On graphs determined by their k-subgraphs

Yair Caro
Department of Mathematics
School of Education
University of Haifa - Oranim
Tivon, 36006
Israel

ABSTRACT. The following problem is formulated:

Let $P(G)$ be a graph parameter and let k and ℓ be integers such that $k > \ell \geq 0$. Suppose $|G| = n$ and for any two k-subsets $A, B \subset V(G)$ such that $|A \cap B| = \ell$ it follows that $P(\langle A \rangle) = P(\langle B \rangle)$. Characterize G.

We solve this problem for two parameters, the domination number and the number of edges modulo m (for any $m \geq 2$). These solutions extend and are based on an earlier work that dated back to a 1960 theorem of Kelly and Merriell.

1 Introduction

In 1960 Paul Kelly and David Merriell [KM] proved the following theorem, with the convention that $\langle A \rangle$ is the induced subgraph on the vertex set A.

Theorem A. *Let G be a graph on $2n$ vertices such that for every n-subset A, $\langle A \rangle \simeq \langle V \backslash A \rangle$. Then G or $\overline{G}$ belongs to the class*

$$\{K_{2n}, K_{n,n}, nK_2, K_n \times K_2, 2C_4\}.$$

□

Although this theorem carries the flavor of classical graph theory no elaborations of this elegant result can be found in the literature. Recently [CA] closely related problems were solved using connectivity of the Kneser's graphs.

Recall the definition of the Kneser graph $K(n, k, \ell)$ whose vertex set is the set of k-subsets of $\{1, 2, \ldots, n\}$, namely $[n]^k$, two vertices being adjacent if the corresponding k-subsets intersect in exactly ℓ elements. Clearly $n >$

$k > \ell \geq 0$ and we shall call a triple (n, k, ℓ) trivial if either $2k - \ell > n$ in which case $K(n, k, \ell)$ contains no edges, or $(n, k, \ell) = (2k, k, 0)$ in which case $K(2k, k, 0)$ is a matching, and this is the exceptional case related to the Kelly-Merriell theorem.

Theorem B. [CA] *$K(n, k, \ell)$ is connected iff (n, k, ℓ) is not a trivial triple.* □

The main contribution of theorem B is that it allows one to deduce that a property that holds for any two subsets $A, B \subset V$ such that $|A| = |B| = k$, $|A \cap B| = \ell$, $|V| = n$ and (n, k, ℓ) is not trivial, holds for all k-subsets of V. Thus using theorem B it was possible to prove a "completion" to the Kelly-Merriell theorem, (here $e\langle A \rangle$ denoted the number of edges in the induced subgraph on vertex set A).

Theorem C. [CA] *Let $n > k > \ell \geq 0$ be integers such that $2k - \ell \leq n$. Let G be a graph on n vertices such that for $A, B \subset (G)$, $|A| = |B| = k \geq 2$, $|A \cap B| = \ell$ it follows that $e\langle A \rangle = e\langle B \rangle$, then the following hold:*

(i) *if $(n, k, \ell) \in \{(k+1, k, k-1), (2k, k, 0)\}$ then G is regular.*

(ii) *if $(n, k, \ell) \notin \{(k+1, k, k-1), (2k, k, 0)\}$ then $G \in \{K_n, \overline{K}_n\}$.* □

Theorem C suggests consideration of a larger class of graph invariant called complete parameters, which we define below.

Let $P(G)$ be a graph parameter (e.g., number of edges, chromatic number, independence number, domination number). We say that $P(G)$ is a complete parameter if for every $k \geq 2$ there exists two real numbers $a_k \leq b_k$ such that if $|V(G)| = k$ then $P(G) \in \{a_k, b_k\}$ iff $G \in \{K_k, \overline{K}_k\}$. Thus $e(G)$ is a complete parameter with $a_k = 0$ and $b_k = \binom{k}{2}$. The chromatic number is a complete parameter with $a_k = 1$ and $b_k = k$, and so are the independence number and the clique number. On the other hand the domination number $\gamma(G)$ is not a complete parameter, nor is the number of edges (mod m) for certain values of m.

A weak version of Theorem C for complete parameters is given by:

Theorem D. [CA] *Let $P(G)$ be a complete parameter, and let $k \geq 2$ and $\ell \geq 0$ be fixed integers such that $k > \ell$. Suppose H is a graph on n vertices such that for every pair of k-subsets A and B satisfying $|A \cap B| = \ell$ the following equality holds: $P\langle A \rangle = P\langle B \rangle$. Then for $n \geq N(k)$, $H \in \{K_n, \overline{K}_n\}$.* □

In Theorem D, $N(k) = \max\{R(k, k), 2k + 1\}$ is a valid choice, where $R(k, k)$ is the classical Ramsey number for monochromatic K_k.

The general characterization problem that we state here is:

Problem 1. Let $P(G)$ be a graph parameter (invariant) and let $k > \ell \geq 0$ be integers. Suppose $|G| = n$ and for any two k-subsets $A, B \subset V(G)$ such that $|A \cap B| = \ell$ it follows that $P\langle A \rangle = P\langle B \rangle$. Can we characterize G?

We shall study two incomplete parameters (after Theorem D), which are the number of edges modulo m in the induced subgraphs of order k and the domination number in the induced subgraphs of order k. For the first parameter $e_m(A) \equiv e\langle A\rangle \pmod m$ we give a complete solution providing $|G| \geq \max\{R(k,k), 2k+1\}$. For the second parameter $\gamma(G)$ we shall give a complete solution for the cases $k = 2, 3$ and for $k \geq 4$ a complete solution providing $|G| \geq N(k)$. Lastly our notation will follow that of Bollobás [BO].

2 The nunber of edges modulo m

In this section we consider the incomplete parameter $e_m(A) \equiv e\langle A\rangle \pmod m$ which is the number of edges modulo m in the induced subgraph on the vertex-set A.

We shall denote by $e(v\colon A)$ the number of vertices in A adjacent to the vertex $v \notin A$.

Our main result is that for $|G| \geq \max\{R(k,k), 2k+1\}$ the graphs that satisfy the constraints of problem 1 belong to the family $\{K_n, \overline{K}_n, K_{a,b}, \overline{K}_{a,b}\}$.

We need two lemmas of some interest by their own.

Lemma 1. *Let $k, m \geq 2$ be integers and G be a graph on at least $2k+1$ vertices such that for any two subsets $A, B \subset V(G)$, $|A| = |B| = k$ it follows that $e\langle A\rangle \equiv e\langle B\rangle \pmod m$. Assume further that G contains an induced $\overline{K}_k$. Then one of the following cases occurs:*

1) $G = \overline{K}_n$.

2) $k \equiv 1 \pmod m$ *and* $G = K_{1,n}$.

3) $m = 2$, $k \equiv 1 \pmod 2$ *and* $G = K_{a,b}$.

Proof: Denote by $A = V(\overline{K}_k)$ and observe $e\langle A\rangle \equiv 0 \pmod m$.

(1) Suppose $k \leq m$. Consider $v \in V \backslash A$.

If $e(v\colon A) > 0$ then either v is adjacent to all vertices of A, but then for each $u \in A$ $e\langle B\rangle = e\langle (A - \{u\}) \cup \{v\}\rangle = e\langle A\rangle + k - 1 = k - 1 \not\equiv 0 \pmod m$, or there is a vertex $u \in A$ which is not adjacent to v and then $e\langle B\rangle = e\langle (A - \{u\}) \cup \{v\}\rangle = e\langle A\rangle + e(v\colon A) = e(v\colon A) \not\equiv 0 \pmod m$ and in both cases $e\langle B\rangle \not\equiv e\langle A\rangle \pmod m$. Hence for each $v \in V\backslash A$ $e(v\colon A) = 0$. Suppose now $u, v \in V\backslash A$ are adjacent. Then for $u_1, u_2 \in A$ and $B = (A\backslash\{u_1, u_2\}) \cup \{u, v\}$ we get $e\langle B\rangle \equiv 1 \pmod m$. Hence also $e\langle V\backslash A\rangle = 0$ and we conclude that $G = \overline{K}_n$.

(2) Suppose $k \geq m+1$. Consider $v \in V\backslash A$.

Let $e(v\colon A) = t$, $0 < t < k$ and write $u_1, u_2, \ldots, u_t$ for the vertices in A adjacent to v and $v_1, v_2, \ldots, v_{k-t}$ for the non-adjacent vertices. Set $B_1 = (A \backslash \{u_1\}) \cup \{v\})$, $B_2 = (A \backslash \{v_1\}) \cup \{v\})$, then $e\langle B_1 \rangle = e\langle B_2 \rangle - 1$. Hence $e\langle B_1 \rangle \not\equiv e\langle B_2 \rangle \pmod{m}$.

Hence we may assume $e(v\colon A) = 0$ or $e(v\colon A) = k$.

Denote by $C = \{v \in V \backslash A\colon e(v\colon A) = k\}$, and $D = \{v \in V \backslash A\colon e(v\colon A) = 0\}$.

Observe that $C \cup D = V \backslash A$ and $|C| + |D| \geq k + 1 \geq 3$.

It is clear that if $k \not\equiv 1 \pmod{m}$ then $C = \phi$ for otherwise if $v \in C$ and $u \in A$ then for $B = (A \backslash \{u\}) \cup \{v\})$, $e\langle B \rangle = k - 1 \not\equiv 0 \pmod{m}$. Suppose $|C| \geq 2$ (hence $k \equiv 1 \pmod{m}$) and let $u_1, u_2 \in A$, $v_1, v_2 \in C$, $B = (A \backslash \{u_1, u_2\}) \cup \{v_1, v_2\})$. Then clearly we obtain

$$e\langle B \rangle = 2(k-2) + \begin{cases} 0 & \text{if } (v_1, v_2) \notin E(G) \\ 1 & \text{if } (v_1, v_2) \in E(G) \end{cases} = \begin{cases} 2k-4 \\ 2k-3 \end{cases} \text{respectively.}$$

But we must have $e\langle B \rangle \equiv 0 \pmod{m}$ and also $k \equiv 1 \pmod{m}$ which is possible only if $m = 2$ (since $2k - 2 \equiv 2k - 4 \equiv 0 \pmod{m}$) and C induces an independent set in G.

Suppose $|D| \geq 2$, $u_1, u_2 \in A$, $v_1, v_2 \in D$ and $(v_1, v_2) \in E(G)$. Then for $B = (A \backslash \{u_1, u_2\}) \cup \{v_1, v_2\})$ we get $e\langle B \rangle = 1 \equiv 1 \pmod{m}$, hence D also induces an independent set in G.

Now suppose $v_1 \in C$, $v_2 \in D$ and $(v_1, v_2) \notin E(G)$ (and clearly $k \equiv 1 \pmod{m}$ since $C \neq \phi$) and let $u_1, u_2 \in A$. Then for $B = (A \backslash \{u_1, u_2\}) \cup \{v_1, v_2\})$ we get $e\langle B \rangle = k - 2 \not\equiv 0 \pmod{m}$. Hence for every $u \in C$ and $v \in D$ it follows that $(u, v) \in E(G)$. Now we can conclude lemma 1.

If $|C| = 0$ then as $D \cup A$ forms an independent set in G it follows that $G = \overline{K}_n$.

If $|C| = 1$ then $k \equiv 1 \pmod{m}$, $D \cup A$ is an independent set in G and $G = K_{1,n}$ (for each $B \subset V$, $|B| = k$, $\langle B \rangle \in \{\overline{K}_k, K_{1.k-1}\}$).

If $|C| \geq 2$ then $m = 2$, $k \equiv 1 \pmod{2}$, $D' = D \cup A$ is an independent set in G, C is an independent set in G and G is a complete bipartite $K_{a,b}$ (for each $B \in V$, $|B| = k$ $e\langle B \rangle \equiv 0 \pmod{2}$). □

Lemma 2. *Let $k, m \geq 2$ be integers and G be a graph on at least $2k + 1$ vertices such that for any two subsets $A, B \subset V(G)$, $|A| = |B| = k$ it follows that $e\langle A \rangle \equiv e\langle B \rangle \pmod{m}$. Assume further that G contains a K_k. Then one of the following cases occurs:*

1) $G = K_n$.

2) $k \equiv 1 \pmod{m}$ *and* $G = \overline{K}_{1,n} = K_1 \cup K_n$.

3) $m = 2$, $k \equiv 1 \pmod 2$ and $G = \overline{K}_{a,b} = K_a \cup K_b$.

Proof: Consider the complement $\overline{G}$. Clearly $\overline{G}$ contains an induced $\overline{K}_k$ and also $e\langle\overline{A}\rangle = \binom{k}{2} - e\langle A\rangle$ for each k-subset A of $V(G)$. Hence for any two subsets $\overline{A}, \overline{B} \subset V(\overline{G})$ $e\langle\overline{A}\rangle \equiv e\langle\overline{B}\rangle \pmod m$. Now by lemma 1 the structure of $\overline{G}$ is known and taking complements we are done. □

We can now state and prove the main theorem of this section.

Theorem 1. *Let k, ℓ, m be integers such that $k, m \geq 2$ and $k > \ell \geq 0$ and let G be a graph on n vertices $n \geq R(k,k)$ for $k \geq 4$, $n \geq 7$ for $k = 3$, $n \geq 5$ for $k = 2$.*

Assume for any two k-subsets $A, B \subset V(G)$ such that $|A \cap B| = \ell$ it follows that $e(A) \equiv e(B) \pmod m$. Then one of the following cases occurs.

1) $G \in \{K_n, \overline{K}_n\}$

2) $k \equiv 1 \pmod m$ and $G \in \{K_{1,n-1}, \overline{K}_{1,n-1}\}$

3) $m = 2$, $k \equiv 1 \pmod 2$ and $G \in \{K_{a,b}, \overline{K}_{a,b}\}$, $a + b = n$.

Proof: Observe first that $|G| = n \geq \max(R(k,k), 2k+1)$ hence the conditions of lemma 1 and lemma 2 are satisfied and there exists either K_k or $\overline{K}_k$ in G.

Also since $n \geq 2k+1$ it follows that (n,k,ℓ) is a non-trivial triple and the corresponding Kneser's graph $K(n,k,\ell)$ is connected. But now it follows that the congruence $e\langle A\rangle \equiv e\langle B\rangle \pmod m$ for $|a \cap B| = \ell$ spreads out to all k-subsets of $V(G)$, and combining lemma 1 and lemma 2 we are done. □

Remark. The condition $|G| \geq \max\{R(k,k), 2k+1\}$ in theorem 1 can't be replaced by $|G| \geq 2k$ since e.g., for $m \equiv 1 \pmod 2$ and $k \equiv 0 \pmod 2$ any graph G on $2k$ vertices in which for each vertex u, $\deg u \equiv 0 \pmod m$ satisfies for $(n,k,\ell) = (2k,k,0)$ the congruence $e\langle A\rangle \equiv e\langle V \backslash A\rangle \pmod m$ as can be checked. This holds as well for the Kelly-Merriell graphs.

It is an open problem whether we can replace $R(k,k)$ by $2k+1$ in theorem 1, but there are few evidences that for fixed m and sufficiently large k such that $m \mid \binom{k}{2}$ this might be true, due to some recent results in zero-sum Ramsey theory (see e.g. [AC]).

3 The domination number

Recall first that the domination number, denoted by $\gamma(G)$, is the minimum cardinality of a set $S \subseteq V$ such that any vertex $u \in V \backslash S$ has a neighbor in S.

Unlike the parameter $e(G)$ for which $e\langle A\rangle = e\langle B\rangle \Leftrightarrow e\langle\overline{A}\rangle = e\langle\overline{B}\rangle$ for $A, B \subset V(G)$, $|A| = |B|$, (and hence the implied congruences), for the domination number this complementary relation is far from true, e.g., $\gamma(K_n) = \gamma(K_{1,n-1}) = 1$ but $\gamma(\overline{K}_n) = n$ and $\gamma(\overline{K}_{1,n-1}) = 2$. Observe also that for any graph G containing a spanning star it follows that $\gamma(G) = 1$, hence the domination number is not a complete parameter.

Our main theorem in this section is:

Theorem 2. *Let k, ℓ be integers $k > \ell \geq 0$, $k \geq 2$, and let G be a graph on n vertices such that $n \geq \max\{(k-1)^2+1, 2k+1\}$ and for any k-subsets $A, B \subset V(G)$, $|A \cap B| = \ell$ it follows that $\gamma\langle A\rangle = \gamma\langle B\rangle$. Then the following situations hold:*

1) $k = 2$ $G \in \{K_n, \overline{K}_n, 2K_2, C_4\}$, *the cases $2K_2$ and C_4 are valid only for the triple (4,2,0).*

2) $k = 3$, $G \in \{\overline{K}_n, K_n \backslash tK_2 \colon 0 \leq t \leq \lfloor \frac{n}{2} \rfloor, 2K_2, K_{3,3}, K_3 \times K_2, 2K_3, C_6, 3K_2, K_6\backslash 2K_{1,2}\}$, *the cases $K_{3,3}, K_3 \times K_2, 2K_3, C_6, 3K_2$ and $K_6 \backslash 2K_{1,2}$ are valid only for the triple (6,3,0).*

3) $k \equiv 0 \pmod 2$, *there exists a finite family of graphs $F_0(k)$ such that*

$$G \in \{\overline{K}_n, K_n \backslash E(H) \colon H \in F_0(k)\}$$

4) $k \equiv 1 \pmod 2$, *there exists a finite family of graphs $F_1(k)$ such that*

$$G \in \{\overline{K}_n, K_n \backslash tK_2, 0 \leq t \leq \lfloor \frac{n}{2} \rfloor, K_n \backslash E(H) \colon H \in F_1(k)\}.$$

Proof: For fixed k let $N(k) = \max\{(k-1)^2+1, 2k+1\}$ and let G be a graph on n vertices, $n \geq N(k)$, satisfying the conditions of Theorem 2. Observe that the condition $n \geq (k-1)^2 + 1$ and the celebrated Ramsey type theorem of Chvatal [CH] implies that G contains either $K_{1,k-1}$ or $\overline{K}_k$. Hence by the conditions of theorem 2 and using Theorem B we infer that either for every k-subset A, $\gamma\langle A\rangle = k$ in which case $G = \overline{K}_n$ and we are done, or for every k-subset A, $\gamma\langle A\rangle = 1$.

Hence from now we assume $\gamma\langle A\rangle \equiv 1$ for all induced k-subgraphs of G. Recall now the celebrated theorem of Erdös and Rado [ER] about Δ-systems, which for graphs states that if G is a graph having at least k^2-k+1 edges then G contains either kK_2 or $K_{1,k}$ (see e.g. [BO] p. 87-90). We shall use this result to consider k-subsets in G and its complement $\overline{G}$ in order to obtain more information on the structure of G.

Suppose $G = K_n \backslash E(H)$ for any graph H for which $e(H) > k^2 - k$.

Then $e(\overline{G}) = e(H) \geq k^2 - k + 1$ and, by Erdös-Rado, $\overline{G}$ contains a subgraph $Q \in \{K_{1,k-1}, \lfloor \frac{k}{2} \rfloor K_2\}$, (we need only $\lfloor \frac{k}{2} \rfloor K_2$ and not kK_2).

Now we consider two possible cases according to the parity of k.

Case 1. $k \equiv 0 \pmod 2$.

Then $|V(Q)| = k$ and clearly $\gamma(\overline{Q}) \geq 2$ because in any case the k-subset $V(Q)$ induces in G a subgraph without a spanning star, contradicting the assumption above that for every k-subset A, $A \subset V(G)$, $\gamma\langle A\rangle = 1$.

Hence if $k \equiv 0 \pmod 2$ and G a graph on $n \geq N(k)$ vertices is represented by $G = K_n \backslash E(H)$, we infer that $e(H) \leq k^2 - k$ and $|V(H)| \leq \min\{n, 2(k^2 - k)\}$ and we can now define the required class $F_0(k)$ as follows:

$$\begin{aligned} F_0(k) = \{H\colon & e(H) \leq k^2 - k, |V(H)| \leq 2(k^2 - k) \text{ and for} \\ & N(k) \leq n \leq 2(k^2 - k) \text{ if } G = K_n \backslash E(H) \\ & \text{then for every } k\text{-subset } A \text{ of } V(G), \gamma\langle A\rangle = 1\}. \end{aligned}$$

It follows from the reasoning above that for every $n \geq N(k)$ if $k \equiv 0 \pmod 2$ and G satisfies the conditions of Theorem 2 then $G \in \{\overline{K}_n, K_n \backslash E(H)\colon H \in F_0(k)\}$.

Observe that the actual construction of $F_0(k)$ is a complicated task but it depends only on k from complexity point of view.

Case 2. $k \equiv 1 \pmod 2$

Recall the possibilities for $Q \in \{K_{1,k-1}, \frac{k-1}{2}K_2\}$.

If $\overline{G}$ contains $K_{1,k-1}$ then G contains a k-subset A without a spanning star and $\gamma\langle A\rangle \geq 2$ a contradiction as before. Also if $\overline{G}$ contains the subgraph $\frac{k-3}{2}K_2 \cup P_3$ then again G contains a k-subset A without a spanning star and $\gamma\langle A\rangle \geq 2$, a contradiction.

Moreover, for $k \equiv 1 \pmod 2$ and $G = K_n \backslash tK_2$, $0 \leq t \leq \lfloor \frac{n}{2} \rfloor$, it is easy to verify that G has the property that for every k-subset A of $V(G)$, $\gamma\langle A\rangle = 1$ as required.

Hence if $k \equiv 1 \pmod 2$ and G a graph on $n \geq N(k)$ vertices is represented by $G = K_n \backslash E(H)$, we infer that either $H = tK_2$ or $e(H) \leq k^2 - k$, $|V(H)| \leq 2(k^2 - k)$ and we can now define the required class $F_1(k)$ as follows:

$$\begin{aligned} F_1(k) = \{H\colon & e(H) \leq k^2 - k, |V(H)| \leq 2(k^2 - k) \text{ and for} \\ & N(k) \leq n \leq 2(k^2 - k) \text{ if } G = K_n \backslash E(H) \\ & \text{then for every } k\text{-subset } A \text{ of } V(G), \gamma\langle A\rangle = 1\}. \end{aligned}$$

It follows from the reasoning above that for every $n \geq N(k)$ if $k \equiv 1 \pmod 2$ and G satisfies the conditions of Theorem 2 then $G \in \{\overline{K}_n, K_n \backslash tK_2$ for $0 \leq t \leq \lfloor \frac{n}{2} \rfloor, K_n \backslash E(H)\colon H \in F_1(k)\}$.

Again, constructing $F_1(k)$ is time consuming but depends on k only.

The cases $k = 2, 3$ were completely determined by a little more work. In case $k = 2$ all graphs belong to the Kelly-Merriell class, but for $k = 3$ there exists an exception $G = K_6 \backslash 2K_{1,2}$ which is a valid graph for the triple (6,3,0) but which doesn't belong to the Kelly-Merriell class. It is also easy to see that for each k and any graph H on less than $\frac{k}{2}$ edges $G = K_n \backslash E(H)$ is a valid choice and $H \in F_i(k)$ $i = 0, 1$, showing that $|F_i(k)|$ grows rather fast. □

Concluding Remarks. The decision problem: "Does G satisfy the requirements of Problem 1" can be solved using trivial brute-force method in time $c(k)n^k$, and thus is polynomial for fixed k.

The results mentioned in theorem C and theorems 1-2 show that for these parameters the above mentioned decision problem can be solved, working on $\overline{G}$, in time $O(|E(G)|) + c(k)$, where $c(k)$ a constant that depends on k only. For an important progress see the recent papers [CY1], [CY2].

References

[AC] N. Alon, Y. Caro, On three zero-sum Ramsey-type problems, *J. Graph Theory* **17** (1993), 177–192.

[BO] B. Bollobás, *Extremal graph theory*, Academic Press, New York, 1978.

[CA] Y. Caro, On graphs with prescribed subgraphs of order k, and a theorem of Kelly and Merriell, *Czechoslovakia Journal of Mathematics* **44** (1994), 623–629.

[CY1] Y. Caro and R. Yuster, Recognizing global occurrence of local properties, *Journal of Complexity*, to appear.

[CY2] Y. Caro and R. Yuster, Graphs having the local decomposition property, submitted.

[CH] V. Chvatal, Tree-complete graph Ramsey numbers, *J. Graph Theory* **1** (1977), 93–94.

[ER] P. Erdös, R. Rado, Intersection theorems for systems of sets, *J. London Math. Soc.* **35** (1960), 85–90.

[KM] P. Kelly, D. Merriell, A class of graphs, *Trans.. Amer. Math. Soc.* **96** (1960), 488–492.