Distance irredundance in graphs: complexity issues

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Abstract

Let $n \geq 1$ be an integer. The *closed n-neighborhood* $N_n[u]$ of a vertex u in a graph $G = (V, E)$ is the set of vertices $\{v | d(u, v) \leq n\}$. The *closed n-neighborhood of a set X of vertices*, denoted by $N_n[X]$, is the union of the closed n-neighborhoods $N_n[u]$ of vertices u in X. For $x \in X \subseteq V(G)$, if $N_n[x] - N_n[X - \{x\}] = \emptyset$, then x is said to be *n-redundant in X*. A set X containing no n-redundant vertex is called *n-irredundant*. The *n-irredundance number of G*, denoted by $ir_n(G)$, is the minimum cardinality taken over all maximal n-irredundant sets of vertices of G. The *upper n-irredundance number of G*, denoted by $IR_n(G)$, is the maximum cardinality taken over all maximal n-irredundant sets of vertices of G. In this paper we show that the decision problem corresponding to the computation of $ir_n(G)$ for bipartite graphs G is NP-complete. We then prove that this also holds for augmented split graphs. These results extend those of Hedetniemi, Laskar and Pfaff (see [7]) and Laskar and Pfaff (see [8]) for the case $n = 1$. Lastly, applying the general method described by Bern, Lawler and Wong (see [1]), we present linear algorithms to compute the 2-irredundance and upper 2-irredundance numbers for trees.

1 Introduction

Let $n \geq 1$ be an integer. The *closed n-neighborhood* $N_n[u]$ of a vertex u in a graph $G = (V, E)$ is the set of vertices $\{v | d(u, v) \leq n\}$, i.e., all those vertices which are at distance at most n from u. The *open n-neighborhood* $N_n(u)$ *of* u is defined as $N_n[u] - \{u\}$. The *open (closed) n-neighborhood of a set* X *of vertices*, denoted by $N_n(X)$ $(N_n[X])$ is the union of the open (closed) n-neighborhoods $N_n(u)$ $(N_n[u])$ of vertices u in X. For $x \in X \subseteq V(G)$, if $N_n[x] - N_n[X - \{x\}] = \emptyset$, then x is said to be *n-redundant in* X. Equivalently, x is n-redundant in X if and only if $N_n[x] \subseteq N_n[X - \{x\}]$. A set X containing no n-redundant vertex is called *n-irredundant.* If X is n-irredundant and $x \in X$, the set $N_n[x] - N_n[X - \{x\}]$ is called the set of *private n-neighbors of* x and is denoted by $PN_n[x, X]$. The *n-irredundance number of* G, denoted by $ir_n(G)$, is the minimum cardinality taken over all maximal n-irredundant sets of vertices of G, while the *upper n-irredundance number of* G, denoted by $IR_n(G)$, is the maximum cardinality taken over all maximal n-irredundant sets of vertices of G.

A set D of vertices in a graph G is defined to be an n-dominating set of G if every vertex of $V(G) - D$ is within distance n from some vertex of D. The minimum cardinality among all n-dominating sets of a graph G is called the *n-domination number of* G and is denoted by $\gamma_n(G)$, while the maximum cardinality among all minimal n-dominating sets of a graph G is called the *upper n-domination number of* G and is denoted by $\Gamma_n(G)$. A set D of vertices in a graph G is called *n-independent* if $d(u, v) > n$ for all $u, v \in D$. The *independent n-domination number of* G, denoted by $i_n(G)$, is the minimum cardinality among all maximal n-independent sets of a graph G, while the *n-independence number of* G, denoted by $\beta_n(G)$, is the maximum cardinality among all maximal n-independent sets of a graph

G.

These parameters are related as follows:

Theorem 1 *If G is a graph, then*

$$ir_n(G) \leq \gamma_n(G) \leq i_n(G) \leq \beta_n(G) \leq \Gamma_n(G).$$

∎

The following result of [4] will prove to be useful later.

Theorem 2 *Let X be a maximal n-irredundant set of vertices in a graph G. If u is a vertex of G not n-dominated by X, then for some $x \in X$,*

$$PN_n[x, X] \subseteq N_n(u).$$

∎

A graph G is called a *split graph* if its vertex set can be partitioned into a non-empty clique and a non-empty independent set. If $n \geq 1$ is an integer, then an $(n-1)$*-path augmented split graph* is a graph obtained from a split graph by attaching a path of length $n-1$ to each vertex in the independent set. A *chordal graph* is a graph in which every cycle of length greater than three has a chord, i.e., an edge joining two non-consecutive vertices of the cycle.

Chang and Nemhauser [2] proved that the decision problems corresponding to the computation of $\gamma_n(G)$ and $\beta_n(G)$ for bipartite graphs G are NP-complete. (The latter was proved for $n \geq 2$.) In the same paper, these authors also prove that the decision problems corresponding to the computation of $\gamma_n(G)$ and $\beta_{2n}(G)$ for $(n-1)$-path augmented split graphs G

are NP-complete. Fricke, Hedetniemi and Henning [3] showed that the decision problem corresponding to the computation of $i_n(G)$ for arbitrary graphs G is NP-complete, while Hattingh, Henning and Walters [6] proved that the decision problem corresponding to the computation of $\Gamma_n(G)$ for arbitrary graphs G is NP-complete. Hattingh and Henning [5] showed that the decision problem corresponding to the computation of $IR_n(G)$ for arbitrary graphs is NP-complete. In Section 2, we use a construction of [2] to prove that the decision problem

DISTANCE IRREDUNDANCE SET (DIS)
INSTANCE: A graph G and positive integers n and k.
QUESTION: Is $ir_n(G) \leq k$?

is NP-complete for bipartite graphs G. This extends the result of Hedetniemi, Laskar and Pfaff (see [7]) that the decision problem corresponding to the computation of $ir(G)$ for bipartite graphs G is NP-complete. In Section 3 we show that **DIS** is NP-complete for $(n-1)$-path augmented split graphs by using another construction of [2]. This result extends that of Laskar and Pfaff (see [8]) for the case $n = 1$.

2 NP-completeness of the problem DIS for bipartite graphs

In this section, we show that the decision problem **DIS** for bipartite graphs is NP-complete, by providing a polynomial reduction of the domination problem on general graphs to the problem **DIS** on bipartite graphs.

We start by discussing a construction by Chang and Nemhauser [2] to show that the decision problems corresponding to the computation of $\gamma_n(G)$ and $\beta_n(G)$ for an arbitrary bipartite graph G are NP-complete. (The latter

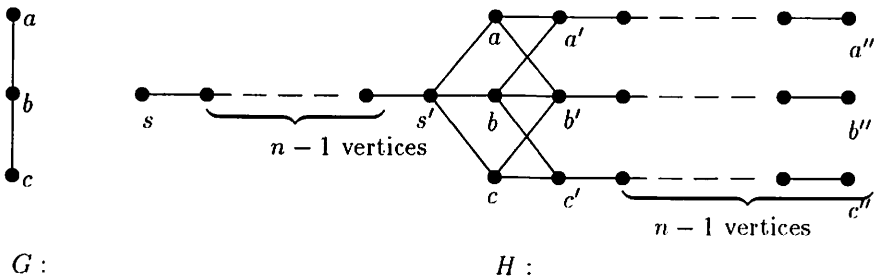


Figure 1: Chang-Nemhauser transformation I

was proved for $n \geq 2$.)

For any graph $G = (V, E)$ and $s \notin V$, construct the bipartite graph $G' = (V_1 \cup V_2, F)$, where $V_1 = V \cup \{s\}, V_2 = \{v' | v \in V_1\}$, and $F = \{xy' | x, y \in V, d_G(x, y) \leq 1\} \cup \{vs' | v \in V_1\}$. If $n \geq 2$, we construct the bipartite graph H from G' as follows: the edge ss' is subdivided $n - 1$ times, while a path P_v of length $n - 1$ is attached to each vertex v' with $v \in V$. For $v \in V$, let v'' be the end-vertex of P_v and let $Y = \{v'' | v \in V\}$. Furthermore, let S be the vertex set of the s-s'-path.

An example of this construction is given in Figure 1.

Lemma 1 (Chang and Nemhauser) $\gamma(G) + 1 = \gamma_n(H)$. ∎

Let I be a maximal irredundant set of H such that $|I| = ir_n(H)$. We show that $ir_n(H) = \gamma_n(H)$.

Lemma 2 $|V(P_v) \cap I| \leq 1$ *for each* $v \in V$.

Proof. If $x, y \in V(P_v) \cap I$ with $d(x, v') < d(y, v')$, then $N_n[y] \subseteq N_n[x]$, which implies that $PN_n[y, I] = \emptyset$, a contradiction. ∎

Lemma 3 $|S \cap I| = 1$.

Proof. Suppose $S \cap I = \emptyset$. Then, since $I \cup \{s\}$ is not n-irredudant, there is a $y \in I \cup \{s\}$ such that $N_n[y] - N_n[I \cup \{s\} - \{y\}] = \emptyset$. Since $s \in N_n[s] - N_n[I]$, we have that $y \neq s$. Further, since $N_n[y] - N_n[I \cup \{s\} - \{y\}] = \emptyset$, it follows that $\emptyset \neq PN_n[y, I] = N_n[y] - N_n[I - \{y\}] \subseteq N_n[s] = S$. Since $y \notin S$, it follows that $\emptyset \neq PN_n[y, I] \subseteq S - \{s\}$. Before proceeding further, we prove three claims:

Claim 1 $|V \cap I| \leq 1$.

Proof. If $v \in V \cap I$ with $v \neq y$, then $PN_n[y, I] \subseteq S - \{s\} \subseteq N_n[v]$, which implies that $PN_n[y, I] = \emptyset$, a contradiction. We deduce that y is the only possible vertex of V in I. ◇

Claim 2 $y \in V$.

Proof. If $n = 1$, then $PN_n[y, I] = \{s'\}$ so that $y \in V$. We assume that $n \geq 2$ and that $y \notin V$. Then y is on P_v for some $v \in V$. By Lemma 2, y is the only vertex on P_v in I. Thus, since $I \cap V = \emptyset$ (cf. Claim 1), it follows that $v'' \in PN_n[y, I]$. But $v'' \notin S - \{s\}$, which is a contradiction. We deduce that $y \in V$. ◇

Claim 3 $N(y) - \{s'\} \subseteq I$.

Proof. Let $v' \in N(y) - \{s'\}$ and assume that $v' \notin I$. If $w \in V(P_v) \cap I$, then, since $w \neq v'$, it follows that $N_n[w] \subseteq N_n[y]$ and so $PN_n[w, I] = \emptyset$, a contradiction. Hence $V(P_v) \cap I = \emptyset$. Thus, since $I \cap V = \{y\}$, it follows that $v'' \in PN_n[y, I]$, so that $v'' \in S$, a contradiction. ◇

In particular, we have that $y' \in I$. This implies that $PN_n[y', I] \neq \emptyset$. Let $w \in PN_n[y', I]$. Since $y \in I - \{y'\}$ and $(S - \{s\}) \cup V(P_y) \subseteq N_n[y]$, it follows that $w \notin S \cup V(P_y)$. Let P be a shortest $y' - w$ path. Note that $y \notin V(P)$ since otherwise $w \in N_n[y]$. Hence the vertex v immediately succeeding y' on P must be in $V - \{y\}$. Since $v \in N(y')$, it follows that $v' \in N(y)$. By Claim 3, $v' \in I$. However, the $v' - w$ path obtained from the $v - w$ section of P by adding v' and the edge $v'v$ has the same length as that of P, whence $w \in N_n[v']$. This contradicts the fact that $w \in PN_n[y', I]$. We conclude that $S \cap I \neq \emptyset$.

If $x, y \in S \cap I$ with $d(x, s') < d(y, s')$, then $N_n[y] \subseteq N_n[x]$, which implies that $PN_n[y, I] = \emptyset$, a contradiction. We conclude that $|S \cap I| = 1$. ∎

By Lemma 3, $\{t\} = S \cap I$. Let U be the set of vertices of Y which are not n-dominated by I. If $U = \emptyset$, then I is an n-dominating set of H. This implies that $\gamma_n(H) \leq |I| = ir_n(H) \leq \gamma_n(H)$, so that $\gamma_n(H) = ir_n(H)$. So let $v'' \in U$. By Theorem 2, there exists $y \in I$ such that $PN_n[y, I] \subseteq N_n(v'')$. For each $v'' \in U$, let $f(v)$ be any such y.

Lemma 4 *$f(v) \notin V$ for all $v'' \in U$.*

Proof. Suppose $f(v) \in V$. Then $f(v) = y$ for some $y \in V - N(v')$. Since $PN_n[y, I] \subseteq N_n(v'')$, it follows that $I \cap V = \{y\}$. Thus, if $V(P_y) \cap I = \emptyset$, then $y'' \in PN_n[y, I]$, so that $d(y'', v'') \leq n$, a contradiction. So let $w \in V(P_y) \cap I$. If $w \neq y'$, then $N_n[w] \subseteq N_n[y]$, so $PN_n[w, I] = \emptyset$, a contradiction. Thus $w = y'$ and $y' \in I$. Proceeding now as in the paragraph immediately following the proof of Claim 3, we arrive at a contradiction, completing the proof. ∎

For each $f(v)$, let $g(f(v))$ be any vertex in $PN_n[f(v), I]$. Note that f, g are chosen to be functions. Then $g(f(v))$ is within distance n from v''. Let $I' = (I - \{f(v) | v'' \in U\}) \cup \{g(f(v)) | v'' \in U\}$. Note that $|I'| \leq |I|$.

Lemma 5 $t \in I'$.

Proof. If not, then $t = f(v)$ for some $v'' \in U$. Since $\{t\} = S \cap I$, it follows that $s \in PN_n[t, I]$. But $s \notin N_n[v'']$ for any $v'' \in U$, producing a contradiction. ∎

Lemma 6 *All vertices of Y are n-dominated by I'.*

Proof. If $v'' \in U$, then, since $g(f(v)) \in I'$, it follows that v'' is n-dominated by I'. Assume, then, that $v'' \notin U$. If $v'' \in I$, then v'' is n-dominated by I', since vertices in I are replaced by vertices within distance n from them. If $v'' \notin I$, then there exists $w \in I$ that n-dominates v''. If $w \in V$, it follows from Lemma 4, that $w \in I'$ and thus v'' is still n-dominated by I'. If, on the other hand, there is no such $w \in I$, then $w \in V(P_v)$ and $v'' \in PN_n[w, I]$. This means that $w \neq f(u)$ for all $u'' \in U$, so that $w \in I'$. Thus v'' is again n-dominated by I'. ∎

Theorem 3 *The problem* **DIS** *is NP-complete for bipartite graphs.*

Proof. Lemmas 5 and 6 imply that I' is an n-dominating set of H, so that $\gamma_n(H) \leq |I'| \leq |I| = ir_n(H) \leq \gamma_n(H)$, so that $\gamma_n(H) = ir_n(H)$. Thus, by Lemma 1, $ir_n(H) = \gamma(G) + 1$. Thus, the problem of determining the domination number of an arbitrary graph G can be transformed to the problem of determining the n-irredundance number of the bipartite graph H. ∎

3 NP-completeness of the problem DIS for chordal graphs

In this section we show that the decision problem **DIS** for $(n-1)$-path augmented split graphs is NP-complete by providing a polynomial reduction of the domination problem on general graphs to the problem **DIS** on $(n-1)$-path augmented split graphs. This result extends that of Laskar and Pfaff (see [8]) for the case $n = 1$.

We start with the following result:

Lemma 7 *Let $n \geq 1$ be an integer. If G is a connected $(n-1)$-path augmented split graph, then $\gamma_n(G) = ir_n(G)$.*

Proof. Let H be the underlying split graph, with the vertex set of H partitioned into the non-empty clique V and the non-empty independent set W. If $w \in W$, a path of length $n-1$ is attached to w; we denote the end-vertex of this path by w'. If $W = \{w\}$, then, since G is connected, there exists a $v \in V$ such that $vw \in E(G)$. Note that $\{v\}$ is both an n-dominating set and a maximal n-irredundant set, so that $\gamma_n(G) = ir_n(G) = 1$. We may assume that $|W| \geq 2$. Let I be a maximal n-irredundant set of cardinality $ir_n(G)$ and let $W' = \{w'|w \in W\}$. If I is an n-dominating set for W', then either $I \neq W'$, in which case I also n-dominates G, or $I = W'$. In the latter case, if we choose $w' \in W'$ arbitrarily and let $v \in V$ be such that $vw \in E(G)$, then $(I - \{w'\}) \cup \{v\}$ is an n-dominating set of G. In both cases, $\gamma_n(G) \leq |I| = ir_n(G)$. If I is not an n-dominating set for W', there exists $w' \in W'$ such that $w' \notin N_n[I]$. By Theorem 2, there exists an $x \in I$ such that $PN_n[x, I] \subseteq N_n(w')$. Before proceeding further, we prove three claims.

Claim 4 $x \in V$.

Proof. If $x \notin V$, then x is on some $t-t'$-path, where $t \in W$ and $t' \in W'$. If $t' \in PN_n[x, I]$, then $d(t', w') \leq n$, which is a contradiction. Hence, there exists a $y \in I - \{x\}$ such that $t' \in N_n[y]$. If y follows x on the $t-t'$-path, we have that $N_n[y] \subseteq N_n[x]$, making y redundant in I. Hence y precedes x on the $t-t'$-path or $y \in V$. However, in both cases, $PN_n[x, I] \subseteq N_n[y]$, which is impossible. This contradiction shows that $x \in V$. ◇

Note that $V \cap I = \{x\}$ since any $y \in V$ has $N_n(w') \subseteq N_n[y]$.

Claim 5 $x \notin N[W]$.

Proof. Since $w' \notin N_n[I]$, it follows that $xw \notin E(G)$. Suppose there exists a $t \in W$ such that $xt \in E(G)$. If $t' \in PN_n[x, I]$, then $d(t', w') \leq n$, which is a contradiction. Hence, there exists a $y \in I - \{x\}$ such that $t' \in N_n[y]$. Since $|V \cap I| = 1$, it follows that y is on the $t-t'$-path, so that $N_n[y] \subseteq N_n[x]$, making y redundant in I, which is a contradiction. ◇

Claim 6 *If $t' \in W' - \{w'\}$, then t' is n-dominated by $I - \{x\}$.*

Proof. Since $x \notin N[W]$, it follows that t' is not n-dominated by x. If t' is not n-dominated by I, then, by Theorem 2, there exists a $y \in I$ such that $PN_n[y, I] \subseteq N_n(t')$. Since $PN_n[y, I] \subseteq N_n(t') \subseteq N_n[x]$, it follows that $PN_n[y, I] = \emptyset$, which is a contradiction. ◇

It now follows from the above claims that if $x^* \in PN_n[x, I]$, then $I - \{x\} \cup \{x^*\}$ is an n-dominating set of cardinality $|I|$ and so $\gamma_n(G) \leq |I| = ir_n(G) \leq \gamma_n(G)$, so that $\gamma_n(G) = ir_n(G)$. ∎

For any graph $G = (V, E)$ consider the split graph $H = (V \cup W, F)$ with $W = \{v' | v \in V\}$ and $F = \{uv | u, v \in V, u \neq v\} \cup \{uv' | u, v \in V, d_G(u, v) \leq$

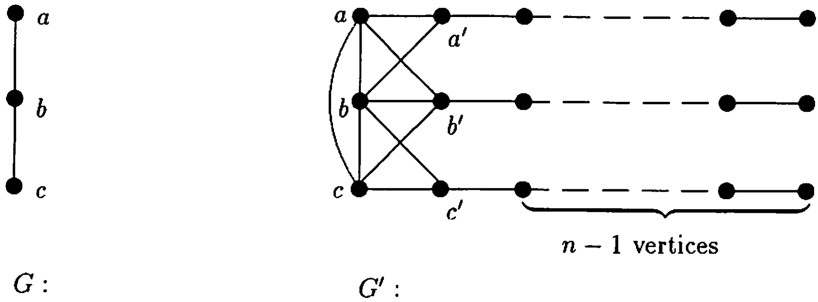


Figure 2: Chang-Nemhauser transformation II

1}. If $n \geq 1$ is an integer, let G' be the $(n-1)$-path augmented split graph with H as the underlying split graph. An example of this construction is given in Figure 2.

Lemma 8 (Chang and Nemhauser) *Let $n \geq 1$ be an integer. If G and G' are the aforementioned graphs, then $\gamma(G) = \gamma_n(G')$.* ∎

Theorem 4 *The decision problem* **DIS** *is NP-complete for $(n-1)$-path augmented split graphs.*

Proof. The NP-completeness of the problem **DIS** follows immediately from Lemmas 7 and 8 and the fact that the domination problem is NP-complete for general graphs. ∎

Since the set of $(n-1)$-path augmented split graphs is a subset of the set of chordal graphs, an immediate corollary now follows.

Corollary 1 *The decision problem* **DIS** *is NP-complete for chordal graphs.*

4 Linear algorithms to compute the 2-irredundance number and upper 2-irredundance number for trees

In this section we develop linear time algorithms to compute the 2-irredundance number and upper 2-irredundance number for trees, by applying the general method described in [1]. We will use the terminology of [1]. We begin with a few definitions.

A composition operation ($\circ$) for trees is defined as follows:

> Let $T_1 = (V_1, E_1)$ and $T_2 = (V_2, E_2)$ be trees with roots v_1 and v_2 respectively. Then $T_1 \circ T_2 = (V_1 \cup V_2, E_1 \cup E_2 \cup \{v_1 v_2\})$ with root v_1.

K_1 is a *primitive graph* for the composition operation in the sense that it cannot be obtained from a composition of two trees.

Let Γ be the set of all tree-subset pairs (T, S), where $S \subseteq V(T)$. We extend the composition operation for trees to Γ as follows:

> Let $(T_1, S_1), (T_2, S_2) \in \Gamma$. Then $(T_1, S_1) \bullet (T_2, S_2) = (T_1 \circ T_2, S_1 \cup S_2)$.

There are two *primitive pairs* (T_1, S_1) and (T_2, S_2) in Γ, with $T_1 = T_2 = K_1$, $S_1 = \{v_1\}$ and $S_2 = \emptyset$.

Let P be a predicate on Γ, with $P(T, S)$ true if and only if S is a 2-irredundant subset of vertices of T. We will define a finite homomorphism h_1 on Γ such that the following holds for all $(T_1, S_1), (T_2, S_2) \in \Gamma$:

(H1) $h_1(T_1, S_1) = h_1(T_2, S_2) \Rightarrow P(T_1, S_1) = P(T_2, S_2)$,

(H2) $h_1((T_1, S_1) \bullet (T_2, S_2)) = h_1(T_1, S_1) \cdot h_1(T_2, S_2)$.

The range of h_1 consists of finite number of equivalence classes C_i, $i = 0, 1, \ldots, n-1$. The composition operation for $h_1(\Gamma)$ is defined with an $n \times n$ composition table as follows:

$C_i \cdot C_j$ is the class C_k with k the entry in row i and column j of the composition table.

There are two *starting classes* in $h_1(\Gamma)$. They are the classes $h_1(T,S)$ such that (T,S) is a primitive pair in Γ. A class C_i is said to be an *accepting class* if $P(T,S)$ is true for a (T,S) pair in Γ with $h_1(T,S) = C_i$.

We will use $N_T[v]$ ($PN_T[v]$) to denote the closed 2-neighborhood (private 2-neighborhood) of v in T.

The following result will be useful in defining the homomorphism h_1.

Let $P(T_1,S_1)$ be false. Then there exists a (T_2,S_2) pair such that $P(T,S)$, with $(T,S) = (T_1,S_1) \bullet (T_2,S_2)$, is true if and only if the following holds:

(i) If $PN_{T_1}[v] = \emptyset$ ($v \in S_1$), then $d(v_1,v) \leq 1$ (where v_1 is the root of T_1),

(ii) $PN_{T_1}[v] = \emptyset$ for exactly one vertex $v \in S_1$.

This result follows from the following observations:

(i)′ If $PN_{T_1}[v] = \emptyset$ ($v \in S_1$), then $PN_T[v]$ can only be non-empty if $N_T[v] \cap V(T_2) \neq \emptyset$.

(ii)′ If $PN_{T_1}[u] = PN_{T_1}[v] = \emptyset$ for $u,v \in S_1$ ($u \neq v$), then either $N_T[u] - N_{T_1}[u] \subseteq N_T[v]$ or $N_T[v] - N_{T_1}[v] \subseteq N_T[u]$. Thus either $PN_T[u] = \emptyset$ or $PN_T[v] = \emptyset$.

An element (T_1,S_1) of Γ will be called *permanently redundant* if $P(T_1,S_1)$ is false and $P((T_1,S_1) \bullet (T_2,S_2))$ is false for all $(T_2,S_2) \in \Gamma$.

The preceding result and definition also applies to a pair on the right in the composition.

Let $(T,S) \in \Gamma$ and let v_1 be the root of T. We define functions D_1 and D_2 on Γ as follows:

$D_1(T_1,S_1) = \min_{v \in S_1}\{d(v_1,v)\}$, and

$D_2(T_1,S_1) = \min_{v \in S_1}\{\max_{u \in PN_{T_1}[v]}\{d(v_1,u)\}\}$.

The following characteristics of a (T_1,S_1) pair will be used to determine its image $h_1(T_1, S_1)$:

(i) For (T_1, S_1) pairs with $P(T_1, S_1)$ true, the number $D_2(T_1, S_1)$ is recorded. This will determine whether a $v \in S$ is a 2-irredundant vertex of $S \subseteq V(T)$, where (T, S) is the resulting pair after a composition involving (T_1, S_1) from the left. This can be seen from the fact that $PN_T[v] = \emptyset$ if and only if a $u' \in PN_{T_1}[v]$ with $d(v_1, u') = \max_{u \in PN_{T_1}[v]} d(v_1, u)$ is 2-dominated in T. (T_1, S_1) pairs with $D_2(T_1, S_1) \geq 2$ are grouped together since a $u \in PN_{T_1}[v]$ with $d(v_1, u) \geq 2$ will be an element of $PN_T[v]$.

(ii) (T_1, S_1) pairs with $P(T_1, S_1)$ false are further characterized by whether (T_1, S_1) is permanently redundant or not.

(iii) For all (T_1, S_1) pairs except those that are permanently redundant, it is determined whether there are vertices $v \in V(T)$ with $d(v_1, v) \leq 1$ and such that $v \notin N_{T_1}[S_1]$. These are candidates for private neighbors in a composition.

(iv) The number $D_1(T_1, S_1)$ is recorded for each (T_1, S_1) pair. This will assist in determining the image of a (T_1, S_1) pair based on the information in (i) to (iii).

The following *rules* will be used to describe the h_1 equivalence classes:

(R1) $P(T, S)$ is true;

(R2) $P(T, S)$ is false and (T, S) is not permanently redundant;

(R3) (T, S) is permanently redundant;

(R4) $d(v_1, v) = 1$ for at least one $v \in S$;

(R5) $v \in N[S]$ for all $v \in V(T)$ with $d(v_1, v) = 1$.

The following are the descriptions of the h_1 equivalence classes:

(C_0) $D_1(T, S) = 0$ and R2 and R4;

(C_1) $D_1(T, S) = 0$ and R2 and not R4;

(C_2) $D_1(T, S) = 0$ and R3;

(C_3) $D_1(T, S) = 0$ and R1 and $D_2(T, S) = 0$;

(C_4) $D_1(T, S) = 0$ and R1 and $D_2(T, S) = 1$;

(C_5) $D_1(T, S) = 0$ and R1 and $D_2(T, S) \geq 2$;

(C_6) $D_1(T, S) = 1$ and R2;

(C_7) $D_1(T, S) = 1$ and R3;

(C_8) $D_1(T,S) = 1$ and R1 and $D_2(T,S) = 0$;

(C_9) $D_1(T,S) = 1$ and R1 and $D_2(T,S) = 1$;

(C_{10}) $D_1(T,S) = 1$ and R1 and $D_2(T,S) \geq 2$;

(C_{11}) $D_1(T,S) = 2$ and R3;

(C_{12}) $D_1(T,S) = 2$ and R1 and R5 and $D_2(T,S) = 0$;

(C_{13}) $D_1(T,S) = 2$ and R1 and R5 and $D_2(T,S) = 1$;

(C_{14}) $D_1(T,S) = 2$ and R1 and R5 and $D_2(T,S) \geq 2$;

(C_{15}) $D_1(T,S) = 2$ and R1 and not R5 and $D_2(T,S) = 0$;

(C_{16}) $D_1(T,S) = 2$ and R1 and not R5 and $D_2(T,S) = 1$;

(C_{17}) $D_1(T,S) = 2$ and R1 and not R5 and $D_2(T,S) \geq 2$;

(C_{18}) $D_1(T,S) \geq 3$ and R3;

(C_{19}) $D_1(T,S) \geq 3$ and R1 and R5 and $D_2(T,S) = 1$;

(C_{20}) $D_1(T,S) \geq 3$ and R1 and R5 and $D_2(T,S) \geq 2$;

(C_{21}) $D_1(T,S) \geq 3$ and R1 and not R5 and $D_2(T,S) = 1$;

(C_{22}) $D_1(T,S) \geq 3$ and R1 and not R5 and $D_2(T,S) \geq 2$.

The composition table for the above classes is given in Table 1.

The Myhill-Nerode theorem of complexity theory is applied to Table 1 to determine which classes are not n-distinguishable (distinguishable henceforth), where $n = 23$ is the number of classes. distinguishability is determined recursively as follows:

(i) C_i and C_j is 0-distinguishable if one is an accepting class and the other not;

(ii) C_i and C_j are m-distinguishable if they are $(m-1)$-distinguishable, if $C_i \cdot C_k$ and $C_j \cdot C_k$ are $(m-1)$-distinguishable for some k, or if $C_k \cdot C_i$ and $C_k \cdot C_j$ are $(m-1)$-distinguishable for some k.

Table 2 gives the resulting composition table after the non-distinguishable classes in Table 1 were coalesced. Class representatives for the classes in Table 2 are given in Figure 3.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
0	2	2	2	2	2	0	2	2	2	2	0	2	2	2	0	2	2	5	2	2	0	2	5
1	2	2	2	2	2	0	2	2	2	2	1	2	2	2	1	2	2	5	2	2	4	2	5
2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
3	2	2	2	2	2	0	2	2	2	2	1	2	2	2	3	2	2	5	2	2	4	2	5
4	2	2	2	2	2	0	2	2	2	2	4	2	2	2	4	2	2	5	2	2	4	2	5
5	2	2	2	2	2	5	2	2	2	2	5	2	2	2	5	2	2	5	2	2	5	2	5
6	7	7	7	7	7	7	7	7	7	6	6	7	7	6	6	7	6	6	7	9	9	9	9
7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7
8	7	7	7	7	7	7	7	7	7	6	6	7	7	8	8	7	8	8	7	9	9	9	9
9	7	7	7	7	7	7	7	7	7	9	9	7	7	9	9	7	9	9	7	9	9	9	9
10	7	7	7	7	10	10	7	7	7	10	10	7	7	10	10	7	10	10	7	10	10	10	10
11	7	7	7	7	7	7	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11
12	7	7	7	7	7	7	11	11	11	11	11	11	12	12	12	12	12	12	11	15	15	15	15
13	7	7	7	7	7	7	11	11	13	13	13	11	13	13	13	13	13	13	11	16	16	16	16
14	6	6	7	9	10	10	11	11	13	14	14	11	13	14	14	13	14	14	11	17	17	17	17
15	7	7	7	7	7	7	11	11	11	11	11	11	15	15	15	15	15	15	11	15	15	15	15
16	7	7	7	7	7	7	11	11	16	16	16	11	16	16	16	16	16	16	11	16	16	16	16
17	9	9	7	9	10	10	11	11	16	17	17	11	16	17	17	16	17	17	11	17	17	17	17
18	7	7	7	7	7	7	11	11	11	11	11	18	18	18	18	18	18	18	18	18	18	18	18
19	7	7	7	7	7	7	12	11	13	13	13	18	19	19	19	19	19	19	18	21	21	21	21
20	6	8	7	9	10	10	12	11	13	14	14	18	19	20	20	19	20	20	18	22	22	22	22
21	7	7	7	7	7	7	15	11	16	16	16	18	21	21	21	21	21	21	18	21	21	21	21
22	9	9	7	9	10	10	15	11	16	17	17	18	21	22	22	21	22	22	18	22	22	22	22

Number of classes: 23.

Starting classes: C_3 and C_{20}.

Accepting classes: C_3, C_4, C_5, C_8, C_9, C_{10}, C_{12}, C_{13}, C_{14}, C_{15}, C_{16}, C_{17}, C_{19}, C_{20}, C_{21} and C_{22}.

Table 1 ($h_1(\Gamma)$ composition table before reduction)

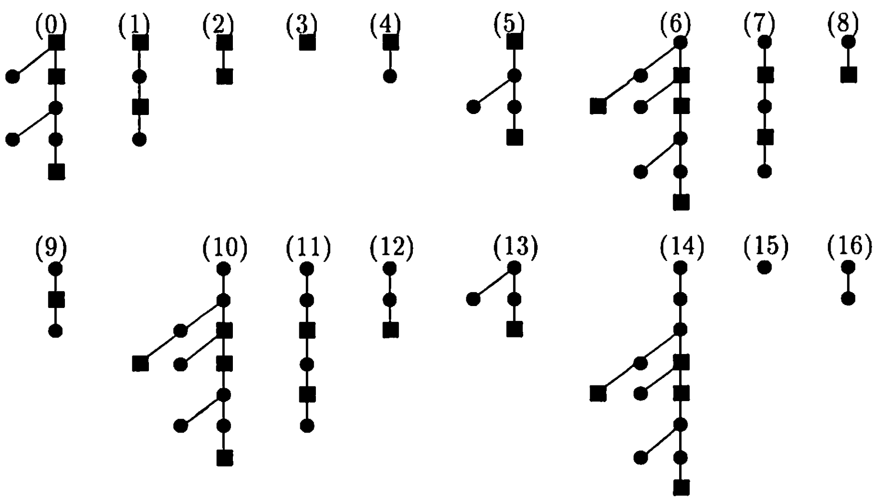
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0	2	2	2	2	2	0	2	2	2	0	2	2	0	5	2	0	5
1	2	2	2	2	2	0	2	2	2	1	2	2	1	5	2	4	5
2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
3	2	2	2	2	2	0	2	2	2	1	2	2	3	5	2	4	5
4	2	2	2	2	2	0	2	2	2	4	2	2	4	5	2	4	5
5	2	2	2	2	2	5	2	2	2	5	2	2	5	5	2	5	5
6	2	2	2	2	2	2	2	2	6	6	2	6	6	6	8	8	8
7	2	2	2	2	2	2	2	2	6	6	2	7	7	7	8	8	8
8	2	2	2	2	2	2	2	2	8	8	2	8	8	8	8	8	8
9	2	2	2	2	9	9	2	2	9	9	2	9	9	9	9	9	9
10	2	2	2	2	2	2	2	2	2	2	10	10	10	10	10	10	10
11	2	2	2	2	2	2	2	11	11	11	11	11	11	11	11	11	11
12	6	6	2	8	9	9	2	11	12	12	11	12	12	12	13	13	13
13	8	8	2	8	9	9	2	11	13	13	11	13	13	13	13	13	13
14	2	2	2	2	2	2	10	11	11	11	14	14	14	14	14	14	14
15	6	7	2	8	9	9	10	11	12	12	14	15	15	15	16	16	16
16	8	8	2	8	9	9	10	11	13	13	14	16	16	16	16	16	16

Number of classes: 17.

Starting classes: C_3 and C_{15}.

Accepting classes: C_3, C_4, C_5, C_7, C_8, C_9, C_{10}, C_{11}, C_{12}, C_{13}, C_{14}, C_{15} and C_{16}.

Table 2 ($h_1(\Gamma)$ composition table after reduction)



(The subset S of $V(T)$ for a (T, S) pair is indicated by square vertices.)

Figure 3 (Class representatives for the equivalence classes in Table 2)

We now define a predicate $maxP$ on Γ such that $maxP(T,S)$ is true if and only if S is a maximal 2-irredundant subset of vertices of T. For $maxP$ we define a mapping h_2 on Γ that satisfies (H1) and (H2) as follows:

> $h_2(T,S)$ is the pair $[C_i, A]$ with $C_i = h_1(T,S)$ and $A = \{C_i | C_i = h_1(T, S \cup \{v\}),\ v \in V(T) - S\}$.

Then h_2 has a finite range consisting of at most $n \cdot 2^n$ equivalence classes, where n is the number of equivalence classes in $h_1(\Gamma)$.

The composition table for $h_2(\Gamma)$ is calculated from the $h_1(\Gamma)$ composition table as follows:

> For $[C_1, A_1]$ and $[C_2, A_2]$ in $h_2(\Gamma)$, $[C_1, A_1] \cdot [C_2, A_2] = [C_1 \cdot C_2, \{C_i \cdot C_2 | C_i \in A_1\} \cup \{C_1 \cdot C_j | C_j \in A_2\}]$.

The starting classes are $[C_{s1}, \emptyset]$ and $[C_{s2}, \{C_{s1}\}]$, where C_{s1} and C_{s2} are the h_1 starting classes. The accepting classes are those $[C_i, A]$ pairs such that C_i is an accepting class and for all $C_j \in A$, C_j is not an accepting class.

The computer program devised to calculate the $h_2(\Gamma)$ composition table from Table 2, executes the following steps:

(1) Indices relating to all $[C_i, A]$ pairs that can be obtained through a sequence of left and right compositions from the starting classes are put in an index set I. (There were 2406 non-empty classes.)

(2) An upper triangular matrix $U = [u_{ij}]$ is used to indicate whether two given classes are distinguishable. For $i, j \in I$, $i < j$, $u_{ij} = 1$ if the classes corresponding to the indices i and j are distinguishable ($u_{ij} = 0$ otherwise). U is initialized by distinguishing between two classes if the one is an accepting class and the other not.

(3) Calculation of U is completed by repeatedly determining all 1-distinguishable classes, using the updated U as input each time. All distinguishable classes are determined as soon as there are no more 1-distinguishable classes.

(4) All the classes that are not distinguishable are coalesced to give a composition table for a minimum number of equivalence classes.

The $h_2(\Gamma)$ composition table (referred to as Table 3) consists of 117 distinquishable classes of which 27 are accepting classes. The table is not presented here, because of its printed size.

The algorithm to calculate the 2-irredundance number of a tree takes a parsed tree and Table 3 as input. Let n be the number of classes, and s_1 and s_2 the starting classes in Table 3. During the postorder traversal of the parsed tree a vector $B = (b_0, b_1, \ldots, b_{n-1})$ is calculated at each vertex as the tree is constructed (using the composition operation for trees).

At each leaf vertex, B is initialized as follows: $b_{s1} = 1$, $b_{s2} = 0$ and $b_i = -1$ for $i \neq s_1, s_2$. Each b_i $(i = 0, 1, \ldots, n-1)$ is the minimum cardinality of a subset $S \subseteq V(T)$, where T is a subtree of the input tree with $h_2(T, S) = C_i$. (Here C_i is an equivalence class in the input table.) A value $b_i = -1$ indicates that no subtree-subset pair (T, S) with $h_2(T, S) = C_i$ was constructed up to that stage (using the composition operation for Γ). The vector B at a vertex v of the parsed tree is calculated as follows:

> Let B_1 and B_2 be the vectors calculated for the left and right child of v respectively. B is initialized as $b_i = -1$, $i = 0, 1, \ldots, n-1$. For $i, j = 0, 1, \ldots, n-1$ such that $b_i^{(1)} \neq -1$ and $b_j^{(2)} \neq -1$, let k be the entry in row i and column j of the input table. Then $b_k = b_i^{(1)} + b_j^{(2)}$, if $b_k = -1$ or if $b_k > b_i^{(1)} + b_j^{(2)}$.

The 2-irredundance number is calculated from the vector B at the root of the parsed tree as the least b_i $(i = 0, 1, \ldots, n-1)$ for which $b_i \neq -1$ and C_i is an accepting class.

The upper 2-irredundance number of a tree is calculated similarly from Table 2 (or Table 3) by maximizing throughout in the above algorithm.

These two algorithms are linear since parsing for trees is linear in the size of a tree and the procedure for calculating the vectors B is $O(1)$ since the size of the 2-irredundance tables are constant with respect to the size of the input tree.

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