Average Distance in Weighted Graphs with Removed Edges*

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ABSTRACT. The average distance in a connected weighted graph G is defined as the average of the distances between the vertices of G. In 1985 P.M. Winkler [5] conjectured that every connected graph G contains an element e, such that the removal of e enlarges the average distance by at most the factor $\frac{4}{3}$.

D. Bienstock and E. Györi proved Winkler's conjecture for the removal of an edge from a connected (unweighed) graph that has no vertices of degree one, and asked whether this conjecture holds for connected weighted graphs. In this paper we prove that any h-edge-connected weighted graph contains an edge whose removal does not increase the average distance by more than a factor of $h/(h-1)$, $h \geq 2$. This proves the edge-case of Winkler's Conjecture for 4-connected weighted graphs

Furthermore, for 3-edge-connected weighted graphs, it has been verified that the four-thirds conjecture holds for every weighted wheel W_p, $p \geq 4$, and for weighted $K_{3,n}$ and $K_{2,n}$ for $n \geq 2$.

1 Introduction

Throughout this paper we consider only finite, undirected, simple graphs. Our terminology and notations will be standard except as indicated. For undefined terms, see [3] and [4].

*This work is based on the Ph.D. Thesis written by the second author under the supervision of the first author, and was submitted to the University of Mosul (1995).

The concept of distance in graphs can be generalized in a most natural manner. By a *weighted graph*, we mean a graph in which each edge e is assigned a positive real number, called the *weight* of e, and denoted by $w(e)$. The *length of a path* P in a weighted graph G is the sum of the weights of the edges of P. For connected vertices u and v of G, the *distance* $d_G(u,v)$ is the minimum of the lengths of $u-v$ paths of G. If all edges of G have unit weight, then G can be regarded as a graph. Let G be a connected graph with (or without) edge weights, the *total distance* $\mathcal{T}(G)$ is defined to be

$$\mathcal{T}(G) = \sum_{u,v \in V(G)} d_G(u,v).$$

The *transmission* of a vertex v of G is defined as

$$\sigma(G,v) = \sum_{u \in V(G)} d_G(u,v).$$

It is clear that $\mathcal{T}(G) = (\frac{1}{2}) \sum_{v \in V(G)} \sigma(G,v)$. The *average distance* of G is then $D(G) = [2\mathcal{T}(G)]/[p(p-1)]$, where p is the order of G.

Define

$$r_0(G) = \min_{u \in V(G)} [D(G-u)]/[D(G)],$$
$$r_1(G) = \min_{e \in E(G)} [D(G-e)]/[D(G)].$$

Winkler's conjecture [5] can be stated as follows: Is it true that in every connected graph G, $r_0(G) \leq 4/3$; while if G is not a tree, $r_1(G) \leq \frac{4}{3}$?

In 1988, D. Bienstock and E. Györi [2] proved that $r_1(G) \leq \frac{4}{3}$ for any (unweighted) graph G which contains no vertex of degree 1, and $r_0(G) \leq (4/3) + O(p^{-1/5})$,for any (unweighted) graph G.

In 1990, I. Althöfer [1] proved that for every h-connected graph G, $r_0(G) \leq h/(h-1)$, $h \geq 2$. This proves Winkler's conjecture for 4-connected graphs without edge weights.

In this paper we prove $r_1(G) \leq h/(h-1)$, for every h-edge-connected weighted graph G. Moreover, it has been verified that the edge version of Winkler's conjecture holds for any weighted wheel W_p, $p \geq 4$, $K_{2,n}$ and $K_{3,n}$ for $n \geq 2$.

It is clear that if G contains an edge e such that $d_G(u,v) < w(e)$, then $d_{G-e}(x,y) = d_G(x,y)$ for every pair x,y of distinct vertices of G. This implies that $r_1(G) = 1$. Hence, we assume throughout this paper that the weighted graphs considered satisfy the following condition.

$$d_G(u,v) = w(e), \tag{1.1}$$

for every edge $e = uv$ of G. This condition is called the *w-condition*. Let K_p be a complete weighted graph of order p. And let $e_0 = uv$ be an edge of maximum weight of K_p. It follows from w-condition that each shortest $u - v$ path of $K_p - e_0$ consists of two edges. Thus there are two edges $e_1 = uw$ and $e_2 = wv$ such that $d_{K_p} - e_0(u,v) = w(e_1) + w(e_2)$, with, say, $w(e_1) \geq w(e_2)$. Therefore $r_1(K_p) \leq 1 + w(e_2)/\sum_{e\in E(K_p)} w(e)$. Thus $r_1(K_p) \leq 4/3$.

2 r_1 for h-edge-connected weighted graphs

In this section .we will give the proof of the main result. For every pair $\{u,v\}$ of vertices of G, choose a shortest path $P(u,v)$ joining u and v, and let $\mathcal{P}(G)$ be the set of all such shortest paths of G. If e is an edge of G, denote by $\mu(e)$ the total number ot paths in $\mathcal{P}(G)$ containing e.

Theorem 2.1. *If G is an h-edge connected weighted graph, $h \geq 2$, then $r_1(G) < h/(h-1)$.*

Proof: Let $k = \min_{e\in E(G)}\{\mu(e)\}$.

An edge $e_0 = uv$ of G is called of *minimum occurrence* in $\mathcal{P}(G)$ if $\mu(e_0) = k$. Since G is an h-edge-connected graph, then by Menger's Theorem [4], there are $h-1$ edge-disjoint paths $P_1, P_2, \ldots, P_{h-1}$ joining u and v in $G-e_0$.

Assume that $d_{G-e_0}(u,v) = w(P_1) \leq w(P_2) \leq \cdots \leq w(P_{h-1})$, in which $w(P_i)$ is the weight of P_i, for all $i = 1,2,\ldots,h-1$. Denote by $Q_1, Q_2, \ldots, Q_k$ the k shortest paths in $\mathcal{P}(G)$ containing e_0, with $Q_1 = e_0$; and let $Q'_i = Q_i - e_0$, for $2 \leq i \leq k$. Then $\mathcal{T}(G) = kw(e_0) + \sum_{i=1}^{k} w(Q'_i) + X$, where X is the sum of the weights of all shortest paths in $\mathcal{P}(G)$ not containing e_0. If Q_i is the shortest path joining the two vertices x and y, then $Q'_i \cup p_1$ is a connected subgraph of G containing x and y. Thus

$$\begin{aligned}\mathcal{T}(G - e_0) &\leq w(P_1) + \sum_{i=1}^{k} w(P_1 \cup Q'_i) + X \\ &\leq kw(P_1) + X + \sum_{i=1}^{k} w(Q'_i).\end{aligned}$$

Hence

$$\begin{aligned}r_1(G) &\leq [kw(P_1) + X + \sum_{i=2}^{k} w(Q'_i)]/[kw(e_0) + X + \sum_{i=2}^{k} w(Q'_i)] \\ &= 1 + k[w(P_1) - w(e_0)]/\mathcal{T}(G).\end{aligned}$$

Since, each edge of G occurs at least k times in $\mathcal{P}(G)$, then $\mathcal{T}(G) \geq$

$k\{\sum_{e\in E(G)} w(e)\}$. Hence

$$r_1(G) \le 1 + [w(P_1) - w(e_0)]/\sum_{e\in E(G)} w(e). \qquad (2.1)$$

Moreover,

$$\sum_{e\in E(G)} w(e) > (h-1)w(P_1).$$

Thus

$$r_1(G) < 1 + [w(P_1)/\{(h-1)w(P_1)\}] = h/(h-1).$$

□

Theorem 2.1 proves the edge case of Winkler's conjecture for 4-connected weighted graphs.

Corollary 2.1. *If there is an edge $e_0 = uv$ of minimum occurrence in an h-edge-connected weighted graph G such that $d_{G-e_0}(u,v) \le \{1+(1/3)h\}w(e_0)$, then $r_1(G) \le 4/3$.*

The proof of this Corollary follows from (2.1) and the fact that $\sum_{e\in E(G)} w(e) \ge hw(e_0)$. □

Corollary 2.2. *If there is an edge e_0 of a weighted cycle C_p of minimum occurrence such that $w(e_0) \ge (\frac{3}{8})\sum_{e\ne e_0} w(e)$, then $r_1(G) \le \frac{4}{3}$.*

Corollary 2.3. *Let e_0 be an edge of maximum weight in a weighted cycle C_p. If $w(e_0) \ge (3/4)\sum_{e\ne e_0} w(e)$, then $r_1(C_p) < \frac{4}{3}$.*

Proof: Let t be the occurrence of e_0 in $\mathcal{P}(C_p)$. Then

$$\mathcal{T}(C_p) \ge A + tw(e_0),$$
$$\mathcal{T}(C_p - e_0) \le A + \sum_{e\ne e_0} w(e),$$

where A is the sum of the weights of the shortest paths in $\mathcal{P}(C_p)$ not containing e_0. Thus

$$\begin{aligned} r_1(C_p) &\le \{A + t\sum_{e\ne e_0} w(e)\}/\{A + tw(e_0)\} \\ &\le \{A + \frac{4}{3}tw(e_0)\}/\{A + tw(e_0)\} \\ &\le 1 + \{\frac{1}{3}tw(e_0)\}/\{A + tw(e_0)\} \\ &< \frac{4}{3}. \end{aligned}$$

□

The 4/3 conjecture for edge case has also been verified by the authors for special 3-edge-connected weighted graphs, such as wheels and complete bipartite graphs $K_{m,n}$, m, $n \geq 2$.

The vertex case of the 4/3 conjecture is more difficult, the authors also considered the removal of a vertex from a connected weighted graph. It is proved that every connected weighted graph contains a vertex whose removal does not increase the average distance by more than a factor of 2. Moreover, we prove that the vertex version of Winkler's conjecture is true for h-connected weighted graphs, $h \geq 4$, weighted wheels and complete bipartite graphs if the ratio of the maximum edge weight y to the minimum edge weight x does not exceed certain upper bounds. Such upper bounds for y/x are imposed to guarantee the existence of a vertex whose removal does not increase the average distance by more than a factor of 4/3.

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