A Lower Bound for the Ramsey Multiplicity of K_4

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Abstract

The Ramsey multiplicity $R(G)$ of a graph G is defined as the smallest number of monochromatic copies of G in any two-coloring of the edges of $K_{r(G)}$, where $r(G)$ is the Ramsey number of G. Here, we prove that $R(K_4) \geq 4$.

In the following, we consider two-colorings of the edges of the complete graph K_n, short *colorings*, the colors used being red and blue.

Let G be a graph. The *Ramsey number* $r(G)$ is the smallest integer n such that in each coloring of K_n a monochromatic copy of G occurs. Harary and Prins [5] introduced the notion of the *Ramsey multiplicity* $R(G)$, the smallest number of monochromatic copies of G in any coloring of $K_{r(G)}$. In their table of the Ramsey multiplicities of all graphs with at most four vertices, the only missing values were $R(K_4-e)$, which was later determined by Schwenk (cited in [4]), and $R(K_4)$. Up to now, the best upper bound for $R(K_4)$ is due to Exoo [1], who showed that $R(K_4) \leq 9$ by giving a coloring of K_{18} with exactly 9 monochromatic K_4's, since $r(K_4) = 18$ (e.g., see [3]).

A nontrivial lower bound for $R(K_4)$ has so far apparently not been available. In this paper, we prove that $R(K_4) \geq 4$.

We will make use of the following result: Define the *multiplicity* $M(G;n)$ of a graph G and a positive integer n as the smallest number of monochromatic copies of G in any coloring of K_n. A well-known theorem of Goodman [2], stated here in a form due to Schwenk [6], says:

$$M(K_3;n) = \binom{n}{3} - \left\lfloor \frac{n}{2} \left\lfloor \left(\frac{n-1}{2}\right)^2 \right\rfloor \right\rfloor . \qquad (1)$$

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Theorem. $R(K_4) \geq 4$.

Proof. Consider an arbitrary coloring of K_{18}. For an edge ij, let $t = t_{ij}$ be the number of monochromatic triangles that contain ij. Then it follows from (1):

$$\sum_{ij} t_{ij} \geq 3\,M(K_3; 18) = 504. \tag{2}$$

We will deduce a contradiction from the following assumption: The given coloring of K_{18} contains at most three monochromatic K_4's.

Let ij be an edge with $t_{ij} \geq 4$, and let ij form a monochromatic triangle with each of four vertices $a_1, \ldots, a_4$.

If one of the edges $a_k a_l$, $1 \leq k, l \leq 4$, has the same color as ij, then i, j, a_k, and a_l form a monochromatic K_4. Let us call it a "(monochromatic) K_4 of type 1" and ij a "type 1 edge". If, on the other hand, all edges $a_k a_l$ do not have the same color as ij, then $a_1, \ldots, a_4$ form a monochromatic K_4. Let us call it a "(monochromatic) K_4 of type 2" and ij a "type 2 edge". Fig. 1 shows a red type 1 and a red type 2 edge ij, where solid lines represent red edges and dashed lines blue edges:

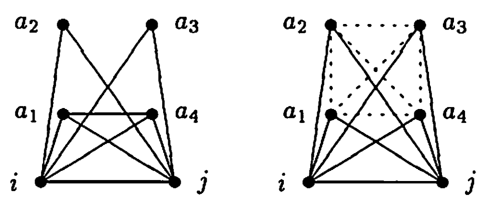


Fig. 1

Thus each edge ij with $t_{ij} \geq 4$ is a type 1 or a type 2 edge and can therefore be assigned a monochromatic K_4 of type 1 or type 2 respectively.

The case that $a_1, \ldots, a_4$ form a monochromatic K_4 in the same color as ij has is impossible, because the six vertices would then form a monochromatic K_6 (containing 15 monochromatic K_4's).

In the following, we assume that the coloring of K_{18} contains exactly three monochromatic K_4's; denote them by K_4^1, K_4^2, and K_4^3. However, all arguments still hold or are dispensable, if there are only one or two monochromatic K_4's. (There has to be at least one since $r(K_4) = 18$.) Let M_1, M_2, and M_3 denote the sets of type 2 edges to which K_4^1, K_4^2, and K_4^3 respectively are assigned as monochromatic K_4's of type 2.

Case 1. There are only edges with $t \leq 4$.

As a K_{18} contains 153 edges, then by (2) there are at least 45 edges with $t = 4$.

Case 1.1. Every two of the three monochromatic K_4's have at most two vertices in common.

Let ij be an edge in one of the monochromatic K_4's, say in K_4^1. W.l.o.g. let ij and hence K_4^1 be red. If ij is a type 1 edge, then $t_{ij} \geq 4$ and so ij has to be contained, apart from the two triangles within K_4^1, in two additional red triangles, say ija_{ij} and ijb_{ij}. Then a_{ij} and b_{ij} cannot be incident with any edge in M_1, since the endvertices of these edges are connected to all four vertices of K_4^1 by blue edges.

Furthermore, in this way different type 1 edges ij and $i'j'$ in K_4^1 are assigned different vertices a_{ij}, b_{ij}, $a_{i'j'}$, and $b_{i'j'}$: Otherwise there would be either a red K_5 and hence five red K_4's (if ij and $i'j'$ are disjoint) or two red K_4's which have three vertices in common (if ij and $i'j'$ are not disjoint):

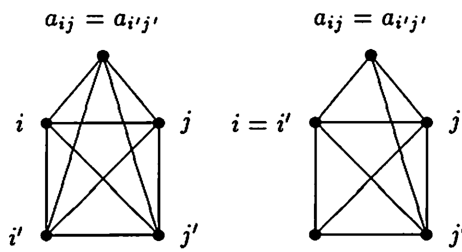


Fig. 2

If all edges in K_4^1, K_4^2, and K_4^3 were type 1 edges, then the edges in M_1, M_2, and M_3 would in each case be incident with at most $18 - 4 - 2 \cdot 6 = 2$ vertices, i.e. $|M_1|, |M_2|, |M_3| \leq 1$. This is a contradiction, since there are at least $45 > 3 \cdot 6 + 3 \cdot 1$ edges with $t = 4$. So at most 17 of the maximum 18 edges in K_4^1, K_4^2, and K_4^3 can be type 1 edges. It follows that $|M_1 \cup M_2 \cup M_3| \geq 45 - 17 = 28$. By the pigeonhole principle one set, say M_1, contains at least ten edges. If K_4^1 is red, then the edges in M_1 are blue.

Among the edges in M_1 there can be no blue triangle, since it would form a blue K_4 with each of the four vertices of K_4^1. So by Turan's theorem the edges in M_1 form a graph with at least seven vertices $b_1, \ldots, b_7$ (a triangle-free graph with six vertices can have at most nine edges; the extremal graph is a $K_{3,3}$).

Apart from the four vertices of K_4^1 and $b_1, \ldots, b_7$, there are seven more vertices $c_1, \ldots, c_7$. Each of them is connected to the vertices of K_4^1 by at least two blue edges, because otherwise there would be a red K_4 which has three vertices in common with K_4^1. So $c_1, \ldots, c_7$ are altogether connected by at least 14 blue edges to K_4^1. Consequently, there is a vertex a of K_4^1 which has at least four blue neighbors c_i, say $c_1, \ldots, c_4$. The set $N = \{b_1, \ldots, b_7, c_1, \ldots, c_4\}$ then contains eleven blue neighbors of a.

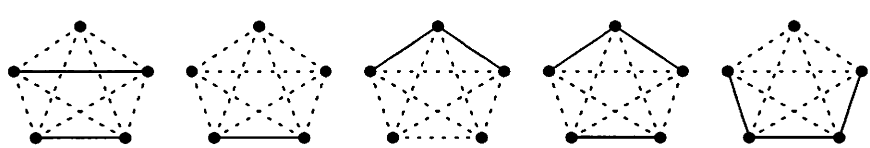
If in N there are at least three vertices of K_4^2 or at least three vertices of K_4^3 respectively, then in each case one such vertex is removed from N. Then we still have $|N| \geq 9$. Since $r(K_3, K_4) = 9$ (see [3]), in the coloring of the $K_{|N|}$ spanned by the vertices in N there is a blue triangle (which together with a forms a blue K_4) or a red K_4. As in both cases the monochromatic K_4 is different from K_4^2 and K_4^3, we obtain a contradiction.

Case 1.2: There are two monochromatic K_4's which have three vertices in common.

Then K_4^1, K_4^2, and K_4^3 together have at most 15 edges, so $|M_1 \cup M_2 \cup M_3| \geq 45 - 15 = 30$. Just like in Case 1.1 it follows that the edges in, say, M_1 form a graph with at least seven vertices $b_1, \ldots, b_7$. Let $c_1, \ldots, c_7$ be defined as above, and let K_4^1 again be red. Since there are no more than two red K_4's different from K_4^1, five of the seven vertices c_i are connected by at least two blue edges and two vertices c_i by at least one blue edge to the vertices of K_4^1. In this way, we find three blue neighbors c_i, say c_1, c_2, and c_3, of a vertex a of K_4^1. The set $N = \{b_1, \ldots, b_7, c_1, c_2, c_3\}$ then contains ten blue neighbors of a. Since there are two monochromatic K_4's having three vertices in common, we obtain the same contradiction as in Case 1.1 after removing at most one vertex from N.

Case 2: There is an edge ij with $t_{ij} = 5$, but no edge with $t \geq 6$.

W.l.o.g. let ij be red, and let ij form a red triangle with each of five vertices $a_1, \ldots, a_5$. Since only three monochromatic K_4's exist, there are the following five possibilities for the edges between $a_1, \ldots, a_5$:



Type 3 Type 4 Type 5 Type 6 Type 7

Fig. 3

Case 2.1: The edge ij is of type 3.

Then two red K_4's are fixed:

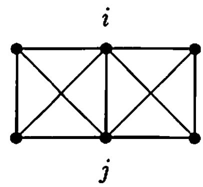
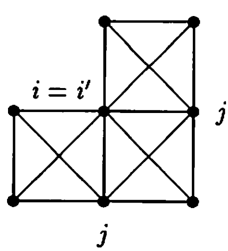


Fig. 4

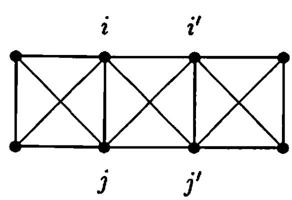
If ij is the only edge with $t = 5$, it follows from (2) that there are at least 43 edges with $t = 4$. Then the three monochromatic K_4's together have at most 17 edges, and the contradiction follows like in Case 1.1 or 1.2 respectively.

If there is exactly one further edge $i'j'$ with $t = 5$, it must also be of type 3, since in the case of types 4 and 5 there are no two monochromatic K_4's having exactly one edge in common, and in the case of types 6 and 7 we would have $ij = i'j'$ (see Fig. 3 or Fig. 6 below).

There are only the following two possibilities (if more vertices were identified, then at least a fourth red K_4 would result):



Case 2.1.1



Case 2.1.2

Fig. 5

It follows from (2) that in addition to the two edges with $t = 5$ there are at least 41 edges with $t = 4$. In both Case 2.1.1 and 2.1.2 the three monochromatic K_4's together have 16 edges, and the contradiction follows like in Case 1.1.

There cannot be a third edge with $t = 5$, because this edge would also have to be contained in two monochromatic K_4's. But the only possibilities in Case 2.1.1 and 2.1.2 for such an edge are ij and $i'j'$.

Case 2.2: All edges with $t = 5$ are of one of the types 4 to 7.

Then by the type of ij all three monochromatic K_4's are fixed:

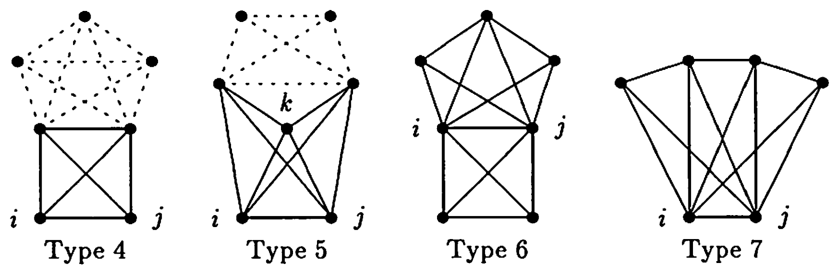


Fig. 6

If there were three or more edges with $t = 5$, then two of them, say ij and $i'j'$, would have to be of the same type (three different types do not fit because of the colors of the monochromatic K_4's). This cannot be type 4, 6, or 7, because this would mean that $ij = i'j'$. It cannot be type 5 either, because then we would have $i'j' = ik$ or $i'j' = jk$, so that the red triangle ijk would form a red K_4 with each vertex of the blue K_4.

Thus there can be at most two edges with $t = 5$. In the case of types 4 to 7, there are each time two monochromatic K_4's having three vertices in common, and the three monochromatic K_4's together have at most 15 edges. (2) implies that there are at least 43 edges with $t \geq 4$, and the contradiction follows like in Case 1.2.

Case 3: There is an edge ij with $t_{ij} = 6$, but no edge with $t \geq 7$.

W.l.o.g. let ij be red, and let ij form a red triangle with each of six vertices $a_1, \ldots, a_6$. We denote the coloring of the K_6 spanned by $a_1, \ldots, a_6$ by C. Suppose ij is a type 2 edge, and a blue K_4 of type 2 is formed by $a_1, \ldots, a_4$, say. Then a_5 and a_6 must be connected by at least one red edge each to $a_1, \ldots, a_4$, because otherwise we would have a blue K_5. Thus all three monochromatic K_4's are fixed. But the remaining six edges from a_5 and a_6 to $a_1, \ldots, a_4$ give rise to at least two more monochromatic K_4's.

So there is no blue K_4 in C, i.e. ij must be a type 1 edge. Then by Turan's theorem, there are at most twelve blue edges in C (the extremal graph is a $K_{2,2,2}$) and consequently at least three red edges. Since every red edge in C gives rise to a red K_4 of type 1, there must be exactly three red edges forming a $\overline{K_{2,2,2}}$, i.e. which are disjoint.

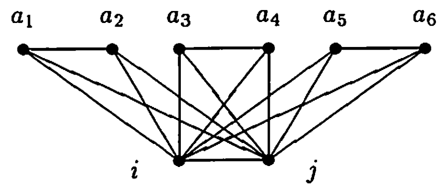


Fig. 7

Again the three monochromatic K_4's are fixed, and there can be no further edge with $t = 6$. Because of the mutual position and the different colors of the three monochromatic K_4's, there cannot be an edge with $t = 5$ either (see the definitions of types 3 to 7).

Then (2) implies that in addition to ij with $t_{ij} = 6$, there are at least 42 edges with $t = 4$, and the three monochromatic K_4's together have 16 edges. Again the contradiction follows like in Case 1.1.

Case 4: There is an edge ij with $t_{ij} \geq 7$.

W.l.o.g. let ij be red, and let ij form a red triangle with each of seven vertices $a_1, \ldots, a_7$. Similar to Case 3 it follows that ij is a type 1 edge. Then by Turan's theorem, there are at most 16 blue edges in the coloring of the K_7 spanned by $a_1, \ldots, a_7$ (the extremal graph is a $K_{3,2,2}$) and thus at least five red edges. But now we already have at least five red K_4's of type 1, which is a contradiction. □

References

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