Note on Whitney's Theorem for k-connected Graphs

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Abstract

In this paper we refine Whitney's Theorem on k-connected graphs for $k \geq 3$. In particular we show the following: Let G be a k-connected graph with $k \geq 3$. For any two distinct vertices u and v of G there are k internally vertex disjoint paths $P_1[u,v]$, $P_2[u,v]$, $\cdots$, $P_k[u,v]$ such that $G - V(P_i(u,v))$ is connected for $i = 1, 2 \cdots, k$, where $P_i(u,v)$ denotes the internal vertices of the path $P_i[u,v]$. Further one of the following properties holds.

- $G - V(P_i[u,v])$ is connected for $i = 1, 2, 3$.
- $G - V(P_i[u,v])$ is connected for $i = 1, 2$ and $G - V(P_i[u,v])$ has exactly two connected components for $i = 3, 4, \cdots, k$.

In addition some other properties will be proved.

1 Introduction

Only finite simple graphs will be considered. In general $G = (V, E)$ will denote a graph with vertex set V and edge set E. Terminology will in general follow that used in the text of Bondy and Murty [1]. Let G be a graph and let $P[u, v]$ be a path of G joining the vertices u and v. We will use $P(u, v)$ to denote $P[u, v] - \{u, v\}$, that is the internal subpath of $P[u, v]$. The orientation of $P[u, v]$ is the direction along the path $P[u, v]$ from u to v. For any two two vertex subsets A and B of G we let $E(A, B)$ denote the set of edges with one vertex in A and the other in B. We use $\omega(G)$ to denote the number of connected components of G.

The *connectivity* (or more precisely the *vertex connectivity*) $\kappa(G)$ of a graph G is the minimum number of vertices that, when deleted, leaves the graph disconnected or with just one vertex. The *edge connectivity* $\kappa_1(G)$ is defined similarly. A graph G is k-connected if $\kappa(G) \geq k$ and a graph G is k-edge-connected if $\kappa_1(G) \geq k$. There are books, for example [12], and survey articles (see [3, 6, 9]) that deal exclusively with various connectivity concepts. A fundamental result on connectivity is due to Menger.

Theorem 1 (Menger [8]) *For any two nonadjacent vertices u and v of a graph G, the maximum number of internally vertex disjoint paths between the vertices u and v is equal to the minimum number of vertices that separate u and v.*

A consequence of this is a result by Whitney.

Theorem 2 (Whitney [13]) *A graph G is k-connected if and only if there are k internally vertex disjoint paths between each pair of distinct vertices of G.*

There are edge versions of both Menger's and Whitney's Theorems. There are numerous proofs using a variety of approaches of these analogues; in particular, proofs using the theory of flows can be found in [2] and [4].

Theorem 3 *For nonadjacent vertices u and v of a graph G, the maximum number of edge disjoint paths from u to v is equal to the minimum number of edges that separate u and v. A graph G is k-edge-connected if and only if there are k edge disjoint paths between each pair of distinct vertices of G.*

Mader [7] recently obtained the following result.

Theorem 4 (Mader[7]) *Let G be a $(k + 1)$-edge-connected graph and u, v be two distinct vertices of G. Then there is a path $P[u, v]$ joining u and v such that $G - E(P[u, v])$ is k-edge-connected.*

Concerning the (vertex) connectivity, Tutte [11] showed that every 3-connected graph has a chordless circuit whose deletion leaves a connected graph. The following result is slightly stronger.

Theorem 5 (Thomassen and Toft [10]) *If G is a connected graph and the minimum degree $\delta(G) \geq 3$, then G has a chordless cycle C such that $G - V(C)$ is connected.*

Lovász made the following conjecture.

Conjecture 1 (Lovász [5]) *For each natural number k, there exists a natural number $\beta(k)$ such that for any two vertices u, v in any $\beta(k)$-connected graph G, there is a path P between the vertices u and v such that $G - V(P)$ is k-connected.*

In this paper, we will refine Whitney's theorem as follows.

Theorem 6 *Let G be a k-connected graph with $k \geq 3$. For any two distinct vertices u and v of G there are k internally vertex disjoint paths $P_1[u, v]$, $P_2[u, v]$, $\cdots$, $P_k[u, v]$ such that $G - V(P_i(u, v))$ is connected for $i = 1, 2 \cdots, k$ and further one of the following properties holds.*

- *$G - V(P_i[u, v])$ is connected for $i = 1, 2, 3$.*
- *$G - V(P_i[u, v])$ is connected for $i = 1, 2$ and $G - V(P_i[u, v])$ has exactly two connected components for $i = 3, 4, \cdots, k$*

The proof of Theorem 6 will be placed in next section, as well as that of the following theorem.

Theorem 7 *Let G be a k-connected graph with $k \geq 3$. For any two distinct vertices u and v, there are k internally vertex disjoint paths $R_1[u, v]$, $R_2[u, v]$, $\cdots$, $R_k[u, v]$ between u and v such that*

$$\sum_{i=1}^{k} \omega(G - V(R_i[u, v])) \leq 2(k - 1)$$

There are many two connected graphs which have two vertices u and v such that deleting any path between them will disconnect the graph. So that $k \geq 3$ is best possible in some sense. The following two results are immediate consequences of Theorem 6.

Theorem 8 *Let G be a k-connected graph with $k \geq 3$. For any two distinct vertices u and v of G, there are k internally vertex disjoint paths $P_1[u, v]$, $P_2[u, v]$, $\cdots$, $P_k[u, v]$ such that $G - V(P_i(u, v))$ is connected for every $i = 1$, $2, \cdots, k$. Further, both $G - V(P_1[u, v])$ and $G - V(P_2[u, v])$ are connected.*

Theorem 9 *Let G be a 3-connected graph and u, v be two distinct vertices of G. Then there are two internally-disjoint paths $P_1[u,v]$ and $P_2[u,v]$ such that both $G - V(P_1[u,v])$ and $G - V(P_2[u,v])$ are connected.*

Corollary 1 *Let G be a 3-connected graph. For any edge $e \in E(G)$ there is a cycle C containing the edge such that $G - V(C)$ is connected.*

2 Proof of Theorems 6 and 7

The proofs of Theorem 6 and 7 will be dependent on the following basic Lemma.

Lemma 1 *Let G be a 3-connected graph and F be a subgraph of G. Let H be a connected component of $G - V(F)$. Then, for any two distinct vertices x and y in H, there is a path $Q[x,y]$ in H such that each connected component C of $H - V(Q[x,y])$ is adjacent to F, that is $E(F,C) \neq \phi$.*

This implies $\omega(G - V(Q[x,y])) \leq \omega(F)$. In particular, $G - V(Q[x,y])$ is connected if F is connected.

Proof: The result is trivial when $x = y$ or $xy \in E(G)$. We now assume that $xy \notin E(G)$.

Since H is connected, there are paths connecting x and y in H. For any such path $Q[x,y]$ in H, a connected component of $H - Q[x,y]$ is called *a good component of* $Q[x,y]$ if it is adjacent to F. Otherwise it is called *a bad component of* $Q[x,y]$. The vertices in a good component of $P[x,y]$ are called the good vertices of $P[x,y]$, and the vertices in a bad component of $P[x,y]$ are called the bad vertices of $P[x,y]$. For brevity we will call them good components, bad components, good vertices, and bad vertices respectively.

Pick a path $Q[x,y]$ in H such that:

1. The total number of bad vertices is as small as possible;

2. Subject to condition **1**, the total number of good vertices is as large as possible.

We shall show that the number of bad vertices of $Q[x,y]$ is zero after we prove the following claim.

Claim 1 *Let s and t be two non-consecutive vertices in $P[x,y]$ wiht t succeeding s in the orientation of $P[x,y]$. If there is a path $R(s,t)$ with all of its internal vertices (possibly empty) bad vertices, then there is no path joining $Q(s,t)$ to F with all its internal vertices (possibly empty) good.*

Proof: Suppose, to the contrary, there is a such path $R[a,b]$ with $a \in Q(s,t)$ and $b \in F$. Then let

$$Q^*[x,y] = Q[x,s]R(s,t)Q[t,y].$$

Note that all bad vertices of $Q^*[x,y]$ are bad vertices of $Q[x,y]$ and all vertices in $Q(s,t)$ become good vertices of $Q^*[x,y]$, which contradicts the choice of $Q[x,y]$. □

Now, we return to our proof of Lemma 1. Suppose, to the contrary, the number of bad vertices is not empty. Let B be a bad component of $H - Q[x,y]$. Since G is 3-connected, $|N(B) \cap Q[x,y]| \geq 3$. In particular, $Q(x_1,y_1) \neq \phi$, where x_1 is the first vertex of $N(B)$ in $Q[x,y]$ in the orientation of $Q[x,y]$ and y_1 be the last vertex of $N(B)$ in $Q[x,y]$ in the orientation of $Q[x,y]$.

From Claim 1, we see that there is no path connecting $Q(x_1,y_1)$ to F with all of its internal vertices good vertices of $Q[x,y]$. Since $\{x_1,y_1\}$ is not a cut set, there are two vertices s_1 and t_1 and a path $R(s_1,t_1)$ such that $s_1 \in Q(x_1,y_1)$ and $t_1 \in Q[x,x_1) \cup Q(y_1,y]$, and all the internal vertices of $R(s_1,t_1)$ (possibly empty) are bad vertices. With no loss of generality, we can assume that $t_1 \in Q[x,x_1)$. Again, by Claim 1, there is no path joining $Q(t_1,s_1)$ to F with all its internal vertices good vertices of $Q[x,y]$. Let $x_2 = t$, $y_2 = y_1$. Then we obtain a subpath $Q[x_2,y_2]$ of $Q[x,y]$ such that

- $Q[x_2,y_2] \supset Q[x_1,y_1]$,
- No path joining $Q(x_2,y_2)$ to F has all its internal vertices good vertices of $Q[x,y]$.

Recall that G is a 3-connected graph. Then $\{x_2,y_2\}$ is not a cut set. In the same manner, we can show that there is a subpath $Q[x_3,y_3]$ such that

- $Q[x_3,y_3] \supset Q[x_2,y_2]$,
- No path joining $Q(x_3,y_3)$ to F has all its internal vertices good vertices of $Q[x,y]$.

Continuing in the same manner, we see that there is an infinite sequence of subpaths $Q[x_i,y_i] \subseteq Q[x,y]$, $i = 1, 2, \cdots$, such that

- $Q[x_{i+1},y_{i+1}] \supset Q[x_i,y_i]$ for all $i \geq 1$.
- No path joining $Q(x_i,y_i)$ to F has all its internal vertices good vertices of $Q[x,y]$ for all $i \geq 1$.

Clearly, the above statements contradict that G is finite. □

Proof of Theorem 6: Let G be a k-connected graph with $k \geq 3$ and u, v be two distinct vertices of G. By Whitney's Theorem, there are k internally vertex disjoint paths

$$Q_1[u,v],\ Q_2[u,v],\ \cdots\ Q_k[u,v]$$

between the vertices u and v. We first assume that $uv \notin E(G)$.

Note that G is 3-connected since $k \geq 3$. For any connected component C of $G - V(\cup_{i=1}^{k} Q_i[u,v])$ there is a path $Q_i[u,v]$, such that there is an edge between $Q_i(u,v)$ and C; that is, $E(Q_i(u,v), C) \neq \phi$. We randomly assign C to one of the $Q_i(u,v)$ if $E(Q_i(u,v), C) \neq \phi$ for more than one i. Thus, we have a partition of the vertex set $V(G) - \{u, v\}$, say

$$V(G) - \{u,\ v\} = V_1 \cup V_2 \cup \cdots \cup V_k.$$

such that the induced subgraph $C_i = G(V_i)$ is connected and $Q_i(u,v) \subseteq C_i$ for each $i = 1, 2, \cdots, k$.

Since $E(\{u,v\}, V_i) \neq \phi$, $G - V_i$ is connected for each $i = 1, 2, \cdots, k$. For each $i = 1, 2, \cdots, k$, pick u_i, $v_i \in V_i$ such that uu_i and $vv_i \in E(G)$. Using Lemma 1, there is a path $P^*[u_i, v_i] \subseteq C_i$ such that $G - P^*[u_i, v_i]$ is connected for each $i = 1, 2, \cdots, k$. Let $P_i = uu_iP^*[u_i, v_i]v_iv$ for each $i = 1, 2, \cdots, k$.

Now we define a new graph H with the vertex set $V(H) = \{p_1, p_2, \cdots, p_k\}$ such that $p_ip_j \in E(H)$ if and only if $E(V_i, V_j) \neq \phi$. Note that, $\omega(G - (V_i \cup \{u,v\})) = \omega(H - p_i)$ for each $i = 1, 2, \cdots, k$. Using Lemma 1 again, there is a path $R_i[u,v] \subseteq G(V_i \cup \{u,v\})$ such that $\omega(G - R_i[u,v]) \leq \omega(H - p_i)$ for each $i = 1, 2, \cdots, k$. We pick such $R_i[u,v]$ such that $\omega(G - R_i[u,v])$ is minimum for each $i = 1, 2, \cdots, k$.

Since $G - \{u,v\}$ is connected, H also is a connected graph. Thus, one of the following two properties for H must hold.

A there are three vertices, say p_1, p_2, and p_3, such that none of them is a cut vertex of H, that is, $H - p_i$ is connected for $i = 1, 2, 3$.

B H is a path $p_1p_2 \cdots p_k$. In this case, we have that both $H - p_1$ and $H - p_k$ are connected.

If **A** holds, replace the path $P_i[u,v]$ by the path $R_i[u,v]$ for each $i = 1, 2, 3$.

If **B** holds, replace the path $P_i[u,v]$ by the path $R_i[u,v]$ for $i = 1$, or k. If there is an $i_0 \neq 1, k$ such that $G - V(R_i[u,v])$ is also connected, we replace $P_{i_0}[u,v]$ by $R_{i_0}[u,v]$. Otherwise we have $\omega(G - V(R_i(u,v)) = 2$ for each $2 \leq i \leq k-1$ since $\omega(H - p_i) = 2$ for every $2 \leq i \leq k-1$. Then we replace $P_i[u,v]$ by $R_i[u,v]$ for each $1 \leq i \leq k$.

So that in any case, we have k internally vertex disjoint paths between the vertices u and v satisfying Theorem 6.

If $uv \in E(G)$, we assume without loss of generality that $Q_1[u,v] = uv$. In the same manner as above, this time working only with $Q_2[u,v]$, $\cdots$, $Q_k[u,v]$, we can show that Theorem 6 holds. □

Proof of Theorem 7: REasoning as in the proof of Theorem 6, recall that $\omega(G - V(R_i[u,v])) \leq \omega(H - p_i)$ for each $1 \leq i \leq k$. Since H is connected, $\sum_{i=1}^{k} \omega(H - p_i) \leq 2(k-1)$. Combining them, we have

$$\sum_{i=1}^{k} \omega(G - V(R_i[u,v])) \leq \sum_{i=1}^{k} \omega(H - p_i) \leq 2(k-1).$$

□

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