A Ramsey Goodness Result for Graphs with Many Pendant Edges

Yusheng Li and Cecil C. Rousseau
Department of Mathematical Sciences
The University of Memphis
Memphis, Tennessee 38152

ABSTRACT. Burr has shown that if G is any graph without isolates and H_0 is any connected graph, every graph H obtained from H_0 by subdividing a chosen edge sufficiently many times to create a long suspended path satisfies $r(G,H) = (\chi(G) - 1)(|V(H)| - 1) + s(G)$, where $s(G)$ is the largest number such that in every proper coloring of $V(G)$ using $\chi(G)$ colors, every color class has at least $s(G)$ elements. In this paper, we prove a companion result for graphs obtained from H_0 by adding sufficiently many pendant edges.

Let G and H be graphs without isolates. The *Ramsey number* $r(G,H)$ is the smallest positive integer p such that for any two-coloring of the edges of the complete graph K_p with colors red and blue, there is either a monochromatic red G or a monochromatic blue H. It is easy to show that if H is connected and $|V(H)| \geq s(G)$, then

$$r(G,H) \geq (\chi(G) - 1)(|V(H)| - 1) + s(G), \tag{1}$$

where $s(G)$ is the largest number such that in every proper vertex coloring of G using $\chi(G)$ colors, every color class has at least $s(G)$ members. This number is called the *chromatic surplus* of G. In case equality holds in (1), we say that H is *G-good*. For a survey of results involving this concept, see [3]. In an early result, Burr proved that if H_0 is any connected graph and j is sufficiently large, any graph H obtained from H_0 by adding j vertices to subdivide a chosen edge and create a long suspended path is G-good [1]. An edge in a graph is *pendant* if one of its vertices has degree 1. Starting with a connected graph H_0 of order n, we can introduce j new vertices and j pendant edges by joining each of the new vertices to some vertex in $V(H_0)$.

We do not expect to obtain a G-good graph from H_0 by adding (arbitrarily) a large number of such pendant edges. For example, large stars are *not* C_4-good, since the bound for $r(C_4, K_{1,n})$ given by (1) is $r(C_4, K_{1,n}) \geq n+2$, whereas it is shown in [2] that $r(C_4, K_{1,n}) \geq n + \sqrt{n} - 6n^{11/40}$ for all sufficiently large n. Nevertheless, we shall show that the collection of graphs obtained this way *contains* some which are G-good.

Before giving the theorem and its proof, we need two more definitions. The *upper chromatic surplus* of a graph G is the smallest integer $\overline{s}(G)$ so that in every proper vertex coloring of G using $\chi(G)$ colors, every color class has at most $\overline{s}(G)$ vertices. Clearly $\overline{s}(G) \geq s(G)$. Let $\mathcal{G}$ and $\mathcal{H}$ denote two classes of graphs. The *class Ramsey number* $r(\mathcal{G}, \mathcal{H})$ is the smallest integer p so that in every two-coloring of the edges of K_p there is a monochromatic red copy of at least one member of $\mathcal{G}$ or a monochromatic blue copy of at least one member of $\mathcal{H}$. If $\mathcal{G} = \{G\}$, we denote $r(\mathcal{G}, \mathcal{H})$ as $r(G, \mathcal{H})$. Clearly

$$r(\mathcal{G}, \mathcal{H}) \leq \min\{r(G, H) \colon G \in \mathcal{G}, H \in \mathcal{H}\},$$

but the equality does not necessarily hold. The following result gives examples to show that the difference between $\min\{r(G, H) \colon G \in \mathcal{G}, H \in \mathcal{H}\}$ and $r(\mathcal{G}, \mathcal{H})$ can be arbitrarily large.

Proposition 1 *Let $\mathcal{C}$ be the class of all cycles and let $\mathcal{C}_1$ be the class of all odd cycles.*

(1) *For any n, $r(K_n, \mathcal{C}) = r(K_n, \mathcal{C}_1) = 2n - 1$.*

(2) *For any $M > 0$, there is $N > 0$ such that if $n \geq N$, $\min\{r(K_n, C) \colon C \in \mathcal{C}\} \geq Mn$, and $\min\{r(K_n, C) \colon C \in \mathcal{C}_1\} \geq Mn$.*

Proof: (1). The two-coloring of $E(K_{2n-2})$ in which $\langle R \rangle$ is isomorphic to $(n-1)K_2$ shows $r(K_n, \mathcal{C}) \geq 2n - 1$. Now consider any red-blue coloring of edges of K_{2n-1} without any blue odd cycles. It is well known that a nontrivial graph is bipartite if and only if it does not contain any odd cycles. Thus the graph $\langle B \rangle$ induced by all blue edges is a bipartite graph, and the larger part induces a red complete graph of order at least n. This proves $r(K_n, \mathcal{C}_1) \leq 2n - 1$. The remaining part follows the fact $r(K_n, \mathcal{C}) \leq r(K_n, \mathcal{C}_1)$ since $\mathcal{C}_1 \subset \mathcal{C}$.

(2). Given $M > 0$, take a positive integer m such that $m(n-1)+1 \geq Mn$ for $n \geq 2$. In [4] it is shown that for fixed m there is a constant $c > 0$ such that

$$\min_{3 \leq k \leq m} r(K_n, C_k) \geq c \left(\frac{n}{\log n} \right)^{(m-1)/(m-2)}.$$

Therefore there is $N \geq 2$ so that $n \geq N$ implies $\min_{3 \leq k \leq m} r(K_n, C_k) \geq Mn$. By the inequality (1) we know that for $k \geq m+1$

$$r(K_n, C_k) \geq (n-1)(k-1) \geq m(n-1) + 1 \geq Mn.$$

Thus $\min\{r(K_n, C) \colon C \in \mathcal{C}\} \geq Mn$ and $\min\{r(K_n, C) \colon C \in \mathcal{C}_1\} \geq Mn$ if $n \geq N$.

Lemma 1 *Let $\mathcal{H}$ be a class of connected graphs. If all graphs in $\mathcal{H}$ have the same order and there is a $G-good$ graph in $\mathcal{H}$, then $r(G, \mathcal{H}) = \min\{r(G, H) \colon H \in \mathcal{H}\}$.*

Proof: Let n be the order of graphs in $\mathcal{H}$. Then the graph $(\chi(G)-1)K_{n-1} \cup K_{s(G)-1}$ defines a two-coloring of edges of complete graph K_p, where $p = (\chi(G)-1)(n-1)+s(G)-1$, to show $r(G, \mathcal{H}) \geq (\chi(G)-1)(n-1)+s(G)$. Since $\min\{r(G, H) \colon H \in \mathcal{H}\} \geq r(G, \mathcal{H})$ and there is a $G-good$ graph $H' \in \mathcal{H}$, we have $r(G, H') \geq \min\{r(G, H) \colon H \in \mathcal{H}\} \geq r(G, \mathcal{H}) \geq (\chi(G)-1)(n-1)+s(G) = r(G, H')$, so equality holds throughout. This proves the lemma.

Theorem 1 *Let G be any graph without isolates and suppose that H is a connected graph of order $n \geq \overline{s}(G)$. For any $V \subset V(H)$ with $|V| = \overline{s}(G)$, let $\mathcal{H}_j$ denote the class of all graphs obtained from H by adding j pendant edges joining new vertices to V. If j is sufficiently large,*

$$r(G, H_j) = (\chi(G)-1)(n+j-1)+s(G)$$

for some $H_j \in \mathcal{H}_j$. Thus $r(G, \mathcal{H}_j) = \min\{r(G, H) \colon H \in \mathcal{H}_j\}$.

Proof: Since every member of $\mathcal{H}_j$ is connected graph of order $n+j \geq n \geq \overline{s}(G) \geq s(G)$, we have

$$r(G, H_j) \geq (\chi(G)-1)(n+j-1)+s(G)$$

for each $H_j \in \mathcal{H}_j$. To prove

$$r(G, H_j) \leq (\chi(G)-1)(n+j-1)+s(G),$$

for some $H_j \in \mathcal{H}_j$, we use induction on $\chi(G)$.

If $\chi(G) = 1$, then $G = \overline{K}_s$ and the result is trivial. [Although this case violates the convention under which neither G nor H have isolated vertices, it does provide a valid basis for the induction.]

Suppose $\chi(G) \geq 2$ and color the vertices of G with $\chi(G)$ colors such that the color classes $C_1, C_2, \ldots, C_\chi$ satisfy

$$s(G) = |C_1| \leq |C_2| \leq \cdots \leq |C_\chi| \leq \overline{s}(G).$$

Set $G' = G - C_\chi$. Then $\chi(G') = \chi(G)-1$ and $s(G') = s(G)$. By the induction hypothesis, there is $N > 0$ such that if $j \geq N$ we can find a specific $H' \in \mathcal{H}_j$ that is $G'-good$, i.e,

$$r(G', H') = (\chi(G)-2)(n+j-1)+s(G).$$

Take $N_1 \geq N$ such that $(\chi(G)-1)(n+N_1-1)+s(G) \geq r(G,H)$. Now for $j \geq N_1$ set $p = (\chi(G)-1)(n+j-1)+s(G)$ and let (R,B) be a two-coloring of the edges of K_p with colors red and blue. We want to show $G \subset \langle R \rangle$ or $H'' \subset \langle B \rangle$ for some $H'' \in \mathcal{H}_j$.

Suppose that $G \not\subset \langle R \rangle$ and $\langle B \rangle$ contains no member of $\mathcal{H}_j$. Since $p \geq r(G,H)$ and $G \not\subset \langle R \rangle$, we have $H \subset \langle B \rangle$. We can thus suppose that there is some i satisfying $0 \leq i < j$ such that $\langle B \rangle$ contains some member of $\mathcal{H}_i$ but no member of $\mathcal{H}_{i+1}$. Then there is a partition $V(K_p) = (X,Y)$ where $|X| = n+i$, $|Y| = p-n-i \geq (\chi(G)-2)(n+j-1)+s(G)$ such that $\langle X \rangle_B$ (the blue graph spanned by X) contains some member of $\mathcal{H}_i$ and all the edges xy where $x \in V \subset V(H) \subset X$ and $y \in Y$ are red. Since $H' \not\subseteq \langle Y \rangle_B$ we have $G' \subseteq \langle Y \rangle_R$. Since $|V| = \overline{s}(G) \geq |C_\chi|$, this gives $G \subseteq \langle R \rangle$. This contradiction completes the proof.

References

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