On two problems about (0,2)-graphs and interval-regular graphs

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ABSTRACT. We give operations on graphs preserving the property of being a (0,2)-graph. In particular, these operations allow the construction of non vertex-transitive (0,2)-graphs. We also construct a family of regular interval-regular graphs which are not interval monotone, thus disproving a weaker version of a conjecture proposed hy H.M. Mulder.

1 Introduction

All graphs used in this paper are assumed to be simple and connected. The Cartesian sum of two graphs G and H is the graph $G \square H$ whose vertex-set is the Cartesian product $V(G) \times V(H)$, in which (u, v) is adjacent to (u', v') if and only if either $u = u'$ and $vv' \in E(H)$ or $v = v'$ and $uu' \in E(G)$. The Cartesian product of G and H is the graph $G \times H$ whose vertex-set is $V(G) \times V(H)$, in which (u, v) is adjacent to (u', v') if and only if $uu' \in E(G)$ and $vv' \in E(H)$.

The hypercube Q_n has as vertices the elements of the n dimensional vector space over $\{0, 1\}$; two vertices being adjacent if and only if they differ in exactly one component. We can notice that $Q_n = K_2 \square K_2 \square \dots \square K_2$ (n times).

The Hamming weight $w(x)$ of a vertex x of Q_n is the number of its non-zero components.

For any two vertices u and v of G the interval between u and v is the set:

$$I(u,v) = \{w \in V(G)/w \text{ lies on a shortest } (u,v)\text{-path}\}.$$

The set of neighbors of a vertex u is denoted by $N(u)$.

A graph G is said to be interval-regular if, for any two vertices u and v of G, the number of neighbors of u lying on shortest (u,v)-paths is precisely the distance between u and v. S. Foldes [3] proved that G is interval-regular if and only if for any two vertices u and v there is exactly $d(u,v)!$ shortest (u,v)-path in G and also that the hypercubes are exactly the bipartite interval-regular graphs.

There is only a few known constructions of interval-regular graphs.

The hypercube is also a (0,2)-graph, i.e. a connected graph in which any two distinct vertices have exactly two common neighbors or none at all. Up to now all known (0,2)-graphs are vertex-transitive and finding a counterexample is a natural question. In the next section we will construct non vertex-transitive (0,2)-graphs .

Assuming the convexity of intervals we obtain the notion of interval monotone graph [7] [8]:

A graph is interval monotone if and only if for any u and v,

$$x, y \in I(u,v) \Rightarrow I(x,y) \subset I(u,v).$$

H.M. Mulder proposed the following conjecture [7] [8].

Conjecture. *An interval-regular graph is interval monotone.*

This conjecture is false [6]. Recently H.M. Mulder [9] asks: what could make all intervals convex in an interval-regular graph and can the other interval-regular graphs be determined? For example K. Nomura [10] has some partial results in proving Mulder's conjecture for distance-regular graphs.

In this direction the graph depicted in Figure 1 is a (0,2)-graph and thus a regular counterexample to Mulder's conjecture. In the last section we will introduce a construction of interval-regular graphs which gives a family of such regular counterexamples.

2 Operations on (0.2)-graphs

Various constructions of (0,2)-graphs are in the literature [4] [5] [7]. The basic one is that the Cartesian sum of two (0,2)-graphs is also a (0,2)-graph. The similar property does not exist for the Cartesian product of two (0,2)-graphs G and H. Hence for $H = K_2$, we have:

Proposition 1. *$G \times K_2$ is a (0,2)-graph if and only if G is a (0,2)-graph.*

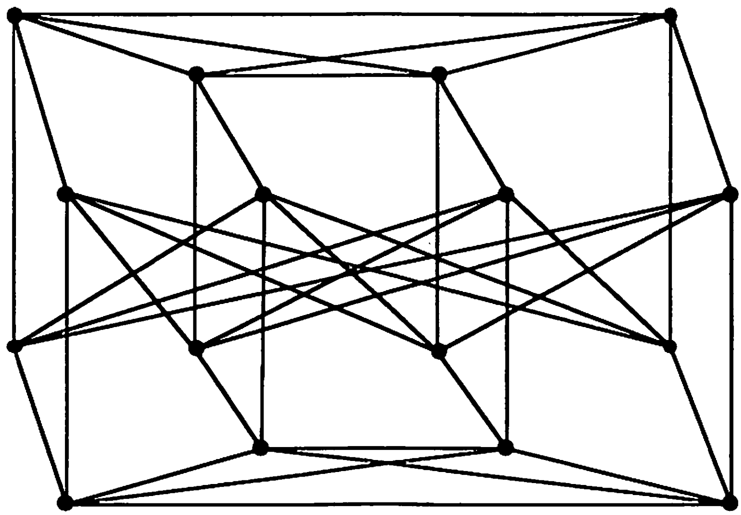

Figure 1

Some properties of the graph $G \times K_2$, also denoted by $B(G)$, can be found in [1].

Proposition 2. *Let G be a (0,2)-graph and f an involutive automorphism of G such that $d(u, f(u)) = 1$ $\forall u \in V(G)$. The graph obtained by deleting the matching $\{uf(u), u \in V(G)\}$ is a (0,2)-graph.*

Let f be an involution of a graph G and $G(f)$ ($G'(f)$ respectively) the graph obtained from G by identifying (joining respectively) in G each pair of vertices $\{u, f(u)\}$.

Proposition 3. [5] *Let G be a (0,2)-graph and f an involutive automorphism of G. If for every $u \in V(G)$ we have $d(u, f(u)) \geq 4$, then $G'(f)$ is a (0,2)-graph.*

This result can be easily generalized by the following theorem:

Theorem 1. *Let G be a (0,2)-graph and f an involutive automorphism of G without fixed point. If for every u and v in G such that $N(u) \cap N(v) \neq \emptyset$ we have $f(v) \notin N(u)$ then $G'(f)$ is a (0,2)-graph.*

The following theorem give a similar property for $G(f)$.

Theorem 2. *Let G be a (0,2)-graph and f an involutive automorphism of G without fixed point. If for every u and v in G such that $N(u) \cap N(v) \neq \emptyset$ we have $N(u) \cap N(f(v)) = \emptyset$ then $G(f)$ is a (0,2)-graph.*

Proof: Let c be a common neighbor of two vertices a and b of $G(f)$. Then clearly there exist three vertices in G say u, v, w such that $a = \{u, f(u)\}$,

$b = \{v, f(v)\}$, $c = \{w, f(w)\}$ and w is a common neighbor of u and v. As G is a (0,2)-graph, let z be the second common neighbor in G of u and v, so $\{z, f(z)\}$ is a second common neighbor in $G(f)$ of the vertices a and b. Assume that a and b have in $G(f)$ a third common neighbor $\{t, f(t)\}$, as w and z are two common neighbors in G of u and v, it follows that neither t nor $f(t)$ is a common neighbor in G of u and v. We have in G, without loss of generality t adjacent to u so t is also adjacent to $f(v)$ thus $t \in N(u) \cap N(f(v))$, whereas $N(u) \cap N(v) \neq \emptyset$, a contradiction, then $G(f)$ is a (0,2)-graph.

Corollary 1. *Let G be a (0,2)-graph and f an involutive automorphism of G. If for every $u \in V(G)$ we have $d(u, f(u)) \geq 5$, then $G(f)$ is a (0,2)-graph.*

For a graph G, let G^* be the graph obtained from G as follows: $V(G^*) = V(G) \times \{1, 2, 3, 4\}$ and (u, i) and (v, j) are adjacent in G^* if and only if one of the following three conditions is satisfied:

(a) $uv \in E(G)$ and $i = j$ and are odd

(b) $uv \in E(G)$ and $i \neq j$ and both are even

(c) $u = v$ and the parities of i and j are different.

The reader can easily check the following result:

Theorem 3. *For every (0,2)-graph G, G^* is a (0,2)-graph.*

Theorems 1, 2 and 3 allow the construction of classes of (0,2)-graphs; some of which are non vertex-transitive. For instance the graph G of Figure 1 is a non vertex-transitive (0,2)-graph obtained in three ways:

(a) G is K_4^*

(b) G is $Q_5(f)$ where f is the automorphism defined by

$$f(u) = \begin{cases} u + 00111 & \text{if } u_1 = u_2 \\ u + 11111 & \text{otherwise} \end{cases}$$

(c) G is $Q'_4(f)$ where f is the automorphism defined by

$$f(u) = \begin{cases} u + 0011 & \text{if } u_1 = u_2 \\ u + 1111 & \text{otherwise} \end{cases}$$

Furthermore this regular graph is interval-regular but not interval monotone. In the next section we will give the construction of a family of such graphs.

3 A construction of interval-regular graphs

Theorem 4. *Let f be an involutive automorphism without fixed points of Q_n such that $w(u)$ and $w(f(u))$ have the same parity for every vertex u. Then $Q'_n(f)$ is an interval-regular (0,2)-graph.*

Notice that the edges of $Q'_n(f)$ are of two kinds: the edges of Q_n (say blue edges) and the perfect matching $\{uf(u), u \in V(Q_n)\}$ (red edges).

The proof of theorem 4 is an immediate consequence of the four following propositions:

Proposition 4. *A shortest path in $Q'_n(f)$ uses at most one red edge.*

Let Π be the set of the shortest paths in $Q'_n(f)$ using at least two red edges.

Let p be a (x, y)-path in Π such that the distance between x and y is minimum among all elements of Π. Clearly $d(x,y) \geq 3$. Let $\mathrm{p} = x, a, b, \mathrm{p}', y$. By minimality of p the edge xa is red and thus ab is blue. Now consider the vertex $f(b)$. We have $f(a) = x$ and $ab \in E(Q_n)$. Thus $xf(b) \in E(Q_n)$ and $x, f(b), b, \mathrm{p}', y$ is a path of Π. Then $d(f(b), y) = d(x,y) - 1$ and the path $f(b), b, \mathrm{p}', y$ is also in Π, contradicting the minimality of $d(x, y)$.

Proposition 5. *If there is a shortest path in $Q'_n(f)$ between x and y using a red edge then every geodesic (or shortest path) between x and y uses a red edge.*

If the edge uv is red then $|w(u) - w(v)|$ is even else $|w(u) - w(v)| = 1$. Then if there exists a path using a red edge between x and y with length L, the parity of $|w(x) - w(y)|$ is the parity of $L - 1$ and if there is a path between x and y of length L using only blue edges then $|w(x) - w(y)|$ and L have the same parity.

Proposition 6. *If there is a shortest path in $Q'_n(f)$ between x and y using only blue edges then x and y are joined by exactly $d(x,y)!$ geodesics.*

This is an immediate consequence of Proposition 5 because the shortest paths between x and y use only the edges of Q_n and there is exactly $d(x, y)!$ geodesics in Q_n.

Proposition 7. *If there is a shortest path in $Q'_n(f)$ between x and y using a red edge then x and y are joined by exactly $d(x,y)!$ geodesics.*

Let S_i be the set of (x, y)-shortest paths such that, starting from x, the ith edge is red. The set S of all (x, y)-shortest paths is the disjoint union of the S_i for $i = 1, \ldots d = d(x, y)$.

Let $\mathrm{p} = x_0, \ldots x_{i-1}, x_i, \ldots, x_d$ be a path of S_i (with $x_0 = x$, $x_d = y$, $i \geq 2$). We have $f(x_{i-1}) = x_i$ and $x_{i-2}x_{i-1} \in E(Q_n)$. Then we have $f(x_{i-2})x_i \in E(Q_n)$ and the path $g(\mathrm{p}) = x_0, \ldots x_{i-2}, f(x_{i-2}), x_i, \ldots, x_d$

is in S_{i-1}. Thus g is clearly a one to one mapping from S_i to S_{i-1} and $|S_i| = |S_{i-1}| = |S_1|$ for $i \geq 2$.

Let S_0 be the number of shortest $(f(x), y)$-path and let $\mathrm{p} = x, f(x), \mathrm{p}', y$ be a path of S_1. The subpath $h(\mathrm{p}) = f(x), \mathrm{p}', y$ is in S_0 and uses only blue edges, but h is a one to one mapping and $|S_1| = |S_0| = (d(x,y) - 1)!$.

The proof that $Q'_n(f)$ is a (0,2)-graph is left to the reader. $\square$

Corollary 2. *For all $n \geq 5$ there exists a n-regular graph, interval-regular but not interval monotone.*

Let f be the mapping from $V(Q_{n-1})$ to $V(Q_{n-1})$ defined by

$$f(u) = \begin{cases} u + c & \text{if } u_1 \neq u_2 \\ u + d & \text{otherwise} \end{cases}$$

where $c = 11110\ldots0$ and $d = 00110\ldots0$.

1) f is clearly an involution such that for all u, u and $f(u)$ have the same parities.

2) f is an automorphism. Since f is involutive, it suffices to prove that if x and y are any adjacent vertices of Q_{n-1} then $f(x)$ and $f(y)$ are adjacent.

Let i and j be two integers such that $f(x) = x + c$, $f(y) = y + d$.
first case: $i = j$ then $f(x) + f(y) = x + y$ then $f(x)$ and $f(y)$ are adjacent.
second case: $i \neq j$ then x and t differ at one of the two first components. But $f(x) + f(y) = x + y + 11000\ldots0$. Thus $f(x) + f(y)$ is of weight 1 so $f(x)$ and $f(y)$ are adjacent.

By Theorem 4 the graph $G_n = Q'_{n-1}(f)$ is interval-regular. Consider in G_n the interval $I(x,y)$ where $x = 0\ldots0$ and $y = 1110\ldots0$. It is easy to check that x and y cannot be at distance less than 3 in G_n thus $d(x,y) = 3$ and $I(x,y) = \{abc0\ldots0\}$. Let $u = 100000\ldots0$ and $v = 01100\ldots0$ two vertices of $I(x,y)$ then $I(u,v) = \{u, v, 01110\ldots0, 10010\ldots0\} \not\subset I\{x,y\}$. So G_n is not interval monotone. $\square$

In fact G_n is isomorphic to $K_4^* \square Q_{n-5}$.

For $n \geq 7$ an interesting other family of interval-regular graphs obtained from Theorem 4 is $Q'_{n-1}(f)$ where f is defined by:
$f(u_1u_2u_3U_4\ldots u_{2i-1}u_{2i}\ldots u_{2k-1}u_{2k}) = \overline{u}_1\overline{u}_2u_4u_3\ldots u_{2i}\,u_{2i-1}\ldots u_{2k}u_{2k-1}$ if $n = 2k+1$ $f(u_1u_2u_3u_4\ \ldots u_{2i-1}u_{2i}\ldots u_{2k-3}u_{2k-2}u_{2k-1}) = \overline{u}_1\overline{u}_2u_4u_3\ldots$ $u_{2i}u_{2i-1}\ldots u_{2k-2}u_{2k-3}\ u_{2k-1}$ if $n = 2k$.

These graphs are not isomorphic to $K_4^* \square Q_{n-5}$ and are also not interval monotone (the interval $I(0\ldots0, 10110\ldots0)$ is not convex).

To conclude we can notice that all known counterexamples to Mulder's conjecture are not vertex-transitive.

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