Isomorphism classes of bipartite cycle permutation graphs

Jin Ho Kwak*
Department of Mathematics
Pohang University of Science and Technology
Pohang 790-784, Korea
email: jinkwak@postech.ac.kr

Jaeun Lee†
Department of Mathematics
Yeungnam University
Kyongsan 712-749, Korea

ABSTRACT. In this paper, we count the number of isomorphism classes of bipartite n-cyclic permutation graphs up to positive natural isomorphism and show that it is equal to the number of double cosets of the dihedral group D_n in the subgroup B_n of the symmetric group S_n consisting of parity-preserving or parity-reversing permutations.

1 Introduction

Let C_n denote an n-cycle with consecutively labeled vertices $1, 2, \cdots, n$. For a permutation α in the symmetric group S_n of n elements, an *α-cycle permutation graph* $P_\alpha(C_n)$ consists of two copies of C_n, say C_x and C_y, with vertex sets $V(C_x) = \{x_1, x_2, \cdots, x_n\}$ and $V(C_y) = \{y_1, y_2, \cdots, y_n\}$, along with edges $x_i y_{\alpha(i)}$ for $1 \le i \le n$. When we wish to specify n, we will call $P_\alpha(C_n)$ *n-cyclic*: with neither α nor n mentioned, it is simply a *cycle permutation graph*. The copies of C_n labeled $x_1, x_2, \cdots, x_n$ will be called the *outer cycle*, the copies of C_n labeled $y_1, y_2, \cdots, y_n$ will be called the *inner cycle*, and the edges of the form $x_i y_{\alpha(i)}$ will be called *permutation edges.* Given two permutations α and β in S_n, $P_\alpha(C_n)$ is isomorphic to $P_\beta(C_n)$

*Supported by BSRI-96-1430
†Supported by TGRC-KOSEF

by a *positive natural isomorphism* Θ if $\Theta(C_x) = C_x$ and $\Theta(C_y) = C_y$. The graph $P_\alpha(C_n)$ is isomorphic to $P_\beta(C_n)$ by a *negative natural isomorphism* Θ if $\Theta(C_x) = C_y$ and $\Theta(C_y) = C_x$. A *natural isomorphism* is either of these. Ringeisen [10] counted the number of distinct cycle permutation graphs isomorphic to a k-twisted prism by a natural isomorphism. Also, Stueckle [11] found the number of permutations which yield cycle permutation graphs isomorphic to a given cycle permutation graph by a natural isomorphism. The authors [9] constructed a cycle permutation graph as a covering graph of the dumbbell graph and by using it, counted the isomorphism classes of n-cyclic permutation graphs up to positive natural isomorphism. It was also shown that the number of isomorphism classes of n-cyclic permutation graphs up to positive natural isomorphism is equal to the number of double cosets of the symmetric group S_n by the dihedral group D_n. The authors and some others recently counted the isomorphism classes of several kinds of graph coverings, see [6]–[8].

In this paper, we count the isomorphism classes of *bipartite* n-cyclic permutation graphs up to positive natural isomorphism, and show that it is equal to the number of double cosets the dihedral group D_n in the subgroup B_n of S_n consisting of parity-preserving or parity-reversing permutations.

2 An algebraic characterization

Let Σ_n denote the conjugacy class of $\rho = (1\,2 \cdots n)$ in the symmetric group S_n, *i.e.*, Σ_n is the set of all n-cycles in S_n. For each $\sigma \in \Sigma_n$, we construct a graph G_σ as follows. The vertex set of the graph G_σ is $\{x_1, \cdots, x_n, y_1, \cdots, y_n\}$, and two vertices u and v are joined in G_σ if they satisfy one of the following three conditions:

(1) $u = x_i$ and $v = x_{\rho(i)}$ for $1 \le i \le n$,

(2) $u = x_i$ and $v = y_i$ for $1 \le i \le n$,

(3) $u = y_i$ and $v = y_{\sigma(i)}$ for $1 \le i \le n$.

Then, for each $\alpha \in S_n$, the cycle permutation graph $P_\alpha(C_n)$ is isomorphic to $G_{\alpha^{-1}\rho\alpha}$ by a positive natural isomorphism (see [9]), and $\alpha^{-1}\rho\alpha \in \Sigma_n$. Hence, the set Σ_n can be identified with the set of all n-cyclic permutation graphs.

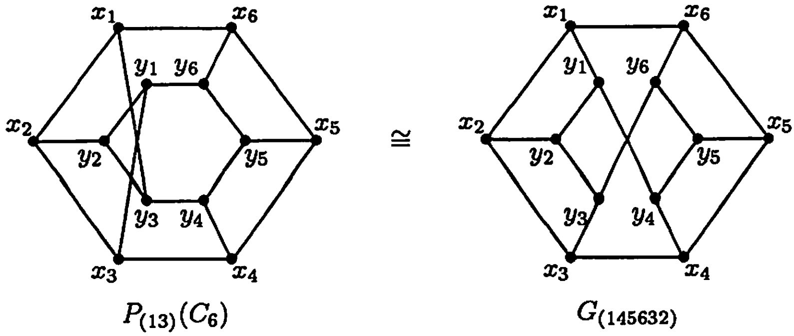


Figure 1. Isomorphic graphs $P_{(13)}(C_6)$ and $G_{(145632)}$

Let $\mathcal{I} : S_n \longrightarrow S_n$ be the map defined by $\mathcal{I}(\sigma) = \sigma^{-1}$ for all $\sigma \in S_n$. Let D_n denote the dihedral group generated by two permutations ρ and τ, where $\tau(i) = n+1-i$ and $\rho(i) = i+1$; that is, the group of automorphisms of the n-cycle C_n. Note that all arithmetic is done modulo n, and the dihedral group D_n is the normalizer of $\{\rho, \rho^{-1}\}$ in S_n. We write $\Gamma = D_n \times \{1, \mathcal{I}\}$ and define an action $\Gamma \times \Sigma_n \longrightarrow \Sigma_n$ by $(d,1)(\sigma) = d\sigma d^{-1}$ and $(d,\mathcal{I})(\sigma) = d\sigma^{-1}d^{-1}$. The following theorem can be found in [9].

Theorem 1 *Let α and β be two permutations in S_n.*

(1) *$P_\alpha(C_n)$ is isomorphic to $P_\beta(C_n)$ by a positivel isomorphism if and only if there exists $\gamma \in \Gamma$ such that $\beta^{-1}\rho\beta = \gamma(\alpha^{-1}\rho\alpha)$.*

(2) *$P_\alpha(C_n)$ is isomorphic to $P_\beta(C_n)$ by a negativel isomorphism if and only if there exists $\gamma \in \Gamma$ such that $\beta^{-1}\rho\beta = \gamma(\alpha\rho\alpha^{-1})$.*

(3) *$P_\alpha(C_n)$ is isomorphic to $P_\beta(C_n)$ by a natural isomorphism if and only if there exists $\gamma \in \Gamma$ such that $\beta^{-1}\rho\beta = \gamma(\alpha^{-1}\rho\alpha)$ or $\beta^{-1}\rho\beta = \gamma(\alpha\rho\alpha^{-1})$.* □

Now, we give a characterization of a bipartite cycle permutation graph. Since no bipartite graph has an odd cycle, a cycle permutation graph $P_\alpha(C_n)$ can be bipartite only for even n. Hence, we consider only even n from now on. For any even n, let B_n denote the set of all α in S_n such that α is either parity-preserving or parity-reversing, *i.e.*, α maps either all odd numbers to odd or all odd to even in $\{1, 2, \cdots, n\}$, and Ξ_n the set of parity-reversing cycles of length n in S_n. Then B_n is a subgroup of S_n. For example, it is the cyclic group $\mathbb{Z}_2$ for $n = 2$, and the dihedral group D_4 for $n = 4$.

Theorem 2 *The following statements are equivalent for $\alpha \in S_n$.*

(1) *The cycle permutation graph $P_\alpha(C_n)$ is bipartite.*

(2) $\alpha^{-1}\rho\alpha \in \Xi_n$.

(3) $\alpha \in B_n$.

Proof: (1)$\Rightarrow$(2). Suppose that $P_\alpha(C_n)$ is bipartite. Then its isomorphic copy $G_{\alpha^{-1}\rho\alpha}$ is bipartite, and any two vertices y_i and y_j are joined if and only if $j = (\alpha^{-1}\rho\alpha)(i)$. A bipartition of $G_{\alpha^{-1}\rho\alpha}$ gives a 2-colouring of $G_{\alpha^{-1}\rho\alpha}$. But ρ reverses parity in $\{1\,2\,\cdots\,n\}$, and the permutation edges are of the form x_iy_i in $G_{\alpha^{-1}\rho\alpha}$. Hence $\alpha^{-1}\rho\alpha$ must reverse parity. That is, $\alpha^{-1}\rho\alpha \in \Xi_n$.

(2)$\Rightarrow$(3). Note that $\alpha^{-1}\rho\alpha = (\alpha^{-1}(1)\,\alpha^{-1}(2)\,\cdots\,\alpha^{-1}(n))$. Suppose that $\alpha^{-1}\rho\alpha \in \Xi_n$. Then α^{-1} preserves parity in $\{1\,2\,\cdots\,n\}$ if $\alpha^{-1}(1)$ is odd, and α^{-1} reverses parity in $\{1\,2\,\cdots\,n\}$ if $\alpha^{-1}(1)$ is even. This implies that $\alpha^{-1} \in B_n$. Since B_n is a subgroup of S_n, $\alpha \in B_n$.

(3)$\Rightarrow$(1). It is easy to construct a 2-colouring of $P_\alpha(C_n)$ for $\alpha \in B_n$. Thus, $P_\alpha(C_n)$ is bipartite. □

By Theorem 2, the set Ξ_n can be identified as the set of all bipartite n-cyclic permutation graphs, which is crucial for the counting of their isomorphism classes. It is known that two cycle permutation graphs $P_\alpha(C_n)$ and $P_\beta(C_n)$ are isomorphic by a positive natural isomorphism if and only if $\beta \in D_n\alpha D_n$, as can be found in [9] and [11]. The following comes from this fact and Theorem 2.

Theorem 3 *The number $Iso_P(BC_n)$ of isomorphism classes of bipartite n-cyclic permutation graphs up to positive natural isomorphism is the number of double cosets of the dihedral group D_n in the permutation group B_n.* □

For convenience, let $|X|$ denote the cardinality of a set X. Note that Ξ_n is an invariant subset of Σ_n under the Γ-action. From Theorem 1 and Burnside's Lemma for the Γ-action on Ξ_n, we can derive

Theorem 4 *The number $Iso_P(BC_n)$ of isomorphism classes of bipartite n-cyclic permutation graphs up to positive natural isomorphism is*

$$\frac{1}{4n}\sum_{\gamma\in\Gamma}|Fix_\gamma|,$$

where $Fix_\gamma = \{\sigma \in \Xi_n : \gamma(\sigma) = \sigma\}$ for any γ in Γ. □

Now, we introduce a lemma which can be found in [9].

Lemma 1 *Let σ and ς be any two n-cycles in Σ_n. Then*

(1) $|\{w \in S_n : w\sigma w^{-1} = \varsigma\}| = n$, *and*
$$\{w \in S_n : w\sigma w^{-1} = \sigma\} = \{\sigma^i : i = 1, 2, \ldots, n\}.$$
(2) *If $w\sigma w^{-1} = \sigma^{-1}$ for some $w \in S_n$, then w^2 is the identity in S_n.* □

Lemma 1 shows that for any $\alpha \in B_n$, there are exactly n permutations ω in S_n such that $\alpha^{-1}\rho\alpha = \omega^{-1}\rho\omega$ in Ξ_n. In fact, such n permutations ω must be in B_n by Theorem 2. It is not difficult to show that

$$|\Xi_n| = \left(\frac{n}{2}\right)! \left(\frac{n}{2} - 1\right)!.$$

Hence, we get

Corollary 1 *The number of bipartite n-cyclic permutation graphs is*

$$|B_n| = n \left(\frac{n}{2}\right)! \left(\frac{n}{2} - 1\right)! = 2 \left(\frac{n}{2}!\right)^2$$

□

3 Counting formulas

In this section, we aim to compute the number $\text{Iso}_P(BC_n)$ of isomorphism classes of bipartite n-cyclic permutation graphs up to positive natural isomorphism. Clearly, $\text{Iso}_P(BC_2) = 1$. For the Γ-action on Ξ_n and any $\gamma \in \Gamma$, let Fix_γ denote the set of fixed points of γ, *i.e.*,

$$\text{Fix}_\gamma = \{\sigma \in \Xi_n : \gamma(\sigma) = \sigma\}.$$

To compute $\text{Iso}_P(BC_n)$ for even $n \geq 4$, we first evaluate $|\text{Fix}_\gamma|$ for the Γ-action on $\Xi_n, n \geq 4$. Let $\mathbb{Z}_n = \{0, 1, \cdots, n-1\}$. For $k \in \mathbb{Z}_n$, let $o(k)$ denote the order of k in the cyclic group $\mathbb{Z}_n$, $\imath(k)$ the index of the subgroup generated by k, and $\phi(k)$ the Euler phi-function, giving the number of integers relatively prime to k between 1 and k.

For each $\alpha \in S_n$, let $j(\alpha) = (j_1, j_2, \cdots, j_n)$ be the cycle type of α, that is, a cycle representation of α has j_k cycles of length k for all $k = 1, 2, \cdots, n$. Now we evaluate $|\text{Fix}_\gamma|$ for even $n \geq 4$.

Lemma 2 *Let n be an even number greater than 4. Then for each $k = 1, 2, \ldots, n$, we have*

$$(1)\ |Fix_{(\rho^k,1)}| = \begin{cases} \phi(o(k))^{\frac{\imath(k)}{2}} \left((\frac{\imath(k)}{2} - 1)! \right)^2 o(k)^{\imath(k)-1} & \text{if } k \text{ is even,} \\ \phi(o(k))\, (\imath(k) - 1)! \left(\frac{o(k)}{2}\right)^{\imath(k)-1} & \text{if } k \text{ is odd.} \end{cases}$$

(2) $|Fix_{(\rho^k\tau,1)}| = \begin{cases} (\frac{n}{2}-1)\,! & \textit{if } k \textit{ is even and } n \neq 0 \pmod 4, \\ 0 & \textit{otherwise.} \end{cases}$

(3) $|Fix_{(\rho^k,\mathcal{I})}| = \begin{cases} (\frac{n}{2})\,! & \textit{if } k = \frac{n}{2} \textit{ and } n \neq 0 \pmod 4, \\ 0 & \textit{otherwise.} \end{cases}$

(4) $|Fix_{(\rho^k\tau,\mathcal{I})}| = \begin{cases} (\frac{n}{2})\,! & \textit{if } k \textit{ is even,} \\ ((\frac{n}{2}-1)(\frac{n}{2}-3)\cdots 2)^2 & \textit{if } k \textit{ is odd, } n \neq 0 \pmod 4, \\ \frac{n}{2}\,((\frac{n}{2}-2)(\frac{n}{2}-4)\cdots 2)^2 & \textit{if } k \textit{ is odd, } n = 0 \pmod 4. \end{cases}$

Proof: (1). For $\rho = (1\,2\,\cdots\,n)$, the cycle type of ρ^k is

$$j(\rho^k) = (0, \cdots, 0, j_{o(k)} = \imath(k), 0, \cdots, 0),$$

and ρ^k is parity-preserving if k is even, and parity-reversing if k is odd. For any k, let $T(k)$ denote the set of all permutations α in B_n which have the same cycle type and the same parity as those of ρ^k. Then any permutation α in $T(k)$ must be a product of $\imath(k)$ cycles of length $o(k)$. If k is even, exactly half of these $\imath(k)$ cycles consist of only odd numbers and the other half consist of only even numbers. If k is odd, each of the $\imath(k)$ cycles must be parity-reversing. Now it is not difficult to show that

$$|T(k)| = \begin{cases} \dfrac{(\frac{n}{2})!(\frac{n}{2}-1)!}{\frac{\imath(k)}{2}\left((\frac{\imath(k)}{2}-1)!\right)^2 o(k)^{\imath(k)-1}} & \text{if } k \text{ is even,} \\ \dfrac{(\frac{n}{2})!(\frac{n}{2}-1)!}{(\imath(k)-1)!\left(\frac{o(k)}{2}\right)^{\imath(k)-1}} & \text{if } k \text{ is odd.} \end{cases}$$

For any k and $\alpha \in T(k)$, we write $\mathrm{fix}_\alpha = \{\sigma \in \Xi_n : \alpha\sigma\alpha^{-1} = \sigma\}$. Then $\rho^k \in T(k)$ and $\mathrm{Fix}_{(\rho^k,1)} = \mathrm{fix}_{\rho^k}$. Since any two elements α and α' in $T(k)$ are conjugate in B_n, we get $|\mathrm{fix}_\alpha| = |\mathrm{fix}_{\alpha'}|$. In particular, $|\mathrm{Fix}_{(\rho^k,1)}| = |\mathrm{fix}_{\rho^k}| = |\mathrm{fix}_\alpha|$ for any $\alpha \in T(k)$. For $\sigma \in \Xi_n$, we write $C_\sigma(k) = \{\alpha \in T(k) : \alpha\sigma\alpha^{-1} = \sigma\}$. For any n-cycle σ in Ξ_n, there are exactly n elements α satisfying $\alpha\sigma\alpha^{-1} = \sigma$ by Lemma 1. But exactly $\phi(o(k))$ elements among them are contained in $T(k)$. Hence, for any $\sigma \in \Xi_n$, we get $|C_\sigma(k)| = \phi(o(k))$. Now, we consider the set of pairs (σ, α) in $\Xi_n \times T(k)$ satisfying the relation $\alpha\sigma\alpha^{-1} = \sigma$. Note that this set can be written in two ways as follows:

$$\bigcup_{\sigma\in\Xi_n} \{(\sigma,\alpha) : \alpha \in C_\sigma(k)\} = \bigcup_{\alpha\in T(k)} \{(\sigma,\alpha) : \sigma \in \mathrm{fix}_\alpha\},$$

where both unions are clearly disjoint. Therefore, we get

$$|\Xi_n|\,\phi(o(k)) = |T(k)|\,|\mathrm{Fix}_{(\rho^k,1)}|$$

and we know $|\Xi_n| = (\frac{n}{2})!\,(\frac{n}{2}-1)!$, which gives the proof of (1).

(2). First, let k be odd, and suppose that $\mathrm{Fix}_{(\rho^k\tau,1)} \neq \emptyset$. Take an element σ in $\mathrm{Fix}_{(\rho^k\tau,1)}$. Then $(\rho^k\tau)\sigma(\rho^k\tau)^{-1} = \sigma$ and $\rho^k\tau = \sigma^\ell$ for some $1 \leq \ell \leq n$. Since $\rho^k\tau$ is of order 2 in S_n and σ is an n-cycle, ℓ must be even. But if k is odd, then $\rho^k\tau$ has two fixed points $\frac{k+1}{2}$ and $\frac{n+k+1}{2}$, while $\sigma^{\frac{n}{2}}$ has no fixed points. This is a contradiction.

Next, let k be even and $n = 0 \pmod 4$. Suppose that $\mathrm{Fix}_{(\rho^k\tau,1)} \neq \emptyset$ and let $\sigma \in \mathrm{Fix}_{(\rho^k\tau,1)}$. Then $\rho^k\tau = \sigma^{\frac{n}{2}}$ as above, and $\rho^k\tau$ is parity-reversing. But $\sigma^{\frac{n}{2}}$ is parity-preserving, because σ is parity-reversing. This is a contradiction.

Finally, let k be even and $n \neq 0 \pmod 4$. With the same notation as (1), we can see that $\rho^k\tau \in T(\frac{n}{2})$ and $|\mathrm{Fix}_{(\rho^k\tau,1)}| = |\mathrm{fix}_{\rho^{\frac{n}{2}}}|$. But $|T(\frac{n}{2})| = (\frac{n}{2})!$ and $|C_\sigma(\frac{n}{2})| = \phi(o(k)) = 1$ for any $\sigma \in \Xi_n$. A similar computation to (1) gives $|\mathrm{Fix}_{(\rho^k\tau,1)}| = (\frac{n}{2}-1)!$.

(3). Let $\sigma = (a_1\, a_2\, \cdots\, a_n) \in \Xi_n$ be an element of $\mathrm{Fix}_{(\rho^k,\mathcal{I})}$, that is, $\sigma = \rho^k\sigma^{-1}\rho^{-k}$. Then $(a_n\, a_{n-1}\, \cdots\, a_1) = \sigma^{-1} = \rho^k\sigma\rho^{-k}$. By Lemma 1, ρ^{2k} is the identity in S_n. Hence, $2k$ must be equal to n and

$$(a_n\, a_{n-1}\, \cdots\, a_1) = \rho^{\frac{n}{2}}\sigma\rho^{-\frac{n}{2}} = \left(a_1{+}\frac{n}{2}\;\, a_2{+}\frac{n}{2}\;\, \cdots\;\, a_n{+}\frac{n}{2}\right).$$

Now, we consider the following two cases. Case i). Let $n = 0 \pmod 4$. Without loss of generality, we can assume that a_1 is odd in $\sigma = (a_1 a_2 \cdots a_n)$. Then a_k is odd if k is odd and a_k is even if k is even, because σ is parity-reversing. Now, let $a_1{+}\frac{n}{2} = a_\ell$ for some $1 \leq \ell \leq n$. Then, $a_1 + \frac{n}{2} = a_\ell$ must be odd, because $\frac{n}{2}$ is even. Hence ℓ must be odd and $a_{\frac{\ell+1}{2}} + \frac{n}{2} = a_{\ell - \frac{\ell+1}{2}+1} = a_{\frac{\ell+1}{2}}$, which is impossible.

Case ii). Let $n \neq 0 \pmod 4$. With the same notation, we can see that $\rho^{\frac{n}{2}} \in T(\frac{n}{2})$. For convenience, we write $I_\alpha = \{\sigma \in \Xi_n : \alpha\sigma\alpha^{-1} = \sigma^{-1}\}$ and $D_\sigma = \{\alpha \in T(\frac{n}{2}) : \alpha\sigma\alpha^{-1} = \sigma^{-1}$ for all $\alpha \in T(\frac{n}{2})$ and $\sigma \in \Xi_n$. It is clear that $I_{\rho^{\frac{n}{2}}} = \mathrm{Fix}_{(\rho^{\frac{n}{2}},I)}$. Since any two members of $T(\frac{n}{2})$ are conjugate in B_n, $|I_{\rho^{\frac{n}{2}}}| = |I_\alpha|$ for each $\alpha \in T(\frac{n}{2})$. It is not difficult to show that $|D_\sigma| = \frac{n}{2}$ for any $\sigma \in \Xi_n$. By a method similar to the proof (1), we have

$$\bigcup_{\sigma \in \Xi_n} \{(\sigma,\alpha) : \alpha \in D_\sigma\} = \bigcup_{\alpha \in T(\frac{n}{2})} \{(\sigma,\alpha) : \sigma \in I_\alpha\}$$

where the both unions are disjoint unions. Therefore, we get

$$|\Xi_n|\,\frac{n}{2} = \left|T(\frac{n}{2})\right|\,\left|\mathrm{Fix}_{(\rho^{\frac{n}{2}},I)}\right|.$$

Recall that $|\Xi_n| = (\frac{n}{2})!\,(\frac{n}{2}-1)!$ and $|T(\frac{n}{2})| = (\frac{n}{2})!$. This gives (3).

(4). Let $\sigma = (a_1\, a_2\, \cdots\, a_n) \in \Xi_n$. First, let k be even, then $\rho^k\tau \in T(\frac{n}{2})$. Using the same method as the proof of the second case of (3), we have $|\mathrm{Fix}_{(\rho^k\tau,\mathcal{I})}| = (\frac{n}{2})!$ for even k.

Next, let k be odd and $n = 0 \pmod 4$. Then $|\mathrm{Fix}_{(\rho^k\tau,\mathcal{I})}| = |\mathrm{Fix}_{(\rho\tau,\mathcal{I})}|$. Note that $\rho\tau$ fixes two points 1 and $\frac{n}{2}+1$, and $\frac{n}{2}+1$ is odd. We can assume that $a_1 = 1$. Let $\sigma \in \mathrm{Fix}_{(\rho\tau,\mathcal{I})}$. Then $(1\, \rho\tau(a_2)\, \cdots\, \rho\tau(a_n)) = (1\, a_n\, a_{n-1}\, \cdots\, a_2)$ and $a_{\frac{n}{2}+1} = \rho\tau(a_{\frac{n}{2}+1}) = \frac{n}{2}+1$. Then there are $\frac{n}{2}$ candidates for a_2 because a_2 is even. If a_2 is given, then $\rho\tau(a_2) = a_n$ is fixed and is even. There are $\frac{n}{2}-2$ candidates for a_3 because a_3 is odd, and if a_3 is given then $\rho\tau(a_3) = a_{n-1}$ is fixed and is odd. There are $\frac{n}{2}-2$ candidates for a_4, and so on. Hence,

$$\left|\mathrm{Fix}_{(\rho^k\tau,\mathcal{I})}\right| = \frac{n}{2}\left((\frac{n}{2}-2)(\frac{n}{2}-4)\cdots 2\right)^2 \text{ if } k \text{ is odd and } n = 0 \pmod 4.$$

Finally, let k be odd and $n \neq 0 \pmod 4$. Note that $1 + \frac{n}{2}$ is even. A similar method gives

$$\left|\mathrm{Fix}_{(\rho^k\tau,\mathcal{I})}\right| = \left((\frac{n}{2}-1)(\frac{n}{2}-3)\cdots 2\right)^2 \text{ if } k \text{ is odd and } n \neq 0 \pmod 4.$$

□

By Theorem 4 and Lemma 2, we have

Theorem 5 *Let $n \geq 4$ be even.*

(1) *If $n \neq 0 \pmod 4$,*

$$\begin{aligned} 4n\, Iso_P(BC_n) = & \sum_{1\leq k=odd\leq n} \phi(o(k))\,(\imath(k)-1)!\left(\frac{o(k)}{2}\right)^{\imath(k)-1} \\ & + \sum_{1\leq k=even\leq n} \phi(o(k))\,\frac{\imath(k)}{2}\left((\frac{\imath(k)}{2}-1)!\right)^2 o(k)^{\imath(k)-1} \\ & +\left(\frac{n}{2}\right)!\left(\frac{n+4}{2}\right) + \frac{n}{2}\left((\frac{n}{2}-1)(\frac{n}{2}-3)\cdots 2\right)^2. \end{aligned}$$

(2) *If $n = 0 \pmod 4$,*

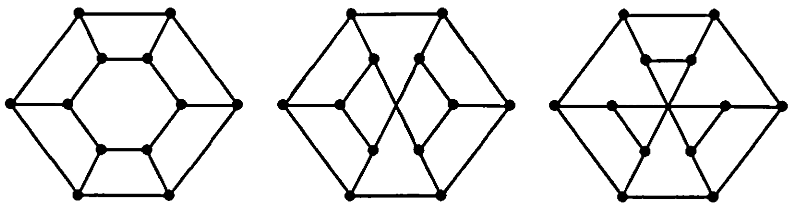
$$\begin{aligned} 4n\, Iso_P(BC_n) = & \sum_{1\leq k=odd\leq n} \phi(o(k))\,(\imath(k)-1)!\left(\frac{o(k)}{2}\right)^{\imath(k)-1} \\ & + \sum_{1\leq k=even\leq n} \phi(o(k))\,\frac{\imath(k)}{2}\left((\frac{\imath(k)}{2}-1)!\right)^2 o(k)^{\imath(k)-1} \\ & +\left(\frac{n}{2}\right)!\,\frac{n}{2} + \left(\frac{n}{2}(\frac{n}{2}-2)(\frac{n}{2}-4)\cdots 2\right)^2. \end{aligned}$$

□

We obtain the following table for $\text{Iso}_P(BC_n)$:

n	2	4	6	8	10	12	...
$\text{Iso}_P(BC_n)$	1	1	3	11	104	1952	...

For convenience, we denote by id the identity element in S_n. Let $n = 6$. Then $\text{Iso}_P(BC_6) = 3$ and the non-isomorphic bipartite 6-cyclic permutation graphs are given in Figure 2 with their representative permutations α.



$P_{id}(C_6)$ $P_{(13)}(C_6)$ $P_{(24)(35)}(C_6)$

Figure 2. Three non-isomorphic bipartite 6-cyclic permutation graphs

In [9], it was shown that a representative $P_\alpha(C_n)$ of a positive natural isomorphism class also represents a natural isomorphism class if and only if $\alpha D_n \alpha \cap D_n \neq \emptyset$. In Figure 2, all α are of order 2 and hence $\alpha D_n \alpha \cap D_n \neq \emptyset$. Thus the three graphs in Figure 2 are all representatives of natural isomorphism classes of bipartite 6-cyclic permutation graphs.

References

[1] G. Chartrand and F. Harary, Planar permutation graphs, *Ann. Inst. Henri Poincaré III* 4 (1967), 433–438.

[2] W. Dörfler, Automorphisms and isomorphisms of permutation graphs, *Colloques Inter.*, C.N.R.S., No. 260, *Problèmes Combinatoires et Théorie des Graphes* (1978),109–110.

[3] W. Dörfler, On mapping graphs and permutation graphs, *Math. Slovaca* 28(3) (1978), 277–288.

[4] J.L. Gross and T.W. Tucker, *Topological Graph Theory* Wiley, New York, 1987.

[5] S. Hedetniemi, On classes of graphs defined by special cutsets of lines, in: *The Many Facets of Graph Theory*, LNM 110, Springer, Berlin, (1969), 171–190.

[6] S. Hong, J.H. Kwak and J. Lee, Regular graph coverings whose covering transformation groups have the isomorphism extension property, *Discrete Math.* **148** (1996), 85–105.

[7] J.H. Kwak, J. Chun and J. Lee, Enumeration of regular graph coverings having finite abelian covering transformation groups, *SIAM J. of Discrete Math.* **11** (1998), 273–285.

[8] J.H. Kwak and J. Lee, Isomorphism classes of graph bundles, *Canad. J. Math.* **XLII** (1990), 747–761.

[9] J.H. Kwak and J. Lee, Isomorphism classes of cycle permutation graphs, *Discrete Math.* **105** (1992), 131–142.

[10] R. Ringeisen, On cycle permutation graphs, *Discrete Math.* **51** (1984), 265–275.

[11] S. Stueckle, On natural isomorphisms of cycle permutation graphs, *Graphs and Combinatorics* 4 (1988), 75–85.