Scenic Graphs I: Traceable Graphs

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Abstract

A path of a graph is *maximal* if it is not a proper subpath of any other path of the graph. The *path spectrum* is the set of lengths of all maximal paths in the graph. A graph is *scenic* if its path spectrum is a singleton set. In this paper we give a new proof characterizing all scenic graphs with a Hamiltonian path; this result was first proven by Thomassen in 1974. The proof strategy used here is also applied in a companion paper in which we characterize scenic graphs with no Hamiltonian path.

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1. Introduction

A well-studied parameter in the field of graph theory is the length of the longest path of a graph. Because computing this parameter enables one to decide, for example, whether a graph has a Hamiltonian path, it is a difficult parameter to compute. Continuing an arbitrary path by simply wandering around in the graph may produce "wrong turns" which hinder or prevent the extension of the path into a longest path. In this paper, we are interested in determining those connected graphs in which "wandering" from either end of an arbitrary path never fails to yield a longest path. We call such graphs *scenic* graphs because every path in such a graph extends to a longest path visiting the same number of vertices as a "scenic tour" of the graph. Note that a longest path of a scenic graph is not necessarily a Hamiltonian path; that is, a scenic graph may not be traceable.

A path in a graph is a sequence of distinct vertices in which consecutive vertices are adjacent. The *length* of a path is the number of edges in the path. A path P is a *subpath* of Q if the sequence corresponding to P appears as a consecutive subsequence of Q. A subpath P of a path Q is *proper* if $P \neq Q$. If P is a proper subpath of Q, then we shall say that P *extends* to Q, Q *extends* P, or Q is an *extension* of P. A path is *maximal* if it is not a proper subpath of any other path, or equivalently, if it has no extension. The *path spectrum* of a connected graph G is the set of lengths of all maximal paths in G. The concept of path spectrum was first introduced by Jacobson et al. [3]. We define a connected graph to be *scenic* if its path spectrum is a singleton. A graph with a Hamiltonian path is called *traceable.*

A traceable graph is scenic if and only if every path is contained in a Hamiltonian path. Graphs with this property were first characterized by Thomassen [6] in 1974. The purpose of this paper is to give a new proof characterizing all scenic graphs with a Hamiltonian path. The proof strategy used here is also applied in a companion paper [4] in which we characterize scenic graphs with no Hamiltonian path.

The notion of scenic graphs extends two avenues of research in graph theory. The first avenue of research investigates contexts in which "maximal implies maximum," as in the concept of well covered graphs for the parameter of independence (see Plummer's paper [5] for a survey of well covered graphs). In a scenic graph, every path extends to a maximum length path, or equivalently, every maximal path is a maximum path. So scenic graphs exhibit this phenomenon. The second avenue of research concerns the idea of being able to find a longest path by randomly extending the path from one of its endvertices. Chartrand and Kronk [1] give the following definition: "A traceable graph G is called *randomly traceable* if a Hamiltonian path always results upon starting at any vertex of G and successively proceeding to any adjacent vertex not yet encountered." Randomly traceable graphs can be viewed as structures in which any maximal 'one-way' extension of any path (with one endvertex fixed) is a Hamiltonian path. Fink [2] has investigated a related notion of randomly near-traceable graphs. Chartrand and Kronk obtain the following result.

Theorem 1.1 (Chartrand and Kronk). *A graph is randomly traceable if and only if it is isomorphic to one of the following graphs:* K_n, $K_{p,p}$ *or* C_n.

We will see that traceable and scenic (i.e.,'two-way' extendible) graphs form a much larger family than randomly traceable (i.e., 'one-way' extendible) graphs. Except for paths P_n $(n \geq 1)$, cycles C_n $(n \geq 3)$, the prism, and the cube, traceable scenic graphs emerge from cliques, K_n $(n \geq 1)$, and from the complete bipartite graphs $K_{p,p}$ and $K_{p,p+1}$ $(p \geq 1)$. We now introduce some notation to explain this more precisely. The union of t disjoint edges (a matching) will be denoted by tK_2. The graph obtained from K_n by removing the edges of a copy of tK_2 $(1 \leq t \leq n/2)$ is denoted by $K_n - tK_2$. The complete $p \times p$ bipartite graph plus (resp. minus) an edge is denoted $K_{p,p} + K_2$ (resp. $K_{p,p} - K_2$). The graph obtained from the complete $p \times p$ bipartite graph by adding one edge into each partite set is denoted $K_{p,p} + 2K_2$. The *prism* $(K_6 - C_6)$ is the graph obtained from K_6 by removing the edges of a six-cycle, the *cube* $(K_{4,4} - 4K_2)$ is the graph obtained from the complete 4×4 bipartite graph by removing four disjoint edges. If $H \in \{K_3, 2K_2, K_{1,q}\}$, then $K_{p,p+1} + H$ denotes the graph obtained from the complete $p \times (p+1)$ bipartite graph by adding all the edges of H to the **largest** partite set containing $p+1$ vertices. In this paper we give a new proof of the following theorem first proven by Thomassen [6].

Theorem 1.2. *A traceable graph is scenic if and only if it belongs to one of the following families:*

$$\begin{aligned}
\Phi[K_n] &= \{K_n,\ K_n - tK_2\ (1 \leq t \leq n/2)\},\\
\Phi[K_{p,p}] &= \{K_{p,p}, K_{p,p} - K_2, K_{p,p} + K_2, K_{p,p} + 2K_2\ \},\\
\Phi[K_{p,p+1}] &= \{K_{p,p+1}, K_{p,p+1} + K_3, K_{p,p+1} + 2K_2,\\
&\qquad K_{p,p+1} + K_{1,q}\ (1 \leq q \leq p)\}\\
\Psi &= \{P_n,\ C_n,\ prism,\ cube\}.
\end{aligned}$$

In Section 2. we show that the graphs defined in Theorem 1.2 are in fact traceable, scenic graphs. The proof of Theorem 1.2 for triangle-free graphs is given in Section 3.. The main tool in characterizing traceable scenic graphs with triangles is the following result:

- *A traceable scenic graph remains traceable and scenic after the removal of the vertices of any maximal clique of order at least three.*

We prove this 'clique removal theorem' in Section 4. (Theorem 4.1). The reverse operation — extending traceable scenic graphs by adding disjoint cliques — is called *clique extension*, and it is discussed in the last two sections.

Let $\omega = \omega(G)$ denote the order of a maximum clique of G. In Section 5. we show that all traceable scenic graphs with $\omega \geq 5$ have the form $K_n - tK_2$, for some $n \geq 5$ and $0 \leq t \leq n/2$, and we describe all traceable scenic graphs with $\omega = 3$ or $\omega = 4$ that can be obtained from triangle-free traceable scenic graphs via clique extension. The proof of Theorem 1.2 concludes in Section 6. where it is shown that any traceable scenic graph can be obtained from a triangle-free traceable scenic graph or from a clique by successively applying at most two clique extensions.

2. Traceable Scenic Graphs

In this section we show that the graphs given in Theorem 1.2 are in fact traceable and scenic. Because every graph G in $\Phi[K_n] \cup \Phi[K_{p,p}] \cup \Phi[K_{p,p+1}] \cup \Psi$ is traceable, G is scenic if every non-Hamiltonian path $P \subset G$ extends to a Hamiltonian path of G. It is obvious by inspection that the graphs in $\Phi[K_n] \cup \Psi$ have this property. The vertices in each partite set of a $K_{p,p}$ or a $K_{p,p+1}$ $(p \geq 1)$ are adjacent to all the vertices in the other partite set. Thus any path P extends until both of the partite sets are exhausted, i.e., P extends to a Hamiltonian path in both cases. To see that $K_{p,p} - K_2$ is scenic, suppose that xy is missing from $K_{p,p}$. If both x and y are vertices of P, then P extends to a Hamiltonian path as in a $K_{p,p}$. If one of x and y, say y, is not covered by P, then P extends to a maximal path Q in the copy of $K_{p-1,p}$ in $K_{p,p}$ not containing y. Both endpoints of Q are in the same partite set (opposite to y); therefore one of them is adjacent to y, and Q extends to a Hamiltonian path in $K_{p,p} - K_2$.

To see that $K_{p,p} + K_2$ is scenic, let x_1 and x_2 be the endvertices of the additional edge (in the same partite set). If the edge x_1x_2 is not covered by

P, then P extends to a Hamiltonian path as in a $K_{p,p}$. If x_1x_2 is an edge of P, then identify x_1 and x_2 in $K_{p,p}$, and consider the resulting graph that is isomorphic to $K_{p-1,p}$. The path P' we obtain from P by contracting x_1x_2 extends to a Hamiltonian path in $K_{p-1,p}$ which in turn defines a Hamiltonian extension of P in $K_{p,p}$. The same argument may be used to show that any path P in $K_{p,p+1} + K_2$ is extendible to a Hamiltonian path. Similarly, any path P in $K_{p,p} + 2K_2$ is extendible to a Hamiltonian path. Hence $K_{p,p+1}$ and each graph in $\Phi[K_{p,p}]$ is scenic.

For $p \geq 2$, consider $K_{p,p+1} + H$, where H is one of the graphs $K_{1,q}$ $(1 \leq q \leq p), 2K_2$ or K_3 added to the $(p+1)$-element partite set of $K_{p,p+1}$. If a non-Hamiltonian path P does not contain an edge of H, then P extends to a Hamiltonian path as in $K_{p,p+1}$. Otherwise, $P \cap H$ is either an edge y_1y_2 or a 3$-$path (y_1, y_2, y_3). Identify y_1 and y_2 in the first case and y_1, y_2 and y_3 in the second case, and remove all edges of H incident with the identified vertex. Thus we obtain either $K_{p-1,p}$ or a scenic graph from $\Phi[K_{p,p}]$. In both cases our path P' extends to a Hamiltonian path of the truncated graph and defines a Hamiltonian extension of P in $K_{p,p+1} + H$. Hence every graph in $\Phi[K_{p,p+1}]$ is scenic.

When discussing triangle-free traceable scenic graphs in the next section we need the following lemma.

Lemma 2.1. *For $p \geq 4$ and $1 \leq t \leq p$, the graph $G = K_{p,p} - tK_2$ is traceable and scenic if and only if G is a cube or a $K_{p,p} - K_2$.*

Proof. Let $\{x_1, \ldots, x_p\}$ and $\{y_1, \ldots, y_p\}$ be the partite sets of $K_{p,p}$ and assume that some $t-$element subset of $\{\ x_iy_i\ :\ 1 \leq i \leq p\ \}$ defines the missing tK_2. First we show that $p \geq 5$ implies $t \leq 1$. If this is not true, say $x_1y_1, x_py_p \notin E(G)$, then $(x_1, y_3, x_2, y_4, x_3, \ldots, y_{p-1}, x_{p-2}, y_2, x_{p-1}, y_p)$ is a maximal non-Hamiltonian path, because it misses y_1 and x_p that are nonadjacent to its endvertices x_1 and y_p. Next we show that for $p = 4$, if $t > 1$ then $t \neq 2$ or 3. Assume that this is not true, and let $x_1y_1, x_4y_4 \notin E(G)$ and $x_2y_2 \in E(G)$. In this case $(x_1, y_3, x_2, y_2, x_3, y_4)$ is a maximal non-Hamiltonian path, because it misses y_1 and x_4 that are nonadjacent to the endvertices x_1 and y_4. □

3. Triangle-free Traceable Scenic Graphs

If a traceable scenic graph G is unicyclic (i.e., it has at most one cycle), then clearly G is either a path or a cycle. In this section we show that if G is triangle-free and not unicyclic, then G is either a cube or one of: $K_{p,p}, K_{p,p} - K_2$ or $K_{p,p+1}$ (Theorem 3.1). An important first step in this proof is to show that G must be bipartite.

We now introduce some nonstandard notation to simplify the presentation. We will say that a vertex $v \in V(G)$ *stars* (resp. *antistars*) a set $X \subset V(G - x)$ if $vx \in E(G)$ (resp. $vx \notin E(G)$), for all $x \in X$. For an x–y path P we will also use the notation (x, P, y) or $(x, \ldots, y)$. If (x, P, y) and (u, Q, v) are disjoint paths with y and u adjacent, then their *concatenation* is a path we denote by either $((x, P, y), (u, Q, v))$, $(x, \ldots, y, (u, Q, v))$, $((x, P, y), u, \ldots, v)$, or $(x, \ldots, y, u, \ldots, v)$. A similar natural extension of this notation is used for concatenations of concatenated paths. In general, if H is a subgraph containing a spanning x–y path, then (x, H, y) denotes an arbitrary such path.

Theorem 3.1. *If G is a triangle-free traceable scenic graph, then G is one of:*

$$P_n,\ C_n\ (n \geq 4),\ \textit{cube},\ K_{p,p},\ K_{p,p} - K_2\ \textit{and}\ K_{p,p+1}.$$

Proof. Let G be a triangle-free traceable scenic graph. If G is a tree, then $G \cong P_n$ because it must contain a Hamiltonian path. If all cycles of G are Hamiltonian, then $G \cong C_n$ $(n \geq 4)$. Hence we may assume that G has a non-Hamiltonian cycle.

Consider a longest non-Hamiltonian cycle C that obviously contains $k \geq 4$ vertices (G is triangle-free). Let x and y be arbitrary consecutive vertices of C. Because G is scenic and traceable, the spanning path (x, C, y) has an extension to a Hamiltonian path. Assume that the path extending C are $P = (x, x_1, \ldots, x^*)$ and $Q = (y, y_1, \ldots, y^*)$ with lengths a and b, respectively $(a \geq b \geq 0)$. By the choice of C, $a > 0$ holds. Because C is a longest non-Hamiltonian cycle of G, the only edges induced between P and Q are xy and possibly x^*y^*.

There are two cases to consider: for some C, x, y, P and Q, we have $a, b > 0$ and $x^*y^* \notin E(G)$ (Case 1); or for every choice of C, x, y, P and Q, either $b = 0$ or $x^*y^* \in E(G)$ (Case 2). Observe that in the second case we may always assume that $b = 0$ by choosing P as a maximal extension of the path (x, C, y) at x. We denote by x' and y' the two consecutive vertices of C such that (x, y, x', y') is a subpath of C.

Case 1. Because $x^*y^* \notin E(G)$, the path $(y^*, \ldots, y_1, (y, C, x'))$ has no extension at y^*. Hence it has an extension at its other endvertex x', say $(x', x'_1, \ldots)$, where x'_1 must be a vertex from P. By the choice of C, the only possibility is $x'_1 = x_1$, that is $x'x_1 \in E(G)$. The same argument repeated for the path $(x^*, \ldots, x_1, (x', C, y'))$ shows that $y'y_1 \in E(G)$. Then $(y', y_1, y, x', x_1, (x, C, y'))$ is a cycle longer than C, therefore it must be a Hamiltonian cycle . Consequently $a = b = 1$, that is C covers every vertex of G different from x^* and y^*. The argument above also implies that all of the vertices of C are alternately adjacent to x^* then y^*. That is, x^* is adjacent to every other vertex of C starting from x, and y^* is adjacent to every other vertex of C starting from y. Because G has no triangles, k must be even, and G must be a $p \times p$ bipartite graph ($p \geq 3$). Observe that any vertex from C can be swapped either for x^* or for y^*, hence G must be a supergraph of $K_{p,p} - pK_2$. For $p = 3$, we obtain $G \cong K_{3,3} - K_2$, and for $k \geq 4$, G is either a cube or a $K_{p,p} - K_2$, by Lemma 2.1.

Case 2. Because $x^*y^* \in E(G)$ holds if $b > 0$, we may assume that $b = 0$ and P is a maximal extension of (x, C, y) at x such that $(x^*, \ldots, x_1, (x, C, y))$ is a Hamiltonian path. If P contains just one edge ($a = 1$), then $G \cong K_{p,p+1}$ follows easily from that fact that G is scenic and triangle-free. Indeed, x^* must be adjacent to every other vertex on C implying that k is even and G is a $p \times (p + 1)$ bipartite graph. Furthermore, every vertex nonadjacent to x^* can be put into C by swapping with x^*. This implies that G must be complete.

Assume that $a \geq 1$. If $x^*y \notin E(G)$, then the path (y, C, x') extends with the edge $x'x_1$. Consider the cycle $C' = (x, x_1, x', y', \ldots, x)$ that has length k, and consider the extension of (x_1, C', x') into a Hamiltonian path with $P' = (x_1, \ldots, x^*)$ and $Q' = (x', y)$. The two endvertices of the extension, x^* and y, are nonadjacent so this reduces to Case 1. Hence, $x^*y \in E(G)$. By the same argument, x^* is adjacent to every other vertex of C starting from y. Similarly, x_1 is adjacent to every other vertex of C starting from x. The cycle $(x, x_1, \ldots, x^*, (y', C, x))$ misses only y and x' from C and includes vertices $x_1, \ldots, x^*$ of P. Therefore, by the choice of C, it follows that $a = 2$. To conclude the proof, apply symmetric arguments by swapping vertices of C for x^* or x_1 to deduce that $G \cong K_{p,p}$. □

4. Clique Removal

Using the earlier notation, if K is a clique and x, y are among its vertices, then (x, K, y) denotes an arbitrary spanning path of K from x to y.

Theorem 4.1. *If G is a traceable scenic graph and K is a maximal clique of G containing at least three vertices, then $G - V(K)$ is also traceable and scenic.*

Proof. Suppose, to the contrary, that $H = G - V(K)$ is either not traceable or not scenic. In particular, H has a maximal path $P = (u, \ldots, v)$ which is not a Hamiltonian path of H. If $u = v$, for every such P, then it follows that H has no edge (Case 1). Otherwise u and v are distinct for some non-Hamiltonian maximal path P (Case 2).

Case 1. Because H contains no edge and is nontraceable, it has at least two (isolated) vertices. Let v be one of these vertices. By the maximality of K, v has a nonneighbor x in K. Let w be a vertex in $K - x$. The path (x, K, w) extends in G to a path containing v. Hence $vw \in E(G)$. Let y be vertex of H different from v. The path $(v, (w, K, x))$ extends to path including y. Thus $xy \in E(G)$. Let z be a vertex of K different from x and w. If $yz \notin E(G)$, then $(v, (w, K, z))$ is a path that does not extend to include y. If $yz \in E(G)$, then $(y, (z, K, x))$ is a path that does not extend to include v.

Case 2. Let $P = (v_0, v_1, \ldots, v_h)$ be a maximal path in H which is not a Hamiltonian path of H $(h > 0)$. Because P extends to a Hamiltonian path in G, one may suppose that $v_h x \in E(G)$ for some $x \in V(K)$. Clearly $Y = V(H) \setminus V(P)$ is nonempty, and since P is maximal in H, neither v_0 nor v_h has a neighbor in Y. Note also that G_Y — the graph induced by the vertices of Y — is a traceable induced subgraph of G, because the path $(v_0, \ldots, v_h, (x, K, y))$ for some $y \in Y$ can be extended to a Hamiltonian path of G.

First we show that some interior vertex of P has a neighbor in Y (in particular, $h \geq 2$). For a contradiction, suppose that there are no edges from P to Y. If v_0 is adjacent to some vertex $z \in V(K) \setminus \{x\}$, then no vertex of Y is contained in any extension of $(v_0, (z, K, x), v_h)$. Hence we may assume that v_0 is nonadjacent to every vertex of $K - x$. If no vertex of Y is adjacent to a vertex in $K - x$, then x is adjacent to some vertex $y \in Y$, and any extension of $(v_0, \ldots, v_h, x, y)$ contains no vertex in $K - x$. Hence some vertex $y \in Y$ has a neighbor $z \in V(K) \setminus \{x\}$. Observe that every vertex $w \in V(K)\backslash\{x,z\}$ has a neighbor from $(v_1,\ldots,v_h)$ since $((w,K,z),y)$ is a path extending to a Hamiltonian path of G. Let $w \in V(K)\backslash\{x,z\}$ and let v_i be the first vertex on the path $(v_1,\ldots,v_h)$ that is a neighbor of w. Now any extension of the path $(w,v_i,\ldots,v_h,(x,K-w,z),y)$ misses v_0, a contradiction. Therefore, some vertex from $(v_1,\ldots,v_{h-1})$ has a neighbor in Y.

Let v_k be the first vertex from the path $P=(v_0,\ldots,v_h)$ with a neighbor $y \in Y$ $(1 \le k \le h-1)$. The proof of the theorem now follows from the following five steps.

Step 1: there are no edges between $K-x$ and $Y \setminus \{y\}$. If $ab \in E(G)$ for some $a \in V(K-x)$ and $b \in Y\backslash\{y\}$, then the path $(y,v_k,\ldots,v_h,(x,K,a),b)$ would not extend in G to include v_0, a contradiction.

Step 2: every vertex $z \in V(K-x)$ has a neighbor v_j, for some $0 \le j < k$. Since otherwise, $(y,v_k,\ldots,v_h,(x,K,z))$ would not extend to include v_0.

Step 3: y is adjacent to every vertex of $K-x$. If $z \in V(K-x)$ and $yz \notin E(G)$, then $(v_0,\ldots,v_h,(x,K,z))$ would not extend to include y.

Step 4: $Y=\{y\}$. If $w \in Y\backslash\{y\}$, then $(v_0,\ldots,v_h,(x,K-z,x'),y,z)$ would not extend to include w (where z and x' are arbitrary distinct vertices of $K-x$).

Step 5: $xy \in E(G)$. Suppose that $xy \notin E(G)$ and let $z \in V(K-x)$. By Step 2, $zv_j \in E(G)$, for some $0 \le j < k$. Any extension of the path $((x,K,z),v_j,\ldots,v_h)$ can not contain y since all of y's neighbors appear in $(v_k,\ldots,v_{h-1})$ or in $K-x$ (which have already been covered by this path).

By steps 3 and 5, $V(K)\cup\{y\}$ is a clique. This contradicts the maximality of K, and the theorem follows. □

In Section 5. we will consider ways that complete subgraphs can and cannot be "added" to traceable scenic graphs. Let G be a traceable scenic graph and let K be a maximal clique of G. By Theorem 4.1, $H=G-V(K)$ is also scenic and traceable, provided that K has at least three vertices. If all these properties are satisfied by G,K and H, then we will say that G is a *(scenic) clique extension* of H. We prove a simple property of clique extensions that will be used frequently later.

Lemma 4.2. *Let G be a traceable scenic graph with a maximum clique K of order at least 3, and let $H=G-V(K)$. If $(x_1,\ldots,x_k)$ is a Hamiltonian* *path of H and $k \geq 2$, then 'parallel edges' exist at the end vertices, that is $x_1a, x_kb \in E(G)$ for some distinct vertices $a, b \in V(K)$.*

Proof. Because the path $(x_1, \ldots, x_k)$ extends in G, we may assume that $x_1a \in E(G)$, for $a \in V(K)$. Observe that some vertex of $K - a$ must have a neighbor in H, for otherwise, any extension of a spanning path of K with endvertices different from a would miss x_2. Define

$$m = \max\{i : 1 \leq i \leq k,\ x_ib \in E(G), \text{ for some } b \in V(K - a)\}.$$

Let $x_mb \in E(G)$ and $c \in V(K - \{a, b\})$. Because the path $P = (c, a, x_1, \ldots, x_m, b)$ extends to a Hamiltonian path, and all neighbors of $K - a$ are covered by P, $m = k$ must hold. Thus the lemma follows. □

The following result enables us to recognize many traceable scenic graphs with no clique extension.

Lemma 4.3. *Suppose that H has distinct vertices x, y, z_1 and z_2 with the following properties:*

- *x and y are nonadjacent,*
- *there are Hamiltonian paths in $H - x$ from y to z_1 and y to z_2,*
- *there is a Hamiltonian path in H from z_1 to z_2.*

If G is a traceable scenic graph with a maximum clique K of order at least 3, then $G - V(K)$ cannot be H.

Proof. Suppose to the contrary that G, K and H are as above with $H = G - V(K)$. Let x and y be vertices with x and y nonadjacent, and z_1, z_2 as above. Since K is a maximum clique of G, x is nonadjacent to some vertex $a \in K$. Also, since z_1 and z_2 are endvertices of a Hamiltonian path of H, by Lemma 4.2, it follows that z_1 and z_2 have parallel edges to K. Consequently, we may assume that z_1 is adjacent to some vertex b of K different from a. If (z_1, P, y) is a Hamiltonian path of $H - x$, then $((a, K, b), (z_1, P, y))$ is a path that cannot be extended to include x. Hence the lemma follows. □

5. Clique Extensions

In this section we describe traceable scenic graphs with no clique extension, and we discuss the structure of clique extensions of other traceable scenic graphs.

Proposition 5.1. *The traceable scenic graphs*

$$prism,\ K_n - tK_2\ (n \geq 5, 1 \leq t \leq n/2), K_{p,p+1} + K_{1,q}\ (p \geq 2, 1 \leq q \leq p),$$

$$\text{and for } p \geq 3,\ K_{p,p} + 2K_2\ ,\ K_{p,p+1} + K_3,\ \text{and } K_{p,p+1} + 2K_2$$

have no clique extension.

Proof. To prove that none of these graphs can be obtained by clique removal from any traceable scenic graph, it is enough to verify that each graph in the proposition satisfies the properties in Lemma 4.3. We omit the details of this straightforward verification. □

Proposition 5.2. *None of the graphs* P_n $(n \geq 4)$, C_n $(n \geq 5)$, *cube and* $K_{p,p} - K_2$ $(p \geq 2)$ *have clique extensions.*

Proof. Let G be a traceable scenic graph and K be a maximum clique of order at least 3 such that $H = G - V(K)$ is a graph from the list.

Case 1: $H = P_n = (x_1, \ldots, x_n)$ $(n \geq 4)$. By Lemma 4.2, distinct vertices $a, b \in V(K)$ exist such that x_1a, $x_nb \in E(G)$. If $x_jc \in E(G)$, for some $c \in V(K) \setminus \{a, b\}$ and $1 < j < n$, then since $n \geq 4$, either $j - 1 > 1$ or $j + 1 < n$. By symmetry, assume $j - 1 > 1$. Then the path $(x_2, x_3, \ldots, x_j, (c, K, a), x_1)$ is maximal and misses x_n, a contradiction. Consequently, we may assume that every $c \in V(K - \{a, b\})$ antistars $\{x_2, \ldots, x_{n-1}\}$. Because K has at least 3 vertices and H has at least 4 vertices, the path $(x_3, x_4, \ldots, x_n, b, a, x_1, x_2)$ is a maximal non-Hamiltonian path of G, a contradiction.

Case 2: $H = C_n = (x_1, \ldots, x_n)$ $(n \geq 5)$. By Lemma 4.2, consecutive vertices of C_n send parallel edges to K. Let $x_1a, x_2b \in E(G)$, for distinct $a, b \in V(K)$, and let $x_3a', x_4b' \in E(G)$, for distinct $a', b' \in V(K)$. Assuming that $a \neq a'$, the path $(x_4, x_3, (a', K, a), x_1, x_n, \ldots, x_5)$ would be maximal and would miss x_2. Therefore $a = a'$ that is $x_1a' \in E(G)$. Thus we obtain $(x_3, x_4, (b', K, a'), x_1, x_2)$ that is a maximal path missing x_5, a contradiction.

Case 3: $H = cube$. For convenience, label the vertices of H as x_1, x_2, x_3, x_4 and y_1, y_2, y_3, y_4 for the vertices in the partite sets, and x_iy_i for $1 \leq i \leq 4$ be the missing edges. By Lemma 4.2, distinct vertices a and b of K exist such that x_4a and y_4b are edges of G. The path $(y_1, x_3, y_4, (b, K, a), x_4, y_3, x_2)$ cannot be extended to include either x_1 or y_2, a contradiction.

Case 4: $H = K_{p,p} - K_2$ $(p \geq 2)$. Let $X = \{x_1, \ldots, x_p\}$ and $Y = \{y_1, \ldots, y_p\}$ be the partite sets of $K_{p,p}$ with $x_py_p \notin E(H)$. By Lemma 4.2, $x_pa, y_pb \in E(H)$, for distinct $a, b \in V(K)$. Consider in H the two disjoint paths $(x_1, y_2, x_3, y_4, \ldots)$ and $(y_1, x_2, y_3, x_4, \ldots)$. Observing that one of these paths terminates at x_p and the other one at y_p. We may combine them to get a path $P = (\ldots, y_{p-1}, x_p, a, b, y_p, x_{p-1}, \ldots)$ with endvertices x_1 and y_1. Because $V(K - \{a, b\}) \neq \emptyset$, P has an extension, so that $x_1c \in E(H)$, for some $c \in V(K - \{a, b\})$. Thus we obtain a maximal path $(x_p, (a, K, c), (x_1, H - \{x_p, y_p\}, y_1))$ that misses y_p, a contradiction. □

Now we prove a technical lemma that is used to examine possible clique extensions of $K_{p,p}$ $(p \geq 2)$ or $K_{p,p} + K_2$ $(p \geq 2)$.

Lemma 5.3. *Let G be a traceable scenic graph with $\omega(G) \geq 3$, and let K be a maximum clique of G. If $G - V(K)$ is either $K_{p,p}$ $(p \geq 2)$ or $K_{p,p} + K_2$ $(p \geq 2)$ and the partite sets of $K_{p,p}$ are X and Y, then every vertex of K stars either X or Y.*

Proof. Suppose, to the contrary, that a vertex $c \in V(K)$ exists that is nonadjacent to $x \in X$ and $y \in Y$. Let $H = G - V(K)$ and assume that X is independent in H. By Lemma 4.2, distinct vertices $a, b \in V(K)$ exist such that $xa, yb \in E(G)$, since H has a Hamiltonian path starting at x and ending at y. It follows that c is star to $X \setminus \{x\}$ (and symmetrically c is star to $Y \setminus \{y\}$), for otherwise, if cx' were not an edge, then the path $((x', K_{p,p} - y, x), (a, K - c, b), y)$ could not be extended to include c. Also, a is antistar to $X \setminus \{x\}$ (symmetrically b must antistar $Y \setminus \{y\}$) since $(y, (b, K - c, a), (x', K_{p,p} - y, x)$ would have to extend to contain c. Also, a must star $Y \setminus \{y\}$ (symmetrically, b must star $X \setminus \{x\}$) because the path $((y', K_{p,p} - x', y), (b, K - a, c), x')$ must extend to include a. Now since X is independent, the path $(a, y', (c, K - a, b), (y, K_{p,p} - \{y', x'\}, x))$ cannot be extended to include x'. This contradiction proves the claim that every vertex of K stars either X or Y for either graph $K_{p,p}$ or $K_{p,p} + K_2$. □

Proposition 5.4. *If G is a traceable scenic graph with $\omega(G) = 3$ or $\omega(G) = 4$ obtained by clique extension from $K_{p,p}$ ($p \geq 2$), then either $G \cong K_{p+1,p+2} + K_{1,q}$ $(1 \leq q \leq p+1)$ or $G \cong K_{p+2,p+2} + 2K_2$.*

Proof. Suppose that $K \subset G$ is a maximal clique such that $G - V(K) \cong K_{p,p}$, and let X and Y be the partite sets of $K_{p,p}$.

Case 1: $V(K) = \{a, b, c\}$. By Lemma 5.3, each of the vertices a, b, and c star either X or Y. If all three star the same set say X, then a larger clique would result contradicting the choice of K. Hence, two vertices, say a and b, star X and c stars Y. Note that it follows that G is a super graph of $K_{p+1,p+2}$. If c were adjacent to a vertex in X, say x, then again a larger complete graph would be in G, contradicting the choice of K; hence c antistars X.

As for additional edges from a and b to Y; if both a and b have an edge to a vertex of Y, then it must be to different vertices, for otherwise G would contain a K_4. Suppose that y and y' are distinct vertices in Y and ay and by' are edges of G. In this case, the path (c, y, a, b, y') cannot be extended to contain all the vertices of X. Consequently, only one of a or b can have edges to Y, and thus $G \cong K_{p+1,p+2} + K_{1,q}$.

Case 2: $V(K) = \{a, b, c, d\}$. Each vertex of K must star X or Y. If all four vertices starred the same set, then a larger clique would result contradicting the choice of K. If three vertices, say a, b, and c star the same set, say X, and d stars Y, then clearly d antistars X for otherwise a larger clique results. However the path (a, d, b, x, c, x') cannot be extended to contain all the vertices of Y, since two vertices of X and no vertices of Y have been used by the path.

Hence we may assume that two vertices star X, say a and b; and c and d star Y. Suppose that there are other edges, say $ay \in E(G)$ for some $y \in Y$. Note that $by \notin E(G)$, since a K_5 would result. Consequently, the extension of the path (y, a, b, c, y', d) would force an edge from d to X and that of (y, a, b, d, y', c) would force an edge from c to X. The vertices in X adjacent to d and c must be distinct for K to be a maximum clique, so assume that $cx,\ dx' \in E(G)$. Now the path $(b, a, x, c, d, (x', K_{p,p} - \{x, y\}, y'))$ missing y has no extension. Hence a and b antistar Y, and c and d antistar X, so $G \cong K_{p+2,p+2} + 2K_2$ follows. □

Proposition 5.5. *If G is a traceable scenic graph with $\omega(G) = 3$ or $\omega(G) = 4$ obtained by clique extension from $K_{p,p} + K_2$ $(p \geq 3)$, then $G \cong K_{p+1,p+2} + 2K_2$.*

Proof. Suppose that K is a maximal clique such that $G - V(K) \cong K_{p,p} + K_2$. Let X and Y be the partite sets of $K_{p,p}$, and assume that $e = xx'$ is the 'extra' edge of $K_{p,p} + K_2$ with $x, x' \in X$.

Case 1: $V(K) = \{a, b, c\}$. By Lemma 5.3, each of the three vertices star either X or Y. If all three star the same set, then a larger clique would result contradicting the choice of K. In addition, if two were to star X, then a K_4 would result; hence two vertices of K, say a and b star Y, and c stars X and antistars Y.

Suppose that a or b had an edge to a vertex in X. If one were adjacent to x or x', say $bx \in E(G)$, then (c, a, b, x, x') could not be extended to cover all of Y. Similarly, if $bx'' \in E(G)$, then the path (c, a, b, x'', y, x, x') could not be extended to cover all of Y. Therefore $G \cong K_{p+1,p+2} + 2K_2$.

Case 2: $V(K) = \{a, b, c, d\}$. Applying Lemma 5.3 one of the following occurs:

- three vertices of K star X, in which case a K_5 results;
- three vertices star Y, say a, b, c. Now at least one of a, b, or c has a nonneighbor in x, x', say $cx' \notin E(G)$. The path $(c, y, b, y', a, d, (x, K_{p,p} - x', x''))$ cannot be extended to include x';
- two vertices, say a and b, star X, and two vertices, say c and d, star Y. In this case $(y, x, (x', (K_{p,p} + K_2) - \{x, x', x'', y, y', y''\}, x^*), a, x'', b, c, d, y'')$ does not extend to a path that includes y'. □

Proposition 5.6. *If G is a traceable scenic graph with $\omega(G) = 3$ or $\omega(G) = 4$ obtained by clique extension from $K_{p,p+1}$ $(p \geq 1)$, then either $G \cong K_{p+2,p+2} + K_2$ or $G \cong K_{p+2,p+3} + K_3$.*

Proof. Let K be a maximal clique such that $G - V(K) \cong K_{p,p+1}$ and denote the partite sets of $K_{p,p+1}$ by X and Y, with $|X| = p$. By the maximality of K, for each vertex of $G - K$ there is at least one vertex of K not adjacent to it. In particular, let $y \in Y$ and suppose y is nonadjacent to $a \in K$. If $xb \in E(G)$ for some $x \in X$ and $b \in V(K - a)$, then the path $((y', K_{p,p+1} - y, x), (b, K, a))$ is not extendible. This implies that each vertex of $K - a$ antistars X. By Lemma 4.2, for every $y' \in Y - y$ parallel edges yb and $y'c$ exist for some $b, c \in V(K)$. If $c \in V(K) - a$, then the path $(y, (b, K - a, c), (y', K_{p+1,p} - y, x))$ extends to a for all $x \in X$, hence a stars X. Furthermore, if some vertex $b \in V(K - a)$ is not adjacent to some vertex $y' \in Y$, then by switching the role of a and b, the above arguments would imply that a both stars and antistars X, a contradiction. Hence, we may assume that each vertex of $K - a$ stars Y. Because K is maximal, a antistars Y. Consequently, if $|V(K)| = 3$, then $G \cong K_{p+2,p+2} + K_2$, and if $|V(K)| = 4$, then $G \cong K_{p+2,p+3} + K_3$.

Thus it only remains to consider the case in which no parallel edges yb and $y'c$ exist with $c \in V(K) - a$. That is, every vertex $y' \in Y \setminus \{y\}$ is adjacent to a and also possibly adjacent to b. But then the path $(y', (a, K - c, b), (y, K_{p+1,p} - y', x))$ misses c and is not extendible. The result follows. □

6. Traceable Scenic Graphs with Triangles

In Section 3. we determined all traceable scenic graphs with $\omega = 2$. Here we deal with the case $\omega \geq 3$.

Theorem 6.1. *If G is a traceable scenic graph with $\omega(G) \geq 5$, then $G = K_n - tK_2$, for some $0 \leq t \leq n/2$ and $n \geq 5$.*

Proof. Suppose that G is a traceable scenic graph with $\omega(G) \geq 5$. To show that $G \cong K_n - tK_2$, it suffices to show that every vertex has at most one nonneighbor in G. Assume that G is not a clique and let K be a maximal clique of G with at least five vertices. By Theorem 4.1, $H = G - V(K)$ is traceable and scenic. Let $P = (x_1, x_2, \ldots, x_k)$ be a Hamiltonian path of H.

If $k = 1$ and x_1 has two nonneighbors a and b in K, then any path (a, K, b) of G cannot be extended to include x_1, a contradiction. Hence, $G \cong K_n - K_2$. For $k \geq 2$, the proof follows from the following steps.

Step 1: Each of x_1 and x_k has at most one nonneighbor in K. By symmetry, it suffices to prove this for x_1. By Lemma 4.2, distinct vertices u and v of K exist such that x_1u and x_kv are edges of G.

First we show that if $y, z \in V(K)$ are distinct nonneighbors of x_1, then one of them must be v (symmetrically, if x_k has two nonneighbors in K, then one of them must be u). Assuming that $y \neq v$, the path $(x_1, (u, K - y, v), x_k, x_{k-1}, \ldots, x_2)$ must extend to include y, hence x_2 is adjacent to y. Now $z = v$, since otherwise $(y, x_2, \ldots, x_k, (v, K - y, z))$ is a path of G that cannot be extended to include x_1.

Suppose now that x_1 has two nonneighbors in K. By the paragraph above, we may assume that these nonneighbors are y and v, and $yx_2 \in E(G)$. Because K has at least five vertices, there are at least two vertices a and b in $K - \{u, v, y\}$. Now x_k is adjacent to at least one of a or b, otherwise x_k has two nonneighbors in K neither of which is u (this contradicts the previous paragraph). Without loss of generality, a is a neighbor of x_k. Now $(y, x_2, \ldots, x_k, (a, K - y, v))$ is a path of G that cannot be extended to include x_1, a contradictiion. Hence x_1 (symmetrically x_k) has at most one nonneighbor in K.

Step 2: No vertex of K is nonadjacent to two consecutive vertices of P. If there were a vertex w of K with two consecutive nonneighbors x_i, x_{i+1} of P, then by Step 1 we could find two distinct vertices a and b from $K-w$ with x_1a and x_kb edges of G. However, this would then imply that the path $(x_{i+1}, \ldots, x_k, (b, K-w, a), x_1, \ldots, x_i)$ could not be extended to include w, a contradiction.

Step 3: Each of $x_2, \ldots, x_{k-1}$ has at most one nonneighbor in K. Suppose that x_i has two nonneighbors y and z in K, for some $2 \leq i \leq k-1$. By Step 2, $x_{i-1}y$ and $x_{i+1}z$ are edges of G. Also, $K - \{y, z\}$ has at least three vertices, so by Step 1, two vertices a and b exist in $K - \{y, z\}$ such that x_1a and x_kb are edges in G. Now the path $(z, x_{i+1}, \ldots, x_k, (b, K - \{z, y\}, a), x_1, \ldots, x_{i-1}, y)$ is a path that cannot be extended to include x_i, a contradiction.

Note: Steps 1 and 3 together with the maximality of K implies that every vertex in H has a unique nonneighbor in K.

Step 4: Every vertex in K has at most one nonneighbor in H. Suppose that w is a vertex in K with two nonneighbors x_i and x_j in H. Assume that $i < j$. By Step 2, x_i and x_j are not consecutive vertices of P. If both x_i and x_j are endpoints of P, then Steps 1 and 3 guarantee that we can find distinct vertices a and b of $K-w$ such that x_1a and x_2b are edges of G. It follows that the path $(x_k, \ldots, x_2, (b, K-w, a), x_1)$ can not be extended to include w, a contradiction.

Hence at least one of x_i and x_j is not an endpoint of P. By symmetry we may assume that $x_i \neq x_1$. Steps 1 and 3 together with $|K| \geq 5$ imply that x_{i-1} and x_{j-1} have a common neighbor in $K-w$, say a. Also, $|K| \geq 5$ and Steps 1 and 3 guarantee that two vertices b and c in $K - \{a, w\}$ exist such that bx_1 and cx_k are edges of G. Now the path $(x_i, \ldots, x_{j-1}, a, x_{i-1}, \ldots, x_1, (b, K - \{a, w\}, c), x_k, \ldots, x_j)$ is a path of G that cannot be extended to include w, a contradiction.

Step 5: H is a clique. Suppose that x_i and x_j are nonadjacent vertices of H with $1 \leq i < j \leq k$. Clearly x_i and x_j are not consecutive vertices of P. As noted above, x_i has a unique nonneighbor in K. Call it w. By Step 2, w is adjacent to x_{i+1}.

Suppose that $i > 1$. Let z be a common neighbor of x_{j-1} and x_{i-1} in $K-w$. Steps 1 and 3 together with $|K| \geq 5$ imply that two distinct vertices

a and b of $K - \{w, z\}$ exist such that $x_1 a$ and $x_k b$ are edges of G. Now the path $(w, x_{i+1}, \dots, x_{j-1}, z, x_{i-1}, \dots, x_1, (a, K - \{w, z\}, b), x_k, \dots, x_j)$ is a path that does not extend to include x_i.

Suppose now that $i = 1$. Let z be a neighbor of x_{j-1} in $K - w$, and let b be a vertex in $K - \{w, z\}$ adjacent to x_k. The path $(w, x_{i+1}, \dots, x_{j-1}, (z, K - w, b), x_k, x_{k-1}, \dots, x_j)$ is a path that does not extend to include x_i. □

Proposition 6.2. *Let G be a traceable scenic graph different from $K_n - tK_2$. If G is obtained by clique extension from K_m, $1 \leq m \leq 4$, then G is isomorphic to one of $K_{2,3} + K_2$, $K_{3,3} + 2K_2$, or the prism.*

Proof. Let $K \subset G$ be a maximal clique such that $G - V(K) = K_m$ and set $V(K_m) = \{v_1, \dots, v_m\}$. Theorem 6.1 implies that K has 3 or 4 vertices. If $m = 1$, then there exist distinct vertices $a, b \in V(K)$ nonadjacent to v_1, since otherwise, $G \cong K_n - tK_2$. Observe that (a, K, b) is a maximal non-Hamiltonian path of G avoiding v_1, a contradiction. Therefore, one may assume that $m = 2, 3$ or 4.

If $m = 2$ then by Lemma 4.2, there are parallel edges $v_1 x_1, v_2 x_2 \in E(G)$, for some $x_1, x_2 \in V(K)$. For every $w \in V(K)$ different from x_1 and x_2, the path $(v_1, (x_1, K - w, x_2), v_2)$ has an extension in G. Hence at least one of wv_1 and wx_2 is an edge of G, for every $w \in V(K)$. It is now straightforward to verify that $G \cong K_{2,3} + K_2$, if $|V(K)| = 3$, and $G \cong K_{3,3} + 2K_2$, if $|V(K)| = 4$.

Finally assume that $m = 3$ or 4. One vertex of K_m, say v_3 has two nonneighbors $y_1, y_2 \in V(K)$, and by Lemma 4.2, there are parallel edges $v_1 x_1, v_2 x_2 \in E(G)$, for some $x_1, x_2 \in V(K)$. If $\{x_1, x_2\} \neq \{y_1, y_2\}$, say $x_1 \notin \{y_1, y_2\}$. then the path $(x_1, (v_1, K_m - v_3, v_2), x_2)$ has an extension P from y_1 to y_2 that covers every vertex of $G - v_3$, a contradiction. Therefore $\{x_1, x_2\} = \{y_1, y_2\}$. By permuting the indices of v_i, x_i and y_i as required, it is straightforward to verify that m must be 3, and either $G \cong K_6 - C_6$ or $G \cong K_{3,3} + 2K_2$. □

Define $\mathcal{G}_0$ to be the class of graphs containing all triangle-free traceable scenic graphs (determined in Theorem 3.1) and all cliques. If $\mathcal{G}_{k-1}$ is defined for some $k \geq 1$, then let $\mathcal{G}_k$ be the class of all traceable scenic graphs $G \notin \cup\{\mathcal{G}_i : i < k\}$ such that for some maximal clique $K \subset G$, $G - V(K) \in \mathcal{G}_{k-1}$. Also define $\mathcal{H}_k \subset \mathcal{G}_k$ to be the class of all those graphs that have a (traceable, scenic) clique extension.

By Theorem 3.1, $\mathcal{G}_0 = \{K_n,\ C_n, P_n, \text{cube}, K_{p,p}, K_{p,p} - K_2, K_{p,p+1}\}$. By Proposition 5.2, P_n $(n \geq 4)$, C_n $(n \geq 5)$, cube, and $K_{p,p} - K_2$ $(p \geq 2)$ are not in $\mathcal{H}_0$, hence $\mathcal{H}_0 = \{K_{p,p}\ (p \geq 2), K_{p,p+1}\ (p \geq 1), K_n (n \geq 1)\}$.

From Propositions 5.4, 5.6, 6.2 and Theorem 6.1, we obtain that $\mathcal{G}_1$ consists of the following graphs: prism, $K_{2,2}+K_2 (= K_4 - K_2)$, $K_n - tK_2 (n \geq 5, 1 \leq t \leq n/2)$, $K_{2,3} + K_2$, $K_{3,3} + 2K_2$ — that is the clique extensions of cliques — $K_{p,p+1} + K_{1,q} (p \geq 3, 1 \leq q \leq p)$, $K_{p,p} + 2K_2$ $(p \geq 4)$, $K_{p,p} + K_2$, $(p \geq 3)$, $K_{p,p+1} + K_3 (p \geq 3)$ — that is the clique extensions of $K_{p,p}$ $(p \geq 2)$ and $K_{p,p+1}$ $(p \geq 1)$.

By Proposition 5.1, the prism, $K_n - tK_2 (n \geq 5, 1 \leq t \leq n/2)$, $K_{2,3} + K_2$, $K_{3,3} + 2K_2$, $K_{p,p+1} + K_{1,q} (p \geq 3, 1 \leq q \leq p)$ and $K_{p,p} + 2K_2$ $(p \geq 4)$, are not in $\mathcal{H}_1$; that is, $\mathcal{H}_1 = \{K_{p,p} + K_2\ (p \geq 2)\}$.

By Propositions 5.5 and 5.1, $\mathcal{G}_2 = \{K_{p,p+1} + 2K_2 (p \geq 3)\}$ and $\mathcal{H}_2 = \emptyset$. This implies that $\mathcal{G}_k = \emptyset$, for every $k \geq 3$. So Theorem 4.1 implies that the union of $\mathcal{G}_0$, $\mathcal{G}_1$, and $\mathcal{G}_2$ contains all traceable scenic graphs. It is easy to check that $\cup_{i=0}^{2} \mathcal{G}_i = \Phi[K_n] \cup \Phi[K_{p,p}] \cup \Phi[K_{p,p+1}] \cup \Psi$. This concludes the proof of Theorem 1.2.

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