3.2.

If $k=4$, then $S_4 \ge m-4+1 \ge 6$ and $S_1 = n-(S_0+S_2+S_3+S_4) \le 31-(1+\binom{4}{2}+\binom{4}{3}+6) = 14$. So $\sum_{j=0}^{k}(j-k/2)S_j = (-2)S_0 + (-1)S_1 + 0 \cdot S_2 + S_3 + 2S_4 \ge (-2)\cdot 1 + (-1)14 + \binom{4}{3} + 2.6 = 0 > -2$, again we are done by Theorem 3.2.

If $k \ge 5$, then

$$n \ge \binom{k}{0} + \binom{k}{1} + \dots \binom{k}{k} = 2^k > 31,$$

a contradiction. □

Theorem 3.5. *Let $\mathcal{F}'$ be a union-closed family over A with $|\mathcal{F}'| = n$ and $|A| = m \ge 12$. Suppose $n \ge 2^m - 2(3/2)^{\lceil m/3 \rceil} - 1/2\binom{m}{3} - \binom{m}{2} - 5m/3 + 44.5$.*

Then for any union-closed family $\mathcal{F} \supseteq \mathcal{F}'$, there exists an element α of A such that

$$|\mathcal{F}_\alpha| \geq \lceil |\mathcal{F}|/2 \rceil .$$

Proof: Assume to the contrary, then by Theorem 3.2, for any $3 \leq k \leq [m/2]$, $\mathcal{F}'$ contains at most $(f_k(m)-1)$ k-subset of A. And by a result of [2, Th.2], we have

$$n \leq \sum_{l=[m/2]+1}^{m} \binom{m}{l} + \sum_{k=3}^{[m/2]} (f_k(m)-1)+1,$$

but by Lemma 1.8 and Lemma 1.9 we have $f_3(m)-1 \leq 1/2\binom{m}{3}$, and

$$\begin{aligned} f_k(m)-1 &\leq \frac{m}{m-k}\frac{m-1}{m-1-k}\cdots\frac{2k-1}{k-1}(\binom{2k-2}{k}-1) \\ &= \binom{m}{k} - \frac{m}{m-k}\frac{m-1}{m-1-k}\cdots\frac{2k-1}{k-1}. \end{aligned}$$

So for $k \leq [m/3]$,

$$f_k(m)-1 \leq \binom{m}{k} - \frac{3k}{2k}\frac{3k-1}{2k-1}\cdots\frac{2k}{k}\frac{2k-1}{k-1} < \binom{m}{k} - \binom{3}{2}^k 4.$$

And for $[m/3]+1 \leq k \leq [m/2]$,

$$f_k(m)-1 \leq \binom{m}{k} - \frac{2k}{k}\frac{2k-1}{k-1} < \binom{m}{k} - 4.$$

Therefore,

$$\begin{aligned} n &< \sum_{l=[m/2]+1}^{m} \binom{m}{l} + \sum_{k=4}^{[m/3]} (\binom{m}{k} - 4(3/2)^k) + \sum_{k=[m/3]+1}^{[m/2]} (\binom{m}{k} - 4) + 1/2\binom{m}{3} \\ &= 2^m - 8((3/2)^{[m/3]+1}-1) + 32.5 - 4([m/2]-[m/3]) - 1/2\binom{m}{3} - \binom{m}{2} - m \\ &\leq 2^m - 12(3/2)^{[m/3]} - 1/2\binom{m}{3} - \binom{m}{2} - (5/3)m + 44.5. \end{aligned}$$

A contradiction. This completes the proof. □

The following corollary indicates that if the size of a union-closed family is large enough with respect to the size of its largest set, then Conjecture holds.

Corollary 3.6. *Let $\mathcal{F}$ be a union-closed family with $m \geq 12$ and $n \geq 2^m - 12(3/2)^{[m/3]} - 1/2\binom{m}{3} - \binom{m}{2} - (5/3)m + 44.5$. Then Conjecture holds for $\mathcal{F}$.*

References

[1] B. Poonen, Union-closed families, *J. Comb. Theory*, **A59** (1992), 253–268.

[2] D.G. Sarvate and J-C. Renaud, On the union-closed sets conjecture, *Ars Combinatoria*, **27** (1989), 149–154.

[3] D.G. Sarvate and J-C. Renaud, Improved bounds for the union-closed sets conjecture, *Ars Combinatoria*, **29** (1990), 181–185.

[4] R.M. Nortona and D.G. Sarvate, A note of the union-closed sets conjecture, *J. Australia. Math. Soc.*, **A55** (1993), 411–413.

[5] A much traveled conjecture, *Australia. Math. Soc. Gaz.* **14** (1987), 63.

[6] I Rival Eds., *Graphs and Order*, Proceedings of the NATO advanced study institute on graphs and order, Reidel, Dordrecht, 1984.

[7] P. Winkler, Union-closed sets conjecture, *Australia. Math. Soc. Gaz.* **14** (1987), 99.

[8] R.P. Stanley, *Enumerative Combinatorics*, Wadsworth, Monterey, 1986.

[9] I. Anderson, *Combinatorics of Finite Sets*, Oxford University Press, 1987.