Asymptotically Optimal (Δ, D', s)-Digraphs

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Abstract

A (Δ, D', s)-digraph is a digraph with maximum out-degree Δ such that after the deletion of any s of its vertices the resulting digraph has diameter at most D'. Our concern is to find large, i.e. with order as large as possible, (Δ, D', s)-digraphs. To this end, new families of digraphs satisfying a Menger-type condition are given. Namely, between any pair of non-adjacent vertices they have $s+1$ internally disjoint paths of length at most D'. Then, new families of asymptotically optimal (Δ, D', s)-digraphs are obtained.

Key words: fault tolerance, diameter vulnerability, dense digraphs.
AMS classification: 05C20, 05C38, 05C12

1 Introduction

Interconnection networks are usually modeled by graphs, directed or not, in which the vertices represent the switching elements or processors. Communication links are represented by edges if they are bidirectional and by arcs if they are unidirectional. We are concerned here with directed graphs only, called digraphs for short. A *digraph* $G = (V, A)$ consists of a set V of *vertices* and a set A of directed edges between vertices called *arcs*. The cardinality of V is called the *order* of the digraph. The set of vertices which are adjacent from (to) a given vertex v is denoted by $\Gamma^+(v)$ ($\Gamma^-(v)$) and its cardinality is the *out-degree* $d^+(v) = |\Gamma^+(v)|$ (*in-degree* $d^-(v) = |\Gamma^-(v)|$). The length of a shortest path from u to v is the *distance from u to v* and is denoted by $d(u, v)$. Its maximum value over all pairs of vertices is the *diameter* of the digraph. The reader is referred to Chartrand and Lesniak [4] for additional graph concepts.

In the design of large interconnection networks several factors have to be taken into account: each processor can be connected just to a few others and communication delays between processors must be short. These requirements lead to the following optimization problems: find digraphs of given maximum out-degree Δ and diameter D which have large order (the (Δ, D)-digraph problem) and find digraphs with given order and maximum out-degree which have small diameter. These problems have been widely studied for graphs (see [1]) as well as for digraphs (see [7]). The case of bipartite graphs (see [2]) and digraphs (see [6]) have been also considered.

An interconnection network must be fault-tolerant. If some processors or communication links cease to function, it is important that the remaining processors can still intercommunicate with reasonable efficiency. One can demand, for example, that the message delay does not increase too much. This means that the (di)graph obtained after deletion of some vertices or edges (arcs) still has a small diameter.

The problem we study in this paper is the (Δ, D, D', s)-digraph problem, that is, to find large digraphs with maximum out-degree Δ and diameter D such that the resulting digraph after the deletion of s vertices has diameter at most D'. This problem has been studied in [12] in the case $D' = D$, giving an optimal family of $(\Delta, 2, 2, s)$-digraphs whenever Δ is a multiple of $s + 1$. In [9] this problem is studied for the bipartite case and a dense family of (Δ, D, D', s)-bipartite digraphs is presented when Δ is a multiple of $s + 1$ and the diameter is $3, 4, 5$ or 6. Such digraphs are denoted as FD Digraphs in Table 2 of the Annex. The analogous problem for graphs has been considered in [3, 8, 14].

2 Some notations and previous results

A digraph with maximum out-degree Δ and diameter D is called a (Δ, D)-*digraph*. A (Δ, D, D', s)-*digraph* is a (Δ, D)-digraph such that the subdigraphs obtained by deleting any set of s vertices have diameter less or equal than D'.

If a (Δ, D)-digraph, $D \geq 2$, verifies that between any pair of non-adjacent vertices there are $s + 1$ internally disjoint paths of length at most D', then it is a (Δ, D, D', s)-digraph.

A Moore-like bound for the number of vertices of a (Δ, D, D', s)-digraph is given in [12]:

$$M(\Delta, D, D', s) = 1 + \Delta + \left\lfloor \frac{\Delta^2 + \Delta^3 + \cdots + \Delta^{D'}}{s + 1} \right\rfloor .$$

In this paper we use two families of dense digraphs: De Bruijn and Kautz digraphs. Let $B(\Delta, D)$ be a De Bruijn digraph. The elements of

$V(G)$ are all strings of D symbols z_i from an alphabet $\mathbf{Z}_\Delta$ and vertex $z_1z_2\ldots z_D$ is adjacent to vertices $z_2\ldots z_Dz_{D+1}$. $B(\Delta, D)$ has diameter D and degree Δ [5]. The Kautz digraph $K(\Delta, D)$ is defined analogously: the elements of $V(G)$ are the strings of D symbols z_i from an alphabet $\mathbf{Z}_{\Delta+1}$ such that two consecutive symbols cannot be equal. Vertex $z_1z_2\ldots z_D$ is adjacent to vertices $z_2\ldots z_Dz_{D+1}$ whenever $z_D \neq z_{D+1}$. $K(\Delta, D)$ has diameter D and degree Δ [11].

A walk of length s from vertex v to vertex w will be denoted as a sequence of length $D+s$ in which the first D symbols correspond to v and the last D ones to w. For instance the walk $abcd \rightsquigarrow bcd\alpha \rightsquigarrow cd\alpha\beta \rightsquigarrow d\alpha\beta\gamma$ is represented as the sequence $abcd\alpha\beta\gamma$.

Another concept used in this paper is the conjunction. Given two graphs G and H, the conjunction $G \otimes H$ is defined as follows: its vertex set is the cartesian product of G and H, i.e.

$$V(G \otimes H) = \{(u, v) \mid u \in V(G), v \in V(H)\}$$

and vertex (u, v) is adjacent to vertex (u', v') if and only if (u, u') is an arc in G and (v, v') is an arc in H.

If G is a digraph with n_1 vertices and degree Δ_1, and H is a digraph with n_2 vertices and degree Δ_2, then the conjunction $G \otimes H$ has n_1n_2 vertices and degree $\Delta_1\Delta_2$.

3 Some results on Kautz and De Bruijn digraphs

3.1 Disjoint paths in Kautz digraphs with $D \leq 3$

Proposition 1 *Let $K(\Delta, 3)$, $\Delta > 1$, be a Kautz digraph with diameter $D = 3$. Between any pair of vertices in $K(\Delta, 3)$ there exist at least $\Delta - 1$ disjoint walks of length exactly 4.*

Proof: . Let $x = x_0x_1x_2$ and $y = y_0y_1y_2$ be the initial and final vertices. We distinguish two cases.

a) $x_2 \neq y_0$.

Without loss of generality, let us suppose $x_2 = d_\Delta$ and $y_0 = d_{\Delta-1}$: $(x_i, y_j \in \{d_0, d_1, \ldots, d_\Delta\})$

$$
\begin{array}{ccccc}
 & & x_0x_1x_2 & & \\
 & \swarrow & \downarrow & & \searrow \\
 & x_1x_2d_0 & x_1x_2d_1 & \cdots & x_1x_2d_{\Delta-2} \\
 & \downarrow & \downarrow & & \downarrow \\
 & x_2d_0y_0 & x_2d_1y_0 & \cdots & x_2d_{\Delta-2}y_0 \\
 & \downarrow & \downarrow & & \downarrow \\
 & d_0y_0y_1 & d_1y_0y_1 & \cdots & d_{\Delta-2}y_0y_1 \\
 & \searrow & \downarrow & & \swarrow \\
 & & y_0y_1y_2 & &
\end{array}
$$

It is obvious that any two labels in the same row cannot be equal because of their construction. Neither can be labels in consecutive rows because of the condition of Kautz labels (two consecutive symbols in a label cannot be equal). Finally, a label in the first row cannot be equal to any label in the last one because of the condition imposed ($x_2 \neq y_0$). Therefore all walks in the scheme are vertex disjoint.

b) $x_2 = y_0$.

In this case, there might be at most one coincidence between a label in the first row and a label in the last one (notice that the first symbol of vertices in the first row is always x_1, while such digit is different in each vertex in the third row). But in this case there is an additional walk of length D, using symbol $d_{\Delta-1}$. □

Similar results can be easily obtained for $D = 1, 2$, that is, in both cases between any pair of vertices there always exist $\Delta - 1$ disjoint walks of length 2 and 3 respectively.

3.2 Disjoint paths in Kautz and De Bruijn digraphs

In [10] it is proved that the number of disjoint paths of length less than or equal to $D+1$ in a Kautz digraph is $\Delta - 2$ and it is $\Delta - 1$ in the De Bruijn one. Nevertheless in order to prove our main results in the next section we need a stronger result.

Let $B(\Delta, D)$ be a De Bruijn digraph. Let us consider Δ walks of length $D+1$ from $x = z_{-D} \ldots z_{-1}$ to $y = z_1 \ldots z_D$, i.e. $W_i = z_{-D} \ldots z_{-1} z_0 z_1 \ldots z_D$ where $z_0 = d_i$.

$$
\begin{array}{ccccc}
 & & z_{-D}\ldots z_{-1} & & \\
 \swarrow & & \downarrow & & \searrow \\
z_{-D+1}\ldots z_{-1}d_0 & & z_{-D+1}\ldots z_{-1}d_1 & \cdots & z_{-D+1}\ldots z_{-1}d_{\Delta-1} \\
\downarrow & & \downarrow & & \downarrow \\
\ldots z_{-1}d_0 z_1 & & \ldots z_{-1}d_1 z_1 & \cdots & \ldots z_{-1}d_{\Delta-1} z_1 \\
\downarrow & & \downarrow & & \downarrow \\
\ldots & & \ldots & \ldots & \ldots \\
\downarrow & & \downarrow & & \downarrow \\
d_0 z_1\ldots z_{D-1} & & d_1 z_1\ldots z_{D-1} & \cdots & d_{\Delta-1} z_1\ldots z_{D-1} \\
 \searrow & & \downarrow & & \swarrow \\
 & & z_1\ldots z_D & &
\end{array}
$$

In this context we will say two walks (or subwalks) are similar if they differ at most in symbol z_0. It is clear that in this particular case it is an equivalence relationship.

Let v_{ij}, $1 \leq i \leq D$, stand for the i-th vertex in walk j. The gap between to vertices v_{ij} and v_{kl} is defined as $k - i$.

Theorem 1 *If there exists a coincidence between two vertices v_{ij} and v_{kl} with gap $s = k - i > 0$, then any other coincidence within two vertices v_{mn} and v_{pq} having gap $t = p - m > 0$ must fulfill one of the following sentences:*
i) $j = n, l = q$ whenever $t = s$.
ii) $n = q$ whenever $t < s$.

Proof:
A coincidence within different vertices means a coincidence within their symbols, and thus:

$$
\begin{array}{cccccccc}
v_{kl} & = & \ldots & d_l & z_1 & \ldots & z_s & \ldots \\
\| & & \| & \| & \| & \| & \| & \| \\
v_{(k-s)j} & = & \ldots & z_{-s} & \ldots & z_{-1} & d_j & \ldots
\end{array}
\qquad (1)
$$

$$
\begin{array}{cccccccc}
v_{(m+t)q} & = & \ldots & d_q & z_1 & \ldots & z_t & \ldots \\
\| & & \| & \| & \| & \| & \| & \| \\
v_{mn} & = & \ldots & z_{-t} & \ldots & z_{-1} & d_n & \ldots
\end{array}
\qquad (2)
$$

Notice that since d_i belongs to all vertices for some i, it always will appear in Equations (1) and (2).

In order to prove this proposition we will distinguish two cases:

i) $t = s$: Because of this condition we have $z_{-t} = z_{-s}$ and $z_t = z_s$. Due to Equations (1) and (2), $d_l = z_{-s}$, $d_q = z_{-t}$, $d_j = z_s$ and $d_n = z_t$. Therefore the four vertices are indeed in two paths: $d_n = d_j$ and $d_q = d_l$ (and thus $n = j$ and $l = q$).

ii) $t < s$: Let us consider walk P_1 (which is a subwalk of W_l) from vertex v_{il} to vertex v_{kl}. Since $v_{ij} = v_{kl}$ we will have by (1) $z_x = z_{x-s}$ for all $x \neq s, 0$, such that z_x and z_{x-s}, $s = k - i$, belong to this subwalk.

Analogously let us consider walk P_2 from v_{mn} to v_{pn}. Again, since $v_{mn} = v_{pq}$ we will have by (2) $z_x = z_{x+t}$ for all $x \neq 0, -t$ such that z_x and z_{x+t}, $t = p - m$, belong to this subwalk.

In order to go on with this proof we need the following result:

Lemma 1 *There exists a subsequence CS in P_1 with length at least $s + t + 1$ which is similar to a subsequence in P_2, and moreover $z_t, z_{-t} \in CS$.*

Proof: Let us recall the following facts:

a) the sequence corresponding to a whole walk W from $x = z_{-D} \dots z_{-1}$ to $y = z_1 \dots z_D$ has length $2D + 1$;

b) none of the vertices we are considering can be x nor y, and

c) P_1 and P_2 will be similar to subsequences of W of length $D + t$ and $D + s$ respectively.

Therefore the intersection has length at least $s + t + 1$ as it is shown in figure 1.

Let us prove $z_t, z_{-t} \in CS$: Since symbols in P_2 are those in Equation 2, both symbols, z_t and z_{-t} belong to P_2. Analogously $z_s, z_{-s} \in P_1$ and thus z_i belongs to P_1 whenever $-s \leq i \leq s$, $i \neq 0$. Since $t < s$, the lemma is proved. □

Because of the first part of this lemma, (i.e. the length of CS is greater than or equal to $t + s + 1$), it is clear that if $z_i \in CS$, then either $z_{i-s} \in CS$ or $z_{i+t} \in CS$. As we showed at the beginning of this proof, if $z_{i-s} \in CS$, then $z_i = z_{i-s}$ whenever $i \neq s, 0$ and if $z_{i+t} \in CS$, then $z_i = z_{i+t}$ whenever $i \neq 0, -t$.

Another lemma is required to finish this proof:

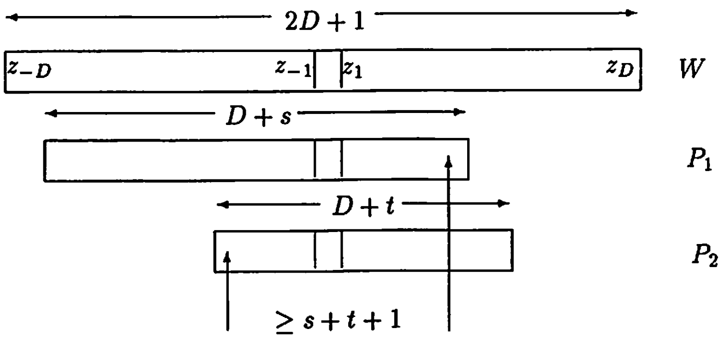

Figure 1: Proof of Lemma 1.

Lemma 2 *If* $z_{at-bs} \in CS$ *and* $z_{a't-b's} \in CS$ *with* $a \le a', b \le b'$ *then* $z_{at-bs} = z_{a't-b's}$ *whenever there do not exist integers* α, β , $a \le \alpha \le a', b \le \beta \le b'$ *such that* $\alpha t - \beta s = 0$.

Proof: As it was shown, $z_{at-bs} = z_{(a+1)t-bs}$ or $z_{at-bs} = z_{at-(b+1)s}$. Let us suppose the first equality holds n times. So we will have $z_{at-bs} = z_{a_n t-bs}$ with $a \le a+n = a_n \le a'$. If a new iteration is not possible, then we will have $z_{a_n t-bs} = z_{a_n t-(b+1)s}$. We will be able to repeat this process until $z_{at-bs} = z_{a't-b_k s}$ or $z_{at-bs} = z_{a_k t-b's}$. Let us suppose that the first equality holds. Since $z_{a't-b_k s} \in CS$ and $z_{a't-b's} \in CS$, then $z_{a't-(b_k+i)s} \in CS$ for all $i \le b' - b_k$, and therefore $z_{at-bs} = z_{a't-b's}$. In the other cases the proof is analogous. □

Since $z_t, z_{-t} \in CS$ (see Lemma 1), the result is now proved as a corollary of Lemma 2:

In case $gcd(t,s) = 1$, because of Lemma 2, $z_t = z_{(s-1)t-ts}$ $(= z_{-t})$. In other case if $\alpha t - \beta s = 0$ then $(\alpha - 1)t - \beta s = -t$ and we can apply the same procedure as in the proof of Lemma 2 to prove $z_t = z_{-t}$.

Finally, as it was shown in Equation (2) $d_q = z_t$ and $z_{-t} = d_n$ and thus $d_n = d_q$ (and hence $n = q$). □

The same result holds for the Kautz digraph, just considering there exist $\Delta - 1$ walks of length $D + 1$.

As a corollary of this theorem we obtain a similar result to the one in [10]:

Corollary 1 *a) Let $B(\Delta, D)$ be a De Bruijn digraph. Between any two vertices there exist Δ paths of length $D+1$, all of them being internally disjoint except at most one.*

b) Let $K(\Delta, D)$ be a Kautz digraph. Between any two vertices there exist at least $\Delta - 1$ paths of length $D+1$, all of them being internally disjoint except at most one.

4 New families of fault tolerant digraphs

Definition 1 *a) Let $KK(\Delta_1, \Delta_2, D)$ be the resulting digraph of the conjunction of two Kautz digraphs of diameter $D-1$, i.e. $K(\Delta_1, D-1) \otimes K(\Delta_2, D-1)$.*

b) Let $BK(\Delta_1, \Delta_2, D)$ be the resulting digraph of the conjunction of a Kautz digraph and a De Bruijn digraph, both of diameter $D-1$, i.e. $B(\Delta_1, D-1) \otimes K(\Delta_2, D-1)$.

The diameter of both digraphs is shown in the next proposition.

Proposition 2 *Both $KK(\Delta_1, \Delta_2, D)$ and $BK(\Delta_1, \Delta_2, D)$ have diameter D.*

Proof: Since both $K(\Delta, D-1)$ and $B(\Delta, D-1)$ are D-reachable i.e. there always exists a walk of length D from any vertex v to any other w (see for instance Corollary 1), and because of construction of the direct product, the diameter of $KK(\Delta_1, \Delta_2, D)$ and $BK(\Delta_1, \Delta_2, D)$ cannot be larger than D.

On the other hand let us recall that in $K(\Delta, D-1)$ there exist vertices which are not $(D-1)$-reachable from some others (see [11]). Let us consider two vertices (v_1, w_1) and (v_2, w_2) in which v_2 is at distance $D-1$ from v_1 in its original digraph (i.e. either Kautz or De Bruijn digraph) and where w_2 is not $(D-1)$-reachable from w_1 in the Kautz digraph. The distance from (v_1, w_1) to (v_2, w_2) must be larger than $D-1$. The proposition is then proved. □

As it is shown in Section 2, $KK(\Delta_1, \Delta_2, D)$ is regular of degree $\Delta_1\Delta_2$ and has order $(\Delta_1\Delta_2)^{D-2}(\Delta_1+1)(\Delta_2+1)$. Analogously $BK(\Delta_1, \Delta_2, D)$ is regular of degree $\Delta_1\Delta_2$ and has order $(\Delta_2\Delta_1)^{D-1}(1+\frac{1}{\Delta_2})$.

Theorem 2 *$KK(\Delta_1, \Delta_2, D)$ is a $(\Delta_1\Delta_2, D, (\Delta_1-1)(\Delta_2-1)-2)$-digraph.*

Proof:

Since between any pair of vertices in a Kautz digraph $K(\Delta, D-1)$ there always exist $(\Delta - 1)$ walks of length D, then there exist $(\Delta_1 - 1)(\Delta_2 - 1)$ walks of length D in $KK(\Delta_1, \Delta_2, D)$.

Let us assume there exists a coincidence within two vertices in different walks, i.e. $(v_{ia}, w_{ib}) = (v_{jc}, w_{jd})$. We will prove there cannot be any other coincidence within vertices in different walks, and therefore, all walks are disjoint except one of them.

Suppose $(v_{kz}, w_{ky}) = (v_{lx}, w_{lv})$ or equivalently in a Kautz digraph, $v_{kz} = v_{lx}$ and $w_{ky} = w_{lv}$. Because of Theorem 1, if $l - k = j - i$ we will have $a = z, b = y, c = x, d = v$. That is, both coincidences are in the same pair of walks.

If $l - k < j - i$ we will have $x = z, v = y$, that is, the second coincidence is in the same walk.

In both cases there only can exist a walk which is not disjoint from the others. □

The next proposition considers digraph $BK(\Delta_1, \Delta_2, D)$.We omit its proof since it is identical to the previous one.

Proposition 3 *$BK(\Delta_1, \Delta_2, D)$ is a $(\Delta_1\Delta_2, D, \Delta_1(\Delta_2 - 1) - 2)$-digraph.*

The next proposition is a consequence of previous results:

Proposition 4 *a) $BK(\Delta_1, \Delta_2, D)$ is a $(\Delta_1\Delta_2, D, \Delta_1(\Delta_2 - 1) - 1)$-digraph whenever $D \leq 4$*

b)$KK(\Delta_1, \Delta_2, D)$ is a $(\Delta_1\Delta_2, D, (\Delta_1 - 1)(\Delta_2 - 1))$-digraph whenever $D \leq 4$.

Proof (sketched): Since the number of disjoint paths of length k between any pair of vertices in a conjunction of digraphs is at least the product of of disjoint paths of the same length in both digraphs, this result is a consequence of Proposition 1 and Theorem 2. □

Let us define a new family of digraphs which is a better solution for the (Δ, D', s) problem than any other known families of digraphs for some values of Δ and s.

Definition 2 *The digraph B^*G is defined as $B(\Delta, D) \otimes G$ where G is any digraph.*

In the next result some parameters of this digraph are presented.

Proposition 5 *Let G be a regular D-reachable digraph with degree Δ_1. The digraph $B^*G = B(\Delta, D) \otimes G$ has $n\Delta^D$ vertices, diameter D, degree $\Delta\Delta_1$. Moreover B^*G has k disjoint walks of length D whenever G has k disjoint walks of length D.*

Proof (sketched): The values for the degree and the diameter are trivially obtained from the definition of the conjunction of digraphs, and from the fact that both digraphs are D-reachable. Moreover, since there always exists one path of length D between any two vertices in $B(\Delta, D)$, the number of disjoint paths of length D in G will be preserved in B^*G. □

5 Conclusions

In Table 1 the properties of Kautz and De Bruijn digraphs are shown as well as the properties of the new families of (Δ, D', s)-digraphs studied in this paper.

digraph (G_i)	degree (d_i)	Order (n_i)	($s_i + 1$)
$K(\Delta, D)$	Δ	$\Delta^D + \Delta^{D-1}$	$\Delta - 2$
$B(\Delta, D)$	Δ	Δ^D	$\Delta - 1$
$K(\Delta_1, D-1) \otimes K(\Delta_2, D-1)$	$\Delta_1\Delta_2$	$(\Delta_1\Delta_2)^{D-2}(\Delta_1+1)(\Delta_2+1)$	$(\Delta_1 - 1)(\Delta_2 - 1) - 1$
$B(\Delta_1, D-1) \otimes K(\Delta_2, D-1)$	$\Delta_1\Delta_2$	$(\Delta_1\Delta_2)^{D-1}(1 + \frac{1}{\Delta_2})$	$\Delta_1(\Delta_2 - 1) - 1$
$B(\Delta, D) \otimes G_i$	Δd_i	$\Delta^D n_i$	$s_i + 1$

Table 1: Families of dense (Δ, D', s)-digraphs.

It is easy to check that all these families are asymptotically optimal when the degree increases, that is,

$$\lim_{(\Delta_1, \Delta_2) \to \infty} \frac{\text{number of vertices}}{\text{Moore-like bound}} = 1$$

Furthermore these new families give most of values in the tables shown in the Annex, which contains the largest known families of dense (Δ, D', s)-digraphs up to date (october 1995).

Acknowledgments

This work has been supported by the Spanish Research Council (Comisión Interministerial de Ciencia y Teccnología, CICYT) under projects TIC 92-1228-E and TIC 94-0592 and by EU-HCM under project ERBCHRX-CT920049.

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Annex: Tables

Symbols and notations used in the tables:

FD	FD bipartite digraph, see [9].
II	Imase-Itoh digraph, see [13].
K	Kautz digraph $K(\Delta, D-1)$, see [10].
B	De Bruijn digraph $B(\Delta, D-1)$, see [10].
BK	$B(\Delta, D-1) \otimes K(\Delta', D-1)$, see Section 4.
KK	$K(\Delta, D-1) \otimes K(\Delta', D-1)$, see Section 4.
*B*B*	$B(\Delta, D) \otimes B(\Delta', D-1)$, see Section 4.
*B*K*	$B(\Delta, D) \otimes K(\Delta', D-1)$, see Section 4.
*B*BK*	$B(\Delta, D) \otimes B(\Delta', D-1) \otimes K(\Delta'', D-1)$, see Section 4.
*B*KK*	$B(\Delta, D) \otimes K(\Delta', D-1) \otimes K(\Delta'', D-1)$, see Section 4.

The number below each graph stands for the equation $\frac{n}{\Delta^D/(s+1)}$, where n is the order of the graph. The larger this number, the closer to the Moore-like bound (notice that this number may be greater than 1).

TABLE 1. LARGEST KNOWN (Δ, D', s) DIGRAPHS WITH $D' = 3$

Δ \ s	1	2	3	4	5	6	7	8	9	10
2	II 1,50									
3	K 0,89	II 1,33								
4	II 1,25	K 0,94	II 1,25							
5			K 0,96	II 1,20						
6	II 1,16	II 1,16	BK 0,89	K 0,97	II 1,16					
7					K 0,98	II 1,14				
8	II 1,12	B*K 0,94	II 1,12		BK 0,94	K 0,98	II 1,12			
9	B*K 0,89	II 1,11	KK 0,79		BK 0,89		K 0,99	II 1,11		
10	II 1,10		B*K 0,96	II 1,10			BK 0,96	K 0,99	II 1,10	
11									K 0,99	II 1,10
12	II 1,08	II 1,08	II 1,08	B*K 0,97	II 1,08		BK 0,89	BK 0,94	BK 0,97	K 0,99
13										
14	II 1,07				B*K 0,98	II 1,07				
15	B*K 0,89	II 1,07	B*K 0,96	II 1,07			KK 0,85		BK 0,89	
16	II 1,06	B*K 0,94	II 1,06		B*BK 0,94	B*K 0,98	II 1,06	KK 0,88		
17										
18	II 1,06	II 1,06	B*KK 0,79	B*K 0,97	II 1,06		B*K 0,99	II 1,06	KK 0,86	
19										
20	II 1,05	B*K 0,94	II 1,05	II 1,05			B*BK 0,96	B*K 0,9	II 1,05	

TABLE 2. LARGEST KNOWN (Δ, D', s) DIGRAPHS WITH $D' = 4$

Δ \ s	1	2	3	4	5	6	7	8	9	10
2	*FD* 1,5									
3	*K* 0,89	*FD* 0,88								
4	*BK* 0,75	*K* 0,94	*FD* 0,62							
5			*K* 0,96	*FD* 0,48						
6	*B***K* 0,89	*BK* 0,75	*BK* 0,89	*K* 0,97	*FD* 0,40					
7					*K* 0,98	*FD* 0,33				
8	*B***BK* 0,75	*B***K* 0,94	*BK* 0,75		*BK* 0,94	*K* 0,98	*FD* 0,29			
9	*B***K* 0,89		*KK* 0,79		*BK* 0,89		*K* 0,99	*FD* 0,24		
10			*B***K* 0,96	*BK* 0,75			*BK* 0,96	*K* 0,99	*FD* 0,22	
11									*K* 0,99	*FD* 0,20
12	*B***K* 0,89	*B***K* 0,94	*B***BK* 0,89	*B***K* 0,97	*KK* 0,83		*BK* 0,89	*BK* 0,94	*BK* 0,97	*K* 0,99
13										
14					*B***K* 0,98	*BK* 0,75				
15	*B***K* 0,89		*B***K* 0,96				*KK* 0,85		*BK* 0,89	
16	*B***BK* 0,75	*B***K* 0,94	*B***BK* 0,75		*B***BK* 0,94	*B***K* 0,98	*BK* 0,75	*KK* 0,88		
17										
18	*B***K* 0,89	*B***BK* 0,75	*B***KK* 0,79	*B***K* 0,97	*B***BK* 0,89		*B***K* 0,99	*BK* 0,75	*KK* 0,86	
19										
20	*B***BK* 0,75	*B***K* 0,94	*B***K* 0,96	*B***BK* 0,75			*B***BK* 0,96	*B***K* 0,9	*BK* 0,75	

TABLE 3. LARGEST KNOWN (Δ, D', s) DIGRAPHS WITH $D' > 4$

s / Δ	1	2	3	4	5	6	7	8	9	10
2										
3	B 0,67									
4	K 0,63	B 0,75								
5		K 0,72	B 0,80							
6	$B*B$ 0,67	BK 0,67	K 0,78	B 0,83						
7				K 0,82	B 0,86					
8	$B*K$ 0,63	$B*B$ 0,75		BK 0,78	K 0,84	B 0,88				
9	$B*B$ 0,67	KK 0,59		BK 0,74		K 0,86	B 0,89			
10		$B*K$ 0,72	$B*B$ 0,80			BK 0,84	K 0,88	B 0,90		
11								K 0,89	B 0,91	
12	$B*B$ 0,67	$B*B$ 0,75	$B*K$ 0,78	$B*B$ 0,83		BK 0,78	BK 0,83	BK 0,87	K 0,90	B 0,92
13										K 0,91
14				$B*K$ 0,82	$B*B$ 0,86					BK 0,90
15	$B*B$ 0,67	$B*K$ 0,72	$B*B$ 0,80			KK 0,75		BK 0,80		BK 0,88
16	$B*K$ 0,63	$B*B$ 0,75		$B*BK$ 0,80	$B*K$ 0,84	$B*B$ 0,88	KK 0,78			BK 0,86
17										
18	$B*B$ 0,67	$B*BK$ 0,67	$B*K$ 0,78	$B*B$ 0,83		$B*K$ 0,86	$B*B$ 0,89	KK 0,77		BK 0,81
19										
20	$B*K$ 0,63	$B*B$ 0,75	$B*B$ 0,80			$B*BK$ 0,84	$B*K$ 0,88	$B*B$ 0,90		KK 0,83