On the Determination Problem for P_3-Transformation of Graphs

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ABSTRACT. Broersma and Hoede studied the P_3-transformation of graphs and claimed that it is an open problem to characterize all pairs of nonisomorphic connected graphs with isomorphic connected P_3-graphs. In this paper, we solve the problem to a great extent by proving that the P_3-transformation is one-to-one on all graphs with minimum degree greater than two. The only cases that remain open are cases where some degree is 1 or 2, and examples indicate that the problem seems to be difficult in these cases. This in some sense can be viewed as a counterpart with respect to P_3-graphs for Whitney's result on line graphs.

1 Introduction

Broersma and Hoede [1] generalized the concept of line graphs and introduced the concept of path graphs. We follow their terminology and give the following definition. Denote by $\Pi_k(G)$ the set of all paths of G on k vertices ($k \geq 1$). Note that a path does not have repeated vertices. The *path graph* $P_k(G)$ of a graph G has vertex set $\Pi_k(G)$ and edges joining pairs of vertices that represent two P_k-paths if and only if the union of which

*Supported by NSFC, the Third World Academy of Sciences and the Institute of Mathematics, National Autonomous University of Mexico.

forms either a path P_{k+1} or a cycle C_k in G. The way of describing a line graph stresses the adjacency concept, whereas the way of describing a path graph stresses the concept of path generation by consecutive paths.

For a graph transformation, Grünbaum [2] refers to two general problems. We state them here for the P_3-transformation.

Characterization Problem: Characterize those graphs that are P_3-graphs.

Determination Problem: Determine which graphs have a given graph as their P_3-graphs.

Broersma and Hoede [1] studied the P_3-transformation and, among other results, presented a solution to the characterization problem. H. Li and Y. Lin [5] found, and corrected, a flaw in that paper. For the Determination Problem, Broersma and Hoede [1] found two pairs of and two classes of nonisomorphic connected graphs with isomorphic connected P_3-graphs. It is not difficult to find more pairs of that kind of graphs. These examples show that Whitney's result [7] on line graphs (i.e. if G and G' are connected and have isomorphic line graphs, then G and G' are isomorphic unless one is $K_{1,3}$ and the other is K_3) has no similar counterpart with respect to P_3-graphs. They claimed that it is an open problem to characterize all pairs of nonisomorphic connected graphs with isomorphic connected P_3-graphs. Recently, we proved [6] that the P_3-transformation is one-to-one on all graphs with minimum degree greater than three as well as on many having minimum degree three. In this paper, we obtain a stronger result that the P_3-transformation is one-to-one on all graphs with minimum degree greater than two, i.e., they are completely determined by their P_3-graphs. This can be regarded as best possible in the sense that $P_3(C_6) \cong C_6 \cong P_3(S(K_{1,3}))$ (see [1]).

2 Preliminaries

In what follows, all graphs are undirected, connected and simple with at least four vertices. As usual, $d(u)$ denotes the degree of a vertex u and $N(u)$ denotes the neighborhood of u. For a non-negative integer d, we denote by $\mathcal{G}_d$ the class of all connected graphs with minimum degree at least d.

We shall follow Beineke-Hemminger [3] treatment of Whitney's Theorem, which in turn reflects Jung's ideas in [4].

A graph isomorphism from G to G' is a bijection $f : V(G) \to V(G')$ such that two vertices are adjacent in G if and only if their images are adjacent in G'. We let $\Gamma(G, G')$ denote the set of all isomorphisms from G to G', for $G = G'$, that is the automorphism group $\Gamma(G)$ of G.

We shorten $\Gamma(P_3(G), P_3(G'))$ to $\Gamma_3(G, G')$ and call the members P_3-isomorphisms from G to G'. One easily sees that under a P_3-isomorphism, two P_3-paths in G form a P_4 if and only if their images do the same (that is, two P_3-paths forming a P_4 cannot map to P_3-paths forming a C_3-cycle in G').

For $f \in \Gamma(G, G')$, define $f^* : \Pi_3(G) \rightarrow \Pi_3(G')$ by $f^*(uvw) = f(u)f(v)f(w)$, and call f^* the mapping induced by f. We let $\Gamma^*(G, G') = \{f^* | f \in \Gamma(G, G')\}$.

Note that f^* is not defined for connected graphs with fewer than three vertices. Also note that the two isomorphisms of the graph P_3 induce the same $*$-function; however, under our assumptions, G is connected with at least four vertices and so the following results are immediate.

Theorem 1 ([6]) *Let $G, G' \in \mathcal{G}_2$. Then*

(1) $\Gamma^*(G, G') \subseteq \Gamma_3(G, G')$;

(2) *the mapping $T : \Gamma(G, G') \rightarrow \Gamma^*(G, G')$ given by $T(f) = f^*$ is one-to-one.*

The following definitions are needed.

If $P_3 = uvw$, then v is called the *middle vertex* of the path. The set of all the P_3-paths with a common middle vertex v is denoted by $S(v)$ and any subset of $S(v)$ is called a *star* at v. A mapping $f : \Pi_3(G) \rightarrow \Pi_3(G')$ is called *star-preserving* if the set $f(S(v))$is a star in G' for every vertex v of G.

From [6], we have the following results.

Theorem 2 *Let $G, G' \in \mathcal{G}_2$ and let $f : \Pi_3(G) \rightarrow \Pi_3(G')$ be a bijective mapping. Then f is induced by an isomorphism from G to G' if and only if both f and f^{-1} are star-preserving P_3-isomorphisms.*

Lemma 3 *Let $G, G' \in \mathcal{G}_2$ and let f be a P_3-isomorphism from G to G'. Then f is star-preserving if and only if for every edge uv of G $f(x_1uv), \cdots, f(x_ruv)$ have a common middle vertex and $f(uvy_1), \cdots, f(uvy_s)$ have a common middle vertex, where $x_1, \cdots, x_r$ and $y_1, \cdots, y_s$ are neighbors of u and v, respectively.*

3 Main Results

Lemma 4 *Let G and G' belong to $\mathcal{G}_3$ and let f be a P_3-isomorphism from G to G'. If f has the property that for some edge uv of G and $N(u) \setminus v = \{x_1, x_2, \cdots, x_r\}$, $f(x_1uv), f(x_2uv), \cdots, f(x_ruv)$ have a common middle vertex in G', then $f(S(u))$ is a star of G'.*

Proof. Since $f(x_1uv), f(x_2uv), \cdots, f(x_ruv)$ have a common middle vertex, we need only show that for any $x_iux_j \in S(u)$, $f(x_iux_j)$ and $f(x_iuv)$ have a common middle vertex.

Since $G \in \mathcal{G}_3$, x_i has at least two neighbors p and q other than u. Since x_iuv is adjacent to both px_iu and qx_iu in $P_3(G)$ and f is a P_3-isomorphism from G to G', we know that $f(x_iuv)$ is adjacent to both $f(px_iu)$ and $f(qx_iu)$ in $P_3(G')$. There are just two possibilities.

Case 1. $f(px_iu)$ and $f(x_iuv)$ have a common edge other than the common edge of $f(qx_iu)$ and $f(x_iuv)$.

Case 2. One of the two edges of $f(x_iuv)$ is the common edge of $f(px_iu)$ and $f(qx_iu)$.

Now we consider x_iux_j, which is adjacent to both px_iu and qx_iu in $P_3(G)$. Since f is a P_3-isomorphism from G to G', $f(x_iux_j)$ must be adjacent to both $f(px_iu)$ and $f(qx_iu)$ in $P_3(G')$. Thus $f(x_iux_j)$ must have a common edge with each of the $f(px_iu)$ and $f(qx_iu)$.

If Case 1 holds, then $f(x_iuv), f(x_iux_j), f(px_iu)$ and $f(qx_iu)$ must form a C_4 in G'. If we let $f(x_iuv) = a'b'$ and $f(x_iux_j) = c'd'$, then $f(px_iu) = a'c'$ and $f(qx_iu) = b'd'$. Consider x_iuv. Since $f(x_juv)$ has a common middle vertex with $f(x_iuv)$, we know that $f(x_juv) = e'a'$ or $e'b'$ for some edge e' of G'. This implies that $f(x_juv)$ is adjacent to $f(px_iu)$ or $f(qx_iu)$ in $P_3(G')$. Since f is a P_3-isomorphism from G to G', x_juv is adjacent to px_iu or qx_iu in $P_3(G)$, which yields a contradiction. Therefore, only Case 2 can hold. Then since $f(x_iux_j)$ has a common edge with each of the $f(px_iu)$ and $f(qx_iu)$, we know that in this case $f(x_iux_j)$ must have a common edge with $f(x_iuv)$, i.e., $f(x_iux_j)$ has a common middle vertex with $f(x_iuv)$, which completes the proof. □

The next lemma is crucial in the following.

Lemma 5 *Let $G, G' \in \mathcal{G}_3$ and let uv be an edge of G. If f is a P_3-isomorphism from G to G', then $f(S(u))$ is a star of G' if and only if $f(S(v))$ is.*

Proof. Let $f(S(v))$ be a star of G' and let $y_1, \cdots, y_s$ be the neighbors of v. Then the set of P_3's $\{f(uvy_i)|i = 1, 2, \cdots, s\}$ have a common middle vertex.

Let $x_1, \cdots, x_r$ be the neighbors of u. Then for $i = 1, 2, \cdots, r$, $f(x_iuv)$ is adjacent to $f(uvy_j)$ for any $j = 1, 2, \cdots, s$. Since $G \in \mathcal{G}_3$, $r, s \geq 2$ and so $f(x_1uv), \cdots, f(x_ruv)$ have a common edge $s't'$, also, $s't'$ is the common edge of $f(uvy_1), \cdots, f(uvy_s)$. By Lemma 4, we know that $f(S(u))$ is a star of G'. The proof is complete. □

Theorem 6 *Let $G, G' \in \mathcal{G}_3$. If f is a P_3-isomorphism from G to G', then f is star-preserving if and only if G has a vertex v such that $f(S(v))$ is a star of G'.*

Proof. The necessity is obvious. For the sufficiency, let u be any vertex of G. If $u = v$, then we already know that $f(S(u)) = f(S(v))$ is a star of G'. Otherwise, $u \neq v$. Since G is connected, we have a path in G connecting v and u, say $v = v_1v_2 \cdots v_k = u$. Since $f(S(v)) = f(S(v_1))$ is a star in G', from Lemma 5 we know that $f(S(v_2)), \cdots, f(S(v_k)) = f(S(u))$ are stars of G', which completes the proof. □

The following results are much stronger than those in [6].

Lemma 7 *Let $G, G' \in \mathcal{G}_3$ and assume that the maximum degree $\Delta(G) \geq 4$. If f is a P_3-isomorphism from G to G', then f is star-preserving.*

Proof. From Theorem 4 of [6] and Theorem 6, this follows immediately. □

Lemma 8 *Let $G, G' \in \mathcal{G}_3$ and G have a triangle. If f is a P_3-isomorphism from G to G', then f is star-preserving.*

Proof. If $\Delta(G) \geq 4$, then from Lemma 7, the conclusion follows.

Now let G be 3-regular. Let the vertices x_1, x_2 and u induce a triangle of G. Since G is 3-regular, we know that u has another neighbor v different from x_1 and x_2. Let $N(v) \setminus u = \{y_1, y_2\}$. The remainder of the proof can be obtained from the proof of Theorem 8 [6] and Theorem 6. □

Theorem 9 *Let $G, G' \in \mathcal{G}_3 \setminus \{K_{3,3}\}$. Then f is a P_3-isomorphism from G to G' if and only if f is induced by an isomorphism of G onto G'. Furthermore, for any $G, G' \in \mathcal{G}_3$, $P_3(G)$ is isomorphic to $P_3(G')$ if and only if G is isomorphic to G'.*

Proof. From Theorem 2, we need to prove that both f and f^{-1} are star-preserving. Since G' has the same property as G, we only need to show that f is.

The "if" part is obvious. For the "only if" part, we distinguish the following cases.

Case 1. $\Delta(G) \geq 4$.

By Lemma 7, f is star-preserving.

Case 2. G is 3-regular.

Subcase 2.1. G has a triangle.

By Lemma 8, f is star-preserving.

Subcase 2.2. G is triangle-free.

Let uv be any edge of G and let $N(u) \setminus v = \{x_1, x_2\}$ and $N(v) \setminus u = \{y_1, y_2\}$. Then x_1, x_2, u, v, y_1 and y_2 must be six different vertices and no two of them are adjacent. Assume that f is not star-preserving. Then by Theorem 6, f does not preserve any stars of G, so, $f(x_1uv), f(x_2uv), f(uvy_1)$ and $f(uvy_2)$ must form a 4-cycle. If $f(x_1uv) = a'b'$ and $f(x_2uv) = c'd'$, then $f(y_1uv) = b'c'$ and $f(y_2uv) = a'd'$. Consider $N(x_1) \setminus u = \{z_1, z_2\}$ and $N(x_2) \setminus u = \{w_1, w_2\}$. Since both $f(z_1x_1u)$ and $f(z_2x_1u)$ are adjacent to $f(x_1uv) = a'b'$, whereas both $f(w_1x_2u)$ and $f(w_2x_2u)$ are adjacent to $f(x_2uv) = d'c'$, we know that, no matter how they are adjacent, there are just two possibilities:

(1). The 4-cycle of G' formed by the four edges a', b', c' and d' must have a vertex with degree at least 4 and so $\Delta(G') \geq 4$. In this case, since f^{-1} is a P_3-isomorphism from G' to G and $\Delta(G') \geq 4$, by Lemma 7, we know that f^{-1} is star-preserving and therefore $\Delta(G) \geq 4$, a contradiction.

(2). Similarly considering $N(y_1) \setminus v = \{s_1, s_2\}$ and $N(y_2) \setminus v = \{t_1, t_2\}$, we obtain that G' must be isomorphic to $K_{3,3}$, again a contradiction to that $G' \in \mathcal{G}_3 \setminus \{K_{3,3}\}$.

The two contradictions show that f must be star-preserving.

For the second part, we need only consider f^{-1}, the P_3-isomorphism from G' to G, and we arrive at the piont that G is also isomorphic to $K_{3,3}$. Hence, we also have $G \cong G'$. □

Corollary 10 *Let $G \in \mathcal{G}_3 \setminus \{K_{3,3}\}$. Then the automorphism group of $P_3(G)$ is isomorphic to that of G.*

Proof. This follows immediately from Theorems 1 and 9. □

Acknowledgement: The author is indebted to Prof. R.L. Hemminger [8] for a correction of the original of Theorem 9. He is also grateful to the referee for helpful comments which improved the original manuscript.

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