EDGE-COLORINGS OF SOME LARGE GRAPHS ON ALPHABETS *

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Abstract

The problem of maximizing the possible number of users of a packet radio network with time division multiplexing, when the number of slots per time frame and the maximum communication delay between users are given, can be modeled by a graph. In this paper a new way of edge-coloring is presented on several families of large graphs on alphabets. This method allows us to obtain a certain improvement of the previous results.

1 Introduction

The problem of maximizing the possible number of users of a packet radio network (PRN) with time division multiplexing, when the number of slots per time frame and the maximum communication delay between users (diameter) are given, can be modeled by a graph ([3, 16]). In such models, the vertices correspond to users and the edges represent the links in such networks. Since several links to the same user are assigned to different time slots, several edges with a common vertex must be denoted by different colors. Thus we are interested in finding edge-colorings of graphs with large numbers of vertices for given values of their diameter and maximum degree (large graphs).

Graphs on alphabets can be easily obtained and have proved to be quite suitable to model large interconnection networks. Moreover, their structure usually provides efficient routing algorithms. These graphs are constructed by labeling the vertices with words on a given alphabet, and specifying a rule that relates pairs of different words to define the edges.With the

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exception of the generalized compound graphs (see [12] and [13]), which require another edge-coloring method, graphs on alphabets are among the largest graphs known when the diameter is large (greater than twenty) (see [15]). In this paper we present edge-colorings for some of these graphs.

2 Basic concepts and known results

A graph $G = (V, E)$ consists of a set V of vertices and a set E of edges that join pairs of vertices of V. The number of vertices $N = |V|$ is the order of the graph. If (x, y) is an edge of E, it is said that x and y are adjacent, and it is usually written $x \sim y$. The degree $\delta(x)$ of a vertex x is the number of vertices adjacent to x, and its maximum value over V is the degree of G, $\Delta = \Delta(G) = max\{\delta(x) : x \in V\}$. The distance between two vertices x and y, $d(x, y)$, is the length in a shortest path between x and y, and its maximum value $D = max\{d(x, y) : x, y \in V\}$ is the diameter of the graph. The order of a graph with maximum degree $\Delta > 2$ and diameter D is easily seen to be bounded above by

$$1 + \Delta + \Delta(\Delta - 1) + ... + \Delta(\Delta - 1)^{D-1} = \frac{\Delta(\Delta - 1)^D - 2}{\Delta - 2} \tag{1}$$

The right hand side is called the Moore bound, and it is known that when $D \neq 1$ it can only be attained for $D = 2$ and $\Delta = 3, 7$ and possibly 57 (see [1] and [7]). Hence it is worthwhile finding graphs which have a large number of vertices, as close as possible to the Moore bound.

In this paper we deal with graphs without selfloops and without parallel edges. An assignment of colors to the edges of a graph G so that adjacent edges are colored differently is an edge-coloring of G. The graph G is n-edge-colorable if it can be edge-colored by using n colors and the edge-coloring is then an n-edge-coloring. The minimum n for which G is n-edge-colorable is its *chromatic index* and it is usually denoted by κ_1. Vizing's theorem states that if G has maximum degree Δ, then $\Delta \leq \kappa_1 \leq 1 + \Delta$ (see for instance [6]). An edge-coloring where the chromatic index coincides with the degree is an optimal edge-coloring.

For small values of the degree and the diameter some edge-colorings are presented in [2]. For general values of the degree and the diameter Bermond and Hell [2] have shown that undirected De Bruijn and Kautz graphs have chromatic index $\kappa_1 = \Delta$. The De Bruijn and the Kautz graphs have even degree and order

$$N = \left(\frac{\Delta}{2}\right)^D \tag{2}$$

$$N = \left(\frac{\Delta}{2}\right)^D + \left(\frac{\Delta}{2}\right)^{D-1} \tag{3}$$

respectively. Moreover, Fiol has presented optimal edge-colorings for Sequence graphs ([9] and [10]). Such graphs have even diameter and order

$$N = \left(\frac{\Delta}{2}\right)^{D} \tag{4}$$

for even degree and

$$N = \left(\frac{\Delta+1}{2}\right)^{\frac{D}{2}} \left(\frac{\Delta-1}{2}\right)^{\frac{D}{2}} \tag{5}$$

for odd degree.

3 Edge-coloring of Bond graphs

Let $|X| = d_1$ and $|Y| = d_2$ be two sets. Vertex u of the Bond graph $B(d_1, d_2, k)$ (see [4, 5]) is a sequence $u = x_1 y_1 x_2 y_2 \dots x_k y_k$, where $x_i \in X$ and $y_i \in Y$, $(1 \leq i \leq k)$, such that successive elements of X must be different; i.e., $x_j \neq x_{j+1}$, $(1 \leq j \leq k-1)$. Vertex u is adjacent to the vertices of the sets:

$$\{u_x \mid u_x = x y_k x_k \dots y_2 x_2 y_1,\ \ x \in X - \{x_k\}\} \tag{6}$$

$$\{u_y \mid u_y = x_k y_{k-1} x_{k-1} \dots y_1 x_1 y,\ \ y \in Y\} \tag{7}$$

They have $N = d_1(d_1-1)^{k-1} d_2^k$ vertices, degree $\Delta = d_1 + d_2 - 1$ and diameter $D = 2k$. If $d_1 = d_2 = d$, the graph $B(d, d, k)$ has odd degree Δ, even diameter D and order

$$N = \left(\frac{\Delta+1}{2}\right)^{\frac{D}{2}+1} \left(\frac{\Delta-1}{2}\right)^{\frac{D}{2}-1} \tag{8}$$

If $d_1 - 1 = d_2 = d$ the graph $B(d+1, d, k)$ has even degree Δ, even diameter $D = 2k$ and order

$$N = \left(\frac{\Delta}{2}\right)^{D} + \left(\frac{\Delta}{2}\right)^{D-1} \tag{9}$$

Now, we define the function:

$$f(a,b) = \begin{cases} b, & if\ \ a > b \\ b-1, & if\ \ a < b \end{cases} \tag{10}$$

where $a, b \in X = \mathbf{Z}_{d_1}$, such that $a \neq b$,. The edge-coloring we propose is the following: If adjacency is according to (6), that is, for edges (u, u_x), give color $f(x_k, x) + f(x_2, x_1) \pmod{(d_1 - 1)}$, and if adjacency is according to (7), that is for edges (u, u_y), give color $y_k + y \pmod{d_2} - d_2$. Notice that the color of (u, u_x), $f(x_k, x) + f(x_2, x_1) \pmod{(d_1 - 1)}$, coincides with the color of (u_x, u), $f(x_2, x_1) + f(x_k, x) \pmod{(d_1 - 1)}$. Also, the color of (u, u_y), $y_k + y \pmod{d_2} - d_2$, is the same as the color of (u_y, u), $y + y_k \pmod{d_2} - d_2$. Thus, this edge-coloring is well defined with the colors of type (u, u_x) belong to the set $\{0, 1, \ldots, d_1 - 2\}$ and different values of x give different colors to edges (u, u_x). On the other hand, the colors of the edges of type (u, u_y) belongs to the set $\{-d_2, -d_2 + 1, \ldots, -2, -1\}$ and distinct values of y give different colors to the edges (u, u_y). Therefore, we have edge-colored the Bond graph with $d_1 + d_2 - 1 = \Delta$ colors.

4 Edge-coloring of Delorme graphs

The Delorme graph $D(d, k)$ has vertex set $V = Z_2 \times X^k$, $X = Z_d$, $d \geq 2$, where k is an odd integer. Vertex $u = a; x_1 x_2 \ldots x_{k-1} x_k$ is adjacent to vertices of the sets:

$$\{u_{k+1} \mid u_{k+1} = a; x_{k+1} x_k x_{k-1} \cdots x_3 x_2\} \tag{11}$$

$$\{u_0 \mid u_0 = a + 1; x_{k-1} x_{k-2} \cdots x_1 x_0\} \tag{12}$$

It has degree $\Delta = 2d$, diameter $D = k$ (see [8]) and order

$$N = 2\left(\frac{\Delta}{2}\right)^D \tag{13}$$

The edge-coloring we propose is the following: If adjacency is according to (11), that is for edges (u, u_{k+1}), give color $x_1 + x_{k+1} \pmod{d} - d$, and if adjacency is according to (12), that is for edges (u, u_0), give color $x_0 + x_k \pmod{d}$. Obviously, the color of (u, u_{k+1}) coincides with the color of (u_{k+1}, u) and the color of (u, u_0) coincides with the color of (u_0, u). Hence, the colors of the edges $(u, u_k + 1)$ belong to the set $\{-d, -d+1, \ldots, -2, -1\}$ and they are all different. Furthermore, the colors of the edges (u, u_0) belong to the set $\{0, 1, \ldots, d\}$ and they are also all different. Therefore, we have edge colored the Delorme graph with $2d = \Delta$ colors.

5 Edge-coloring of T(d,k) graphs

We consider the graph $T(d, k)$, proposed in [15], where k is an odd integer, whose vertices are represented by the sequences $a; x_1 x_2 \ldots x_{k-1} x_k$, with $a \in Z_2$, $x_i \in \mathbf{Z}_{d+1}$, such that $x_i \neq x_{i+1}$, $1 \leq i \leq k - 1$. Vertex $u = 0; x_1 x_2 \ldots x_{k-1} x_k$ is adjacent to the vertices of the sets:

$$\{v_{k+1} \mid v_{k+1} = 0; x_{k+1} x_k x_{k-1} \cdots x_3 x_2\} \tag{14}$$

$$\{\bar{u}_0 \mid \bar{u}_0 = 1; x_0 x_1 \ldots x_{k-2} x_{k-1}\} \tag{15}$$

$$\{\bar{u} \mid \bar{u} = 1; x_1 x_2 \ldots x_{k-1} x_k\} \tag{16}$$

and vertex $\bar{u} = 1; x_1 x_2 \ldots x_{k-1} x_k$ is adjacent to vertices of the sets:

$$\{u_{k+1} \mid u_{k+1} = 0; x_2 x_3 \cdots x_k x_{k+1}\} \tag{17}$$

$$\{\bar{v}_0 \mid \bar{v}_0 = 1; x_{k-1} x_{k-2} \ldots x_1 x_0\} \tag{18}$$

$$\{u \mid u = 0; x_1 x_2 \ldots x_{k-1} x_k\} \tag{19}$$

Such a graph has degree $\Delta = 2d + 1$, diameter $D = k$ (see [15]) and order:

$$N = 2\left(\frac{\Delta + 1}{2}\right)\left(\frac{\Delta - 1}{2}\right)^{D-1} \tag{20}$$

The edge-coloring we propose is the following: If adjacency is according to (14), that is for edges (u, v_{k+1}),we give color $f(x_k, x_{k+1}) + f(x_2, x_1)$ (mod$d$) $- d$; if adjacency is according to (15), that is for edges $(u, \bar{u}_0)$, give color $f(x_1, x_0) + f(x_{k-1}, x_k)$ (mod$d$); and if adjacency is according to (16) or (19), that is for edges $(u, \bar{u})$ or $(\bar{u}, u)$, give the color d. Likewise, if adjacency is according to (17), that is for edges $(\bar{u}, u_{k+1})$, give color $f(x_k, x_{k+1}) + f(x_2, x_1)$ (mod$d$); if adjacency is according to (18), that is for edges $(\bar{u}, \bar{v}_0)$, give color $f(x_1, x_0) + f(x_{k-1}, x_k)$ (mod$d$) $- d$. The edge-coloring is well defined since the color of (u, v_{k+1}), $(u, \bar{u}_0)$, $(u, \bar{u})$, $(\bar{u}, u_{k+1})$, $(\bar{u}, \bar{v}_0)$ is the same as (v_{k+1}, u), $(\bar{u}_0, u)$, $(\bar{u}, u)$, $(u_{k+1}, \bar{u})$, $(\bar{v}_0, \bar{u})$, respectively (the proof is similar to the proof seen in Section 3). Notice that $f(x_k, x_{k+1}) + f(x_2, x_1)(\text{mod}d) - d$ belongs to $\{-d, -d+1, \cdots, -1\}$ while $f(x_1, x_0) + f(x_{k-1}, x_k)(\text{mod}d)$ belongs to the set $\{0, 1, \cdots, d-1\}$. Finally, color d does not belong to either of these two sets. So, we have edge-colored the $T(d, k)$ graph with $2d + 1 = \Delta$ colors.

6 CONCLUSIONS

In this paper, we describe optimal edge-colorings for some graphs that improve almost all of the previous results when D is large (see the previous results in Section 2). Furthermore, some of the results we present here could even be improved by adding vertices without increasing either the degree or the diameter (see [11, 14]). Next, we give a comparative table with the best orders, for the case of large diameters.

D / $\kappa_1 = \Delta$	even	odd
even	$N = \left(\frac{\Delta}{2}\right)^D + \left(\frac{\Delta}{2}\right)^{D-1}$ Kautz graph $N = \left(\frac{\Delta}{2}\right)^D + \left(\frac{\Delta}{2}\right)^{D-1}$ **Bond graph**	$N = 2\left(\frac{\Delta}{2}\right)^D$ **Delorme graph** $N = \left(\frac{\Delta}{2}\right)^D + \left(\frac{\Delta}{2}\right)^{D-1}$ Kautz graph
odd	$N = \left(\frac{\Delta+1}{2}\right)^{\frac{D}{2}+1} \left(\frac{\Delta-1}{2}\right)^{\frac{D}{2}-1}$ **Bond graph** $N = \left(\frac{\Delta+1}{2}\right)^{\frac{D}{2}} \left(\frac{\Delta-1}{2}\right)^{\frac{D}{2}}$ Sequence graph	$N = 2\left(\frac{\Delta+1}{2}\right)\left(\frac{\Delta-1}{2}\right)^{D-1}$ **T(d, k) graph**

Graphs studied in this paper are displayed in boldtype.

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