Generalized de Bruijn multigraphs of rank $\vec{k}$

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ABSTRACT. New class $\mathcal{GBG}_{\vec{k}}$ of generalized de Bruijn multigraphs of rank $\vec{k} \in N^m$ is introduced and briefly characterized. It is shown, among the others, that every multigraph of $\mathcal{GBG}_{\vec{k}}$ is connected, Eulerian and Hamiltonian. Moreover, it consists of the subgraphs which are isomorphic with the de Bruijn graphs of rank $r = \sum_{i=1}^{m}(d_i \cdot k_i)$, for arbitrary nonzero vector $\vec{d} = (d_1, \dots, d_m) \in \{0.1\}^m$. Then, the subgraphs of every multigraph of $\mathcal{GBG}_{\vec{k}}$, called the $\vec{k}$-factors, are distinguished.

An algorithm, with small time and space complexities, for the construction of the $\vec{k}$-factors, in particular the Hamiltonian circuits, is given. At the very end a few open problems are put forward.

1 Introduction

A class $\mathcal{GBG}_{\vec{k}}$ of the generalized de Bruijn multigraphs of rank $\vec{k}$, briefly $\vec{k}$-multigraphs, covers over the class $\mathcal{BG}_k$ of de Bruijn graphs of rank k [4]. It is shown, that every $\vec{k}$-multigraph $G_{\vec{k}}$ of $\mathcal{GBG}_{\vec{k}}$ is connected, Eulerian and Hamiltonian (the same properties are true for graphs of $\mathcal{BG}_k$) The lower bounds of the cardinalities of the classes of Euler lines and of Hamiltonian circuits of $G_{\vec{k}}$ are given. Moreover $G_{\vec{k}}$ consists of the subgraphs which are isomorphic with the de Bruijn graphs of rank $r = \sum_{i=1}^{m}(d_i \cdot k_i)$, for arbitrary nonzero vector $\vec{d} = (d_1, \dots, d_m) \in \{0, 1\}^m$.

One can distinguish the subgraphs of the $\vec{k}$-multigraphs, called the $\vec{k}$-factors, by analogy with the subgraphs of the de Bruijn graphs, called the k-factors.

Every $\vec{k}$-factor (resp. k-factor) is a subgraph of $G_{\vec{k}}$ (resp. of G_k) consisting of all nodes of $G_{\vec{k}}$ (resp. of G_k) and additionaly all its connected components form the cycles.

The $\vec{k}$-factors of $G_{\vec{k}}$ (resp. k-factors) can be realized by the technical devices which are called cascade parallel $\vec{k}$-nets of shift registers [6] (resp. k shift registers).

The problem of construction of Hamiltonian circuits being the factors of de Bruijn graphs is of the great importance with respect to their wide use in technics [3,7,8,9, 11,13].

An algorithm, with small time and space complexities, for the construction of the $\vec{k}$-factors, among the others the Hamiltonian circuits, is presented.

In the opinion of the authors' the class $\mathcal{GBG}_{\vec{k}}$ of $\vec{k}$-multigraphs will be interesting, both for theoretical studies as well as for practical use, especially during the transmission of information. But the theory of the class $\mathcal{GBG}_{\vec{k}}$ is in the beginning stage of developments. A few open problems will be put forwards in Section 6.

The majority of notions related to graphs are the same as in book of Deo [2] and will be not recalled here.

2 Basic definitions

For a finite alphabet A and the positive integers $k_1, \ldots, k_m$ (not necessarily different) let $V = A^{k_1} \times \ldots \times A^{k_m}$. Let $\vec{k} = (k_1, \ldots, k_m)$.

Let us define $\texttt{id}_i \subseteq A^{k_i} \times A^{k_i}$ and $\texttt{L-shift}_i^a \subseteq A^{k_i} \times A^{k_i}$, $1 \le i \le m$ and $a \in A$, called the identity relation and a left-hand side shift relation respectively, as follows.

For arbitrary $t = (t^1, \ldots, t^m)$ and $u = (u^1, \ldots, u^m)$ of V, where $t^i = t_1^i \ldots t_{k_i}^i$ and $u^i = u_1^i \ldots u_{k_i}^i$ for $i = 1, 2, \ldots, m$, we have:

$$(t^i, u^i) \in \texttt{id}_i \quad \text{iff} \quad t^i = u^i$$

$$(t^i, u^i) \in \texttt{L-shift}_i^a \quad \text{iff} \quad u^i = t_2^i, \ldots t_{k_i}^i a.$$

Note that the relations $\texttt{id}_i$ and $\texttt{L-shift}_i^a$ are the functions of A^{k_i} into A^{k_i}.

Now let us define $\texttt{L-trans}_{\vec{k}} \subseteq V \times V$, called a *transition relation*, as follows:

For every $t, u \in V$, $(t, u) \in \texttt{L-trans}_{\vec{k}}$ iff $(t^i, u^i) \in \texttt{id}_i$ or $(t^i, u^i) \in \texttt{L-shift}_i^a$, for all $1 \le i \le m$, but there exists at least one $1 \le j \le m$ such that $(t^j, u^j) \in \texttt{L-shift}_j^a$ for some $a \in A$. If $(t, u) \in \texttt{L-trans}_{\vec{k}}$ then u is said to be an immediate successor of t with respect to $\texttt{L-trans}_{\vec{k}}$ relation.

Example 2.1. For $A = \{0,1\}$, $\vec{k} = (3,2,3,4)$ and $V = A^3 \times A^2 \times A^3 \times A^4$ the following pairs:[1]

$$((011,00,000,1011),(011,00,000,1011)),$$
$$((011,00,000,1011),(011,01,000,1011)),$$
$$((011,00,000,1011),(011,01,001,1011)),$$
$$((011,00,000,1011),(110,01,001,0110))$$

are elements of the relation $\texttt{L-trans}_{\vec{k}}$.

Let $E = V \times B$, where $B = (\{A \cup \odot\}^m \setminus \{(\odot,\ldots,\odot)\})$. Let us define a mapping $\Psi^L\colon E \to V \times V$ such that

$$\Psi^L(t,b) = (t,u), \text{ where } u^i = \begin{cases} \texttt{id}_i(t^i) & \text{if } b_i = \odot \\ \texttt{L-shift}_i^{b_i}(t^i) & \text{otherwise.} \end{cases}$$

for every $(t,b) \in E$, $b = (b_1,\ldots,b_m) \in B$, $t = (t^1,\ldots,t^m)$, $u = (u^1,\ldots,u^m)$ of V and $i = 1,2,\ldots,m$.

Obviously $\Psi^L\colon E \to \texttt{L-trans}_{\vec{k}}$. Then a multigraph $G_{\vec{k}} = (V,E,\Psi^L)$ describes a $\texttt{L-trans}_{\vec{k}}$ relation. Observe that Ψ^L is a mapping assigning to each edge $(t,b) \in E$ a pair (t,u) of nodes. In such a case an edge is directed from t to u. A multigraph $G_{\vec{k}} = (V,E,\Psi^L)$ is called a *generalized de Bruijn multigraph of rank $\vec{k}$* over an alphabet A (briefly $\vec{k}$-multigraph).

If Ψ^L is one-to-one mapping of E into $V \times V$ then $G_{\vec{k}}$ is defined as a pair (V,E). In this case the edges are identified with the pairs of nodes. Additionally, if $\vec{k} = (k)$ (i.e $\vec{k}$ has a unique component k) then $G_{\vec{k}}$ is called *de Bruijn graph of rank k* and denoted by G_k. The set of nodes of this graph G_k is equal to A^k and the set of edges E consists of all pairs (t,u), $t = t_1\ldots t_k$, $u = u_1\ldots u_k$ such that $t_2\ldots t_k = u_1\ldots u_{k-1}$.

For details related to the de Bruijn graphs the reader is referred to [4].

Example 2.2. The de Bruijn graphs of ranks 2 and 3 over the alphabet $A = \{0,1\}$ are shown in Figure 2.1.

Example 2.3. For lack of space let us only signalize an idea of the construction of a generalized de Bruijn $\vec{k}$-multigraph $G_{\vec{k}}$. Let $\vec{k} = (2,3)$ and $A = \{0,1\}$. Let us consider, for example, the node $v = 01,001$. Then all edges which are incident out of v and incident into v are shown in Figure 2.2.

[1] For clarity all components of the nodes of $\mathcal{GBG}_{\vec{k}}$ are separated with commas.

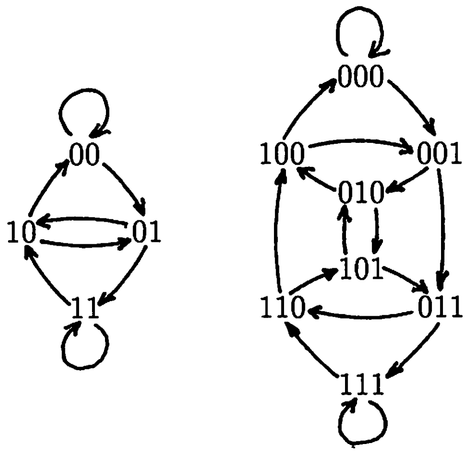

Figure 2.1. De Bruijn graphs of ranks 2 and 3

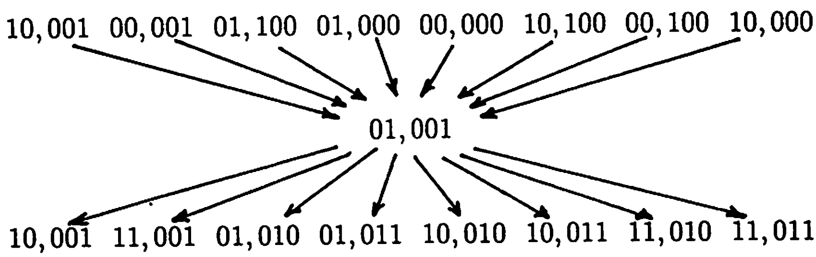

Figure 2.2. Segment of multigraph $G_{(2,3)}$

3 Basic properties of $\vec{k}$-multigraphs

It is shown, among the others, that every $\vec{k}$-multigraph $G_{\vec{k}}$ is connected, Eulerian and Hamiltonian. The lower bounds of the cardinalities of the classes $\mathcal{E}_{\vec{k}}$ and $\mathcal{H}_{\vec{k}}$ of Euler lines and of Hamiltonian circuits are stated.

It is also shown that some subgraphs of $G_{\vec{k}}$ are isomorphic with de Bruijn graphs.

Lemma 3.1. *For the set V of nodes and the set E of edges of generalized de Bruijn multigraph $G_{\vec{k}}$ the following conditions hold:*

$$|V| = |A|^p$$

and

$$|E| = |A|^p \cdot [(|A|+1)^m - 1],$$

where $p = k_1 + \cdots + k_m$.

Proof: Let $G_{\vec{k}} = (V, E, \Psi^L)$ be a generalized multigragh of rank $\vec{k}$. Then we have:

$$|V| = |A^{k_1} \times A^{k_2} \times \ldots \times A^{k_m}| = |A^{k_1}| \cdot |A^{k_2}| \cdot \ldots \cdot |A^{k_m}| = |A|^p$$

and

$$|E| = |V \times B| = |V| \cdot |B| = |A|^p \cdot [(|A| + 1)^m - 1].$$

□

Corollary 3.2. *For every node v of $G_{\vec{k}}$ we have:*

$$d^+(v) = (|A| + 1)^m - 1, \tag{3.1}$$

and

$$d^-(v) = (|A| + 1)^m - 1, \tag{3.2}$$

where $d^+(v)$ and $d^-(v)$ denote the out-degree and in-degree of a node v, respectively.

Proof: To prove (3.1) let us observe that for a fixed $v \in V$ we have:

$$\begin{aligned} &\{(v, b) \in E \colon \Psi^L(v, b) = (v, y), y \in V\} \\ &= \{b \in B \colon \Psi^L(v, b) = (v, y), y \in V\} = B \end{aligned} \tag{3.3}$$

and then

$$d^+(v) = |\{(v, b) \in E \colon \Psi^L(v, b) = (v, y), y \in V\}| = |B| = (|A| + 1)^m - 1.$$

To prove (3.2) we have to introduce some auxiliary notions. Let us define the relation $\texttt{R-shift}_i^a \subseteq A^{k_i} \times A^{k_i}$, $1 \leq i \leq m$ and $a \in A$, called a *right-hand side shift relation*, as follows:
For arbitrary $t = (t^1, \ldots, t^m)$ and $u = (u^1, \ldots, u^m)$ of V, where $t^i = t_1^i \ldots t_{k_i}^i$ and $u^i = u_1^i \ldots u_{k_i}^i$ for $i = 1, 2, \ldots, m$, we have:

$$(t^i, u^i) \in \texttt{R-shift}_i^a \quad \text{iff} \quad t^i = a u_1^i \ldots u_{k_i - 1}^i.$$

Note that a $\texttt{R-shift}_i^a$ relation is a total function of A^{k_i} into A^{k_i}.
Let us define a mapping $\Psi^R \colon E \to V \times V$ as follows:

$$\Psi^R(v, b) = (v, y), \text{ where } y^i = \begin{cases} \texttt{id}_i(v^i) & \text{if } b_i = \odot \\ \texttt{R-shift}_i^{b_i}(v^i) & \text{otherwise} \end{cases}$$

for every $(v, b) \in E$, $b = (b_1, \ldots, b_m) \in B$, $v = (v^1, \ldots, v^m)$, $y = (y^1, \ldots, y^m)$ of V and $i = 1, 2, \ldots, m$.

Thus Ψ^R assigns to each edge $(v,b) \in E$ a unique pair (v,y) of nodes. Then there exists $b' \in B$ such that

$$\Psi^L(v,b) = (v,y) = \Psi^R(v,b').$$

This implies the equality:

$$\begin{aligned} &\{(v,b) \in E\colon \Psi^R(v,b) = (v,y), y \in V\} \\ &= \{(v,b') \in E\colon \Psi^L(v,b') = (v,y), y \in V\}. \end{aligned} \tag{3.4}$$

Since Ψ^R is a total function then we have:

$$\{b \in B\colon \Psi^R(v,b) = (v,y), y \in V\} = B.$$

From (3.3) and (3.4) immediately it follows that

$$d^-(v) = |\{b \in B\colon \Psi^R(v,b) = (v,y), y \in V\}| = |B| = (|A|+1)^m - 1,$$

what finishes the proof. □

Remark 3.3. One can able introduce the relation $\texttt{R-trans}_{\vec{k}}$, called *R-transition relation*, by analogy with $\texttt{L-trans}_{\vec{k}}$ relation which has been introduced in Section 2. It is sufficient to replace in the $\texttt{L-trans}_{\vec{k}}$ relation $\texttt{L-shift}^{a}_{\vec{k}_i}$ relation by $\texttt{R-shift}^{a}_{\vec{k}_i}$ one.

Theorem 3.4. *Every $\vec{k}$-multigraph $\mathcal{GBG}_{\vec{k}}$ over an alphabet A consists of the subgraphs which are isomorphic with the de Bruijn graphs of rank $r = \sum_{i=1}^{m}(d_i \cdot k_i)$ over A, for arbitrary nonzero vector $\vec{d} = (d_1, \ldots, d_m) \in \{0,1,\}^m$.*

Proof: Let $G_{\vec{k}} = (V, E, \Psi^L)$ be a $\vec{k}$-multigraph and $\vec{d} = (d_1, \ldots, d_m)$ a nonzero vector of $\{0,1\}^m$. The proof consists of three parts:

- The construction of a subgraph G' of $G_{\vec{k}}$;
- The construction of the de Bruijn graph G_r, $r = \sum_{j=1}^{m} d_i \cdot k_i$, by means of G';
- The proof that G' and G_r are isomorphic.

Let $i_1, \ldots, i_n$ be an increasing subsequence of $1, \ldots, m$ of all indexes which correspond to nonzero components of $\vec{d}$. Let us construct a subgraph $G' = (V', E', \Psi'^L)$ of $G_{\vec{k}}$ as follows:

For a fixed $b \in A$ let V' be a subset of V of all nodes $v = (v^1, \ldots, v^m)$ such that $v^j = (b \ldots b) \in A^{k_j}$ for every $j \in \{1, \ldots, m\} \setminus \{i_1, \ldots, i_n\}$ Let us define $\texttt{L*-trans}^a_{\vec{k}} \subseteq V' \times V'$ for some $a \in A$, as follows:

$$(u,v) \in \texttt{L*-trans}^a_{\vec{k}} \text{ iff } v^j = \begin{cases} u_2^{i_s} \ldots u_{k_{i_s}}^{i_s} u_1^{i_s+1} & \text{for } j = i_s,\ 1 \le s \le n-1; \\ u_2^j \ldots u_{k_j}^j a & \text{for } j = i_n; \\ u_1^j \ldots u_{k_j}^j & \text{for } j \in \{1, \ldots, m\} \setminus \{i_1, \ldots, i_n\}. \end{cases}$$

Note that $\texttt{L*-trans}^a_{\vec{k}}$ is a function.

Let $E' = \{(u,v) \in V' \times V' \colon v = \texttt{L*-trans}^a_{\vec{k}}(u), a \in A\}$ and $\Psi'^L = \Psi^L_{|E'}$.

Let us define a mapping φ of V' into $V'' = A^{k_{i_1}} \times \ldots \times A^{k_{i_n}}$ and an inverse mapping $\varphi^{-1} \colon V'' \to V'$ as follows:

For arbitrary $t = (t^1, \ldots t^m) \in V'$, $t^i = t^i_1 \ldots t^i_{k_i}$ for $1 \le i \le m$, and $u = (u^1, \ldots, u^n) \in V''$ we have $\varphi(t) = u$ and $\varphi^{-1}(u) = t$ iff

$$t^l = \begin{cases} u^j & \text{for } l = i_j, j \in \{1, \ldots, n\} \\ (b \ldots b) & \text{for the remaining indexes.} \end{cases} \tag{3.5}$$

Let $G'' = (V'', E'')$, where $E'' = \{(\varphi(t), \varphi(u)) \colon (t,u) \in E'\}$.

In other words the digraph $G'' = (V'', E'')$ is obtained from $G' = (V', E')$ in such a way that the set of nodes V'' is obtained from V' by omitting all its components corresponding to zeros components of $\vec{d}$. Then $\texttt{L*-trans}^a_{\vec{k}}$ shifts leftwards all elements of V'' and adds at the end by an element $a \in A$. It follows from the construction that G'' is de Bruijn graph of rank $r = \sum_{i=1}^m (d_i \cdot k_i)$.

It is obvious that φ is isomorphism transforming a subgraph G' of $G_{\vec{k}}$ into de Bruijn graph G''. □

For an illustration of Theorem 3.4 let us see the following example.

Example 3.5. Let us consider the generalized de Bruijn $\vec{k}$-multigraph $G_{\vec{k}} = (V, E, \Psi^L)$ with $\vec{k} = (3,2,4)$ over the alphabet $A = \{0,1\}$. Let $\vec{d} = (1,0,1)$.

Let us construct a subgraph $G' = (V', E', \Psi'^L)$ of $G_{\vec{k}}$ as follows:
$V' = \{t = (t^1, t^2, t^3) \in A^3 \times A^2 \times A^4 \colon t^2 = 00\}$, $E' = \{(u,v) \in V' \times V' \colon v = \texttt{L*-trans}^a_{\vec{k}}(u)$, for some $a = 0$ or $1\}$ and $\Psi'^L = \Psi^L_{|E'}$.

Let us consider a mapping φ of V' into $V'' = A^3 \times A^4$ and an inverse mapping φ^{-1} of V'' into V' as has been defined by means of (3.5).

Then $G'' = (V'', E'')$, where $E'' = \{(\varphi(t), \varphi(u)), (t,u) \in E'\}$, is the de Bruijn graph of rank $d = 3 \cdot 1 + 2 \cdot 0 + 4 \cdot 1$ which is isomorphic with G'. For an illustration of the above considerations let us consider, for example, the node $u = (101, 00, 1010)$. Then the nodes $u_1 = \texttt{L*-trans}^0_{\vec{k}}(u) = (011, 00, 0100)$ and $u_2 = \texttt{L*-trans}^1_{\vec{k}}(u) = (011, 00, 0101)$ are incident out of u in the digraph G'. But the nodes $w_1 = \varphi(u_1) = (0110100)$ and $w_2 = \varphi(u_2) = (0110101)$ are incident out of the node $w = \varphi(u) = (1011010)$ in the de Bruijn graph G''. □

Theorem 3.4 implies the following corollaries.

Corollary 3.6. *For arbitrary generalized de Bruijn graph $G_{\vec{k}}$, $\vec{k} = (k_1, \ldots, k_m)$ and a nonzero vector $\vec{d} = (d_1, \ldots, d_m) \in \{0,1\}^m$ we are able to* *construct (effectively) at least q subgraphs of $G_{\vec{k}}$ with $q = |A^p|$ and $p = \sum_{i=1}^{m}((1-d_i)\cdot k_i)$ which are isomorphic with the de Bruijn graph G_r, $r = \sum_{i=1}^{m}(d_i \cdot k_i)$.*

Proof: One can able modify the proof of Theorem 3.4 in such a way that V' consists of all nodes of V whose all components corresponding to zeros components of $\vec{d}$ are the arbitrary sequences over A instead of constant ones. □

Corollary 3.7. *Every de Bruijn graph G_p, $p = k_1 + \cdots + k_m$, over an alphabet A is a subgraph of a generalized de Bruijn multigraph $G_{\vec{k}}$ consisting of all nodes as $G_{\vec{k}}$.*

Proof: It is sufficient to put (in Theorem 3.4) $d_i = 1$ for all i, $1 \le i \le m$. □

It follows from the previous considerations that every generalized multigraph $G_{\vec{k}}$ over an alphabet A can be obtained by means of the de Bruijn graph G_p, $p = k_1 + \cdots + k_m$, by attaching some additional edges. In particular, every Hamiltonian circuit of G_p is a Hamiltonian circuit of the generalized de Bruijn multigraph of rank $\vec{k} = (k_1, \ldots, k_m)$ such that $p = k_1 + \cdots + k_m$.

Corollary 3.8. *Every generalized de Bruijn multigraph G_k, $\vec{k} \in N^m$ and $m \ge 1$, is connected.*

Corollary 3.9. *Every generalized de Bruijn multigraph $G_{\vec{k}}$ is Hamiltonian.*

Proof of Corollaries 3.8 and 3.9 immediately follows from Corollary 3.7 and the fact that de Bruijn graph G_p is connected and Hamiltonian.

Corollary 3.10. *Every generalized de Bruijn $\vec{k}$-multigraph $G_{\vec{k}}$ is Eulerian.*

Lemma 3.11. *Let $\mathcal{E}_{\vec{k}}$ and $\mathcal{H}_{\vec{k}}$ denote the classes of all Eulerian lines and Hamiltonian circuits of a $\vec{k}$-multigraph $G_{\vec{k}}$. Then we have:*

$$|\mathcal{H}_{\vec{k}}| \ge (|A|-1)!^{|A|^{p-1}} \cdot (|A|^{|A|^{p-1}-p}) \tag{3.6}$$

and

$$|\mathcal{E}_{\vec{k}}| \ge (|A|-1)!^{|A|^{p}-1} \cdot (|A|^{|A|^{p}-p-1} \cdot [(|A|+1)^m - 2]!^{|A|^p}) \tag{3.7}$$

Proof: Let us set $c_1 = (|A|-1)!^{|A|^{p-1}} \cdot (|A|^{|A|^{p-1}-p})$ and $c_2 = (|A|-1)!^{|A|^{p}-1} \cdot (|A|^{|A|^{p}-p-1} \cdot [(|A|+1)^m - 2]!^{|A|^p})$. From Corollary 3.7 it follows that every Hamiltonian circuit of G_p, $p = k_1 + \cdots + k_m$, is a Hamiltonian circuit of $G_{\vec{k}}$. As there exists exactly c_1 Hamiltonian circuits of G_p then the inequality (3.6) holds.

To prove (3.7) let us observe that every Euler line of G_p, $p = k_1 + \cdots + k_m$, determines unique spanning tree of $G_{\vec{k}}$. It follows from the known theorem

(Theorem 9.13 pp. 226 of [2]) that every spanning tree determines an existence of $[(|A|+1)^m - 2]!^{|A|^p}$ Euler lines of $G_{\vec{k}}$, different to each other. Finally, $G_{\vec{k}}$ consists of at least c_2 of Euler lines. □

We leave as an open problem to determine the cardinalities of the sets $\mathcal{E}_{\vec{k}}$ and $\mathcal{H}_{\vec{k}}$

4 The factors of $\vec{k}$-multigraphs

The factors of $\vec{k}$-multigraphs can be analogously introduced as the factors of de Bruijn graphs of rank k.

By a factor of a $\vec{k}$-multigraph $G_{\vec{k}}$ (briefly $\vec{k}$-factor) we mean a subgraph of $G_{\vec{k}}$ consisting of all nodes of $G_{\vec{k}}$ and additionally all its connected components form the cycles.

Every $\vec{k}$-factor $F_{\vec{k}} - = (V, \mathcal{E})$ of $G_{\vec{k}}$, analogously as a factor $F_k = (A^k, E)$ of G_k, determines uniquely a mapping $\Phi \colon V \to V$ (resp. $\varphi \colon A^k \to A^k$), called a transition function of of $F_{\vec{k}}$ (resp. of F_k), such that $\Phi(x) = y$ iff $(x, y) \in \mathcal{E}$ (resp. $\varphi(t) = u$ iff $(t, u) \in E$) for every $x, y \in V$ (resp. $t, u \in A^k$).

Example 4.1. The (2,3)-factor $F_{(2,3)}$ which is constructed by means of the factors F_2 and F_3 (Figure 4.1) is shown in Figure 4.2.

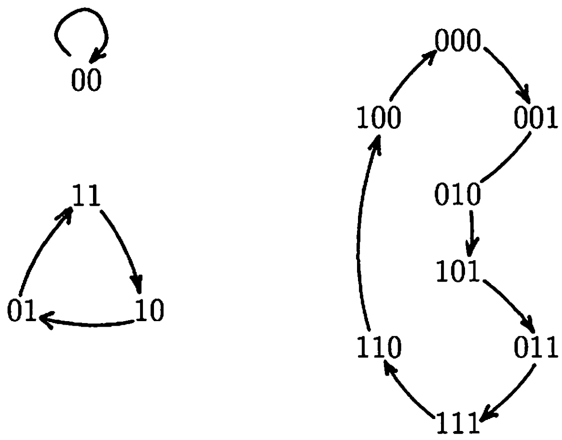

Figure 4.1.
Factors F_2 and F_3 of the de Bruijn graphs G_2 and G_3

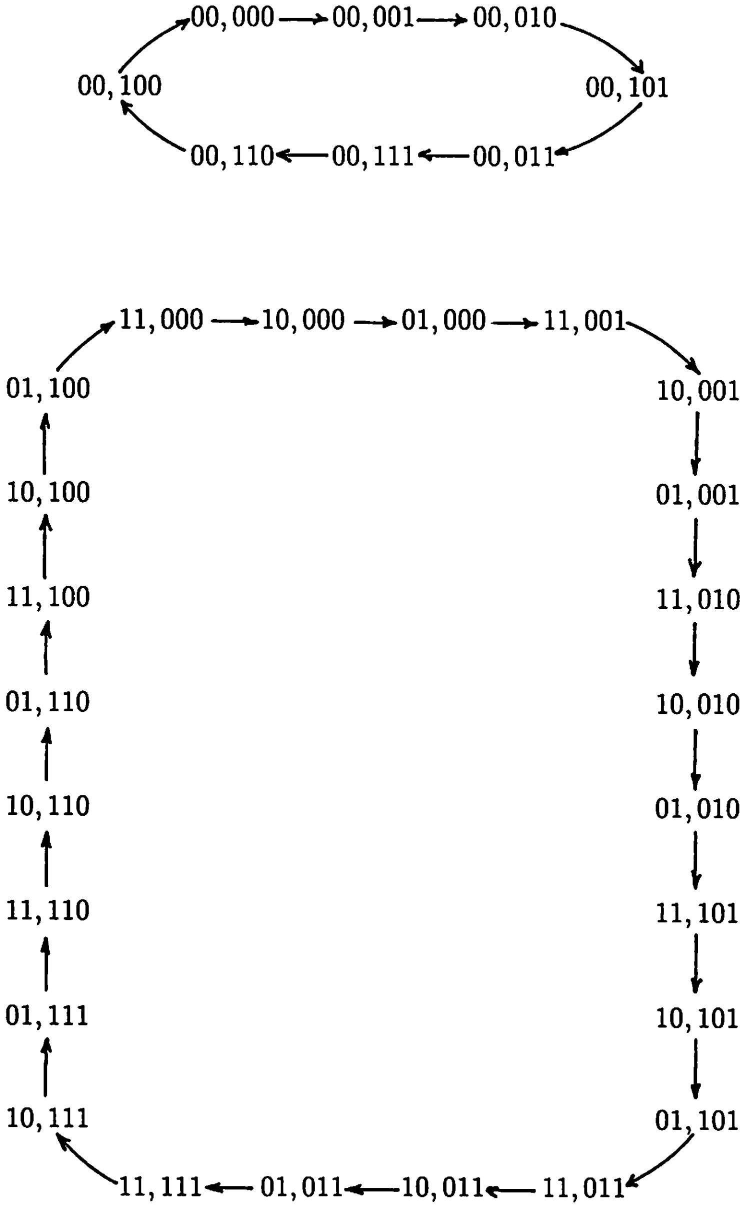

Figure 4.2. A (2,3)-factor $F_{(2,3)}$

Some comments relating to the construction of the $\vec{k}$-factors are necessary. Every cycle of $F_{\vec{k}}$ is constructed according the following rule. We prefer an activity of a unique transition function φ_i of a k_i-factor F_{k_i} with the least number i so long as it is possible, otherwise if it is a need to include the transition function φ_q of F_{k_q}, $q > 1$, (to omit the repetition of the nodes) then all the functions $\varphi_1, \ldots, \varphi_{q-1}$ are included only once.

Now let us define an algorithm allowing to construct the $\vec{k}$-factors of $G_{\vec{k}}$ by means of the k_i-factors of G_{k_i} for all $i = 1, \ldots, m$.

Algorithm.

Input data: Vector $\vec{k} = (k_1, \ldots, k_m)$ and the factors $F_{k_1}, \ldots, F_{k_m}$, (F_{k_i} is a factor of G_{k_i} for $i = 1, 2, \ldots m$).

Let φ_i be a transition function of F_{k_i}.

Result: Factor $F_{\vec{k}}$ of $G_{\vec{k}}$.

1. Let $v = (v^1, \ldots, v^m)$, $v^i \in A^{k_i}$ for $1 \le i \le m$, be an initial node of successively constructed circuit of a factor $F_{\vec{k}}$ which does not occur in the previously constructed circuits; If such a node does not exist then go to 7;
2. Put $j := 1$ and $u := v$;
3. We compute a temporary node u by using the function φ_j to the jth component of the last computed state u;
4. If the value of φ_j which has been computable in the point 3 is equal to v^j then we go to 5, else to 6;
5. If $j < m$ then we put $j := j + 1$ and go to 3, else to 1;
6. If $j > 1$ then we put $j := 1$; we assume u as the succesive node of the constructed circuit and go to 3;
7. STOP.

The correctness of Algorithm is obvious. Time and space complexities of Algorithm are equal to $c \cdot |A|^p$, $p = k_1 + \cdots + k_m$, for some constant $c \in \langle 1, 2 \rangle$.

A brief characterization of the resulting $\vec{k}$-factors, which have been constructed by using Algorithm, in the following theorem is given.

Theorem 4.2. *Let us suppose that the input data of Algorithm are the factors $F_{k_1}, \ldots, F_{k_m}$ and $F_{\vec{k}}$ is the resulting $\vec{k}$-factor. Then we have:*

(1) *The cardinality of the set of all connected components of $F_{\vec{k}}$ is equal to $p_1 \cdot \ldots \cdot p_m$, where p_i, $1 \le i \le m$, is the cardinality of the set of all connected components of F_{k_i}, $1 \le i \le m$;*

(2) *If $v = (v^1, \ldots, v^m)$, $v^i \in A^{k_i}$, is node of any cycle C of $F_{\vec{k}}$ then the length of C is equal to $q_1 \cdot \ldots \cdot q_m$, where q_i is the length of the cycle C_i of F_{k_i} containing v^i, for every $1 \le i \le m$.*

Proof is obvious.

Corollary 4.3. *The method used in Algorithm guarantees obtaining a Hamiltonian circuit of $G_{\vec{k}}$ iff the input data F_{k_i}, $1 \le i \le m$, are the maximal factors of the de Bruijn graphs G_{k_i}.*

5 The maximal $\vec{k}$-factors

The construction of the maximal $\vec{k}$-factors of $G_{\vec{k}}$ will be given.

A $\vec{k}$-factor $F_{\vec{k}}$ of a $\vec{k}$-multigraph $G_{\vec{k}}$ (resp. k-factor of G_k) is called *maximal* if it forms a Hamiltonian circuit.

Example 5.1. Let us recall the Fredricksen's method [3], called "the preference of 1's", of the construction of the maximal k-factors of the de Bruijn graphs of rank k. Assuming $x_1 = 0^k$ we construct the successive nodes of F_k: $x_2 = 0^{k-1}1, x_3 = 0^{k-2}1^2, \ldots, x_{k+1} = 1^k, x_{k+2} = 1^{k-1}0, \ldots, x_{2^k} = 10^{k-1}$ (each x_{i+1} is incident out of x_i, $1 \le i \le 2^k$ in F_k).

The graphical representation of the maximal factors F_2, F_3 and F_4 which are constructed by means of Fredricksen's method in Figure 5.1 is shown.

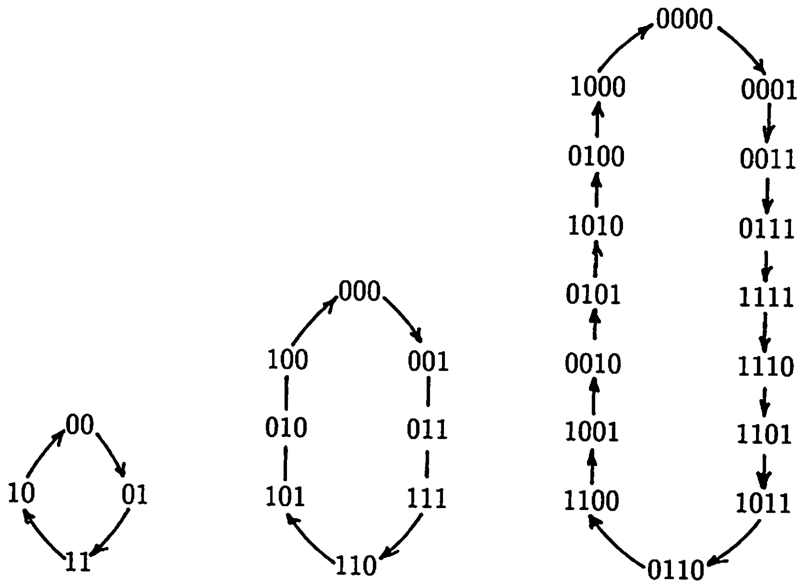

Figure 5.1. Maximal factors F_2, F_3, and F_4 of the de Bruijn graphs

Remark 5.2. The Fredricsen's method "preference of 1's" can be extended on arbitrary finite alphabet A. For an alphabet $A = \{a_1, \ldots, a_n\}$, $n > 2$, we introduce an order $\preceq$. Let be for example $a_1 \preceq a_2 \preceq \cdots \preceq a_n$. We construct a Hamiltonian circuit H_k of rank k as follows:

Starting with a node $(a_1)^k$ we construct the successive nodes of the form $(a_1)^{k-1}a_n, \ldots, (a_n)^k, (a_n)^{k-1}a_{n-1}, (a_n)^{k-1}a_{n-1}a_n, \ldots, (a_2)^{n-1}a_1$.

The successive nodes are obtained from their immediate predecessors by cutting the first elements and adding at the end the greatest element of A with respect to the relation $\preceq$. But we construct only such successive nodes which have been not previously occurred.

Example 5.2. Let $A = \{0, 1, 2, 3\}$ and $0 \preceq 1 \preceq 2 \preceq 3$. The maximal factors F_2 and F_3 are shown in Figure 5.2 and Figure 5.3.

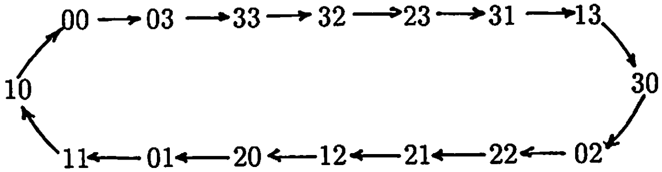

Figure 5.2. A maximal factor F_2 over $A = \{0, 1, 2, 3\}$

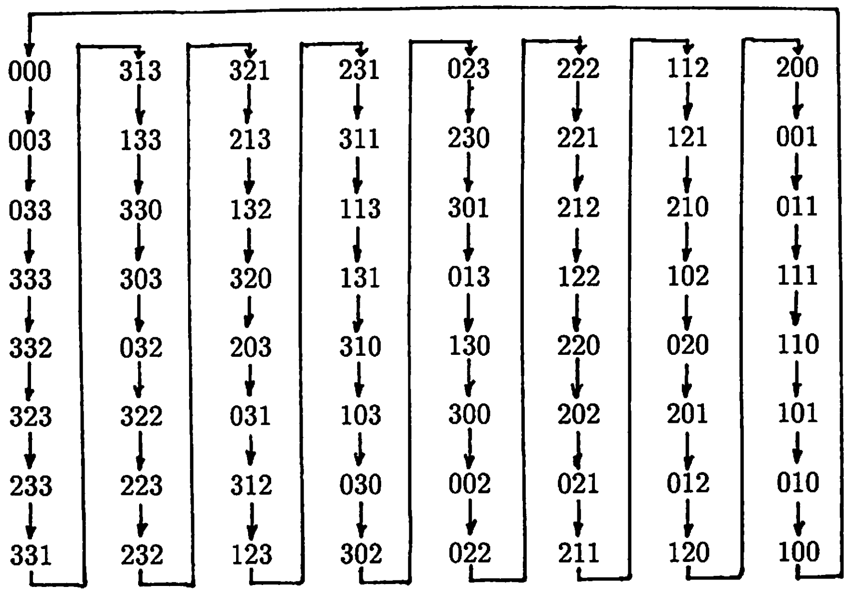

Figure 5.3. A maximal factor F_3 over $A = \{0, 1, 2, 3\}$

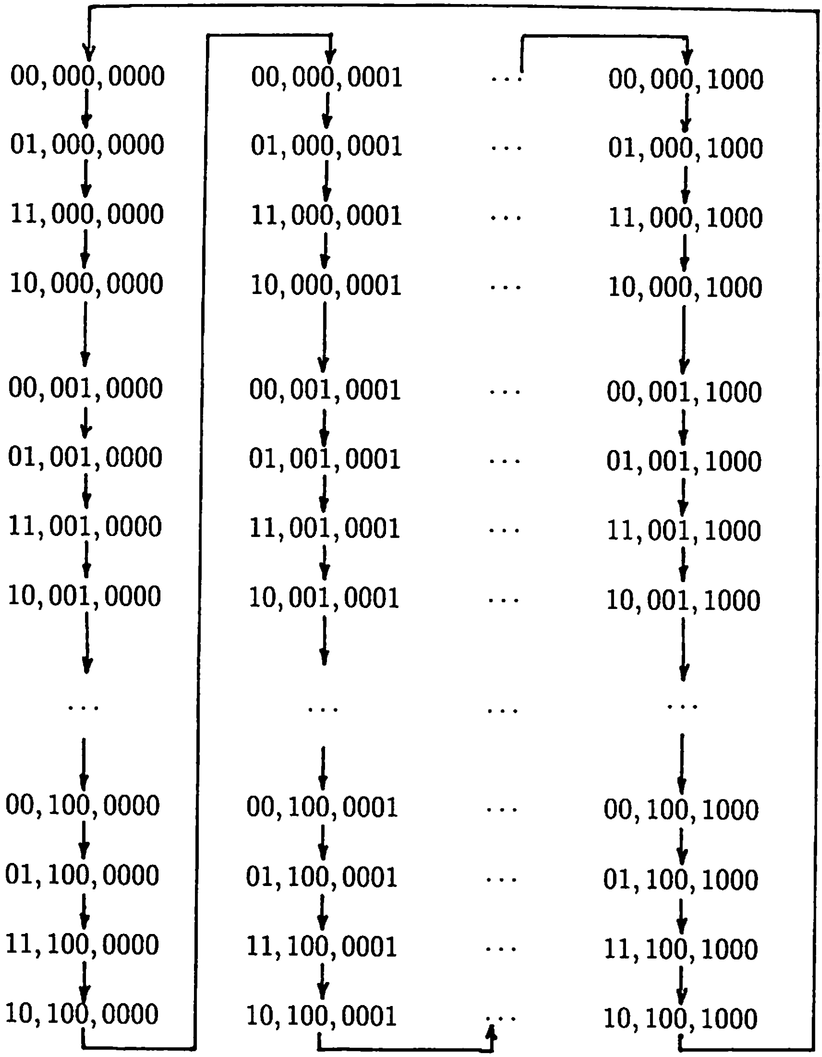

Figure 5.4. A maximal (2,3,4)-factor $F_{(2,3,4)}$

Example 5.3. The maximal (2,3,4)-factor constructed by Algorithm which in Section 4 has been given by means of the factors F_2, F_3 and F_4 of Example 5.1 in Figure 5.4 is given.

Remark 5.5. The proposed order of switching of the feedback functions in Algorithm can be changed. It is sufficient to guarantee the possibility of generating of all nodes of V.

6 Open problems

A theory of the class $\mathcal{GBG}_{\vec{k}}$ of $\vec{k}$-multigraphs is in the initial stage of developments. Only basic properties of this class are given. Many problems remain open. Let us point out a few of them:

(6.1) We have to determine the cardinalities of the classes $\mathcal{E}_{\vec{k}}$ and $\mathcal{H}_{\vec{k}}$ of Euler lines and Hamiltonian circuits of $G_{\vec{k}}$;

(6.1) We have to solve the analogical problems for $\mathcal{GBG}_{\vec{k}}$ as have been solved in [1,9] for the de Bruijn graphs.

(6.3) We have to define an algorithm which allows to construct a class of $\vec{k}$-factors which are similar to à given $\vec{k}$-factor. Two $\vec{k}$-factors $F_{\vec{k}}$ and $F'_{\vec{k}}$ are said to be similar iff there exists a one-to-one mapping $\varphi\colon F_{\vec{k}} \rightarrow F'_{\vec{k}}$ such that corresponding to each other cycles consist of the same nodes (possibly with different ordering). One can understand similarity of the $\vec{k}$-factors in another way, for example the corresponding to each other cycles have the same lengths;

(6.4) We have to study the complexity problem of the $\vec{k}$-factors;

(6.5) We have to decide, if an arbitrary graph G is a $\vec{k}$-multigraph for same vector $\vec{k} \in \mathcal{N}^m$;

(6.6) Given graph G over an alphabet A we have to determine a vector $\vec{k}$ such that G is a $\vec{k}$-factor, if such $\vec{k}$ there exists.

Final remarks. One can able introduce a class $\mathcal{RGBG}_{\vec{k}}$ of right-hand side generalized de Bruijn graphs of rank $\vec{k}$ by analogy with the class $\mathcal{GBG}_{\vec{k}}$ introduced here (this class can be also called the class of left-hand side generalized de Bruijn $\vec{k}$-multigraphs). It is sufficient to replace $\texttt{L-trans}_{\vec{k}}$ relation by $\texttt{R-trans}_{\vec{k}}$ one (this relation in Remark 3.3 has been introduced). The class $\mathcal{RGBG}_{\vec{k}}$ has the same properties as $\mathcal{GBG}_{\vec{k}}$. Both the classes $\mathcal{GBG}_{\vec{k}}$ and $\mathcal{RGBG}_{\vec{k}}$ can be exchangeable considered with respect to technical realization of their $\vec{k}$-factors.

Acknowledgements. The authors are very grateful to the anonymous referee for providing valuable comments and constructive remarks on the first version of this paper.

References

[1] O. Collins, S. Dolliner, A VLSI decomposition of the de Bruijn graphs, *J.A.C.M.*, **39**,No. 4 (1992), 930–948.

[2] N. Deo, *Graph theory with applications to engineering and computer science*, Prentice-Hall, Inc., Englewood Cliffs. 1974.

[3] H. Fredricksen, A survey of full length nonlinear shift register cycle algorithms, *SIAM Rev.*, **24** (1982), 195–221.

[4] S.W. Golomb, *Shift register sequences*, Aegean Park Press, Laguna Hills, CA, rev. ed., 1982.

[5] Z. Grodzki, A. Wronski, Generalized de Bruijn graph, *J. Inf. Processing and Cybernet. EIK* **30** (1994) 5–17.

[6] Z. Grodzki, A cascade parallel $(\vec{k}, \vec{s})$-nets of shift-registers (manuscript).

[7] A. Lempel, On a homomorphism of the de Bruijn graph and its applications to the design of feedback shift registers, *IEEE Trans. Comput.* **C-19** (1970), 1204–1209.

[8] M. Latko, On a construction of all de Bruijn cycles defined by Lempel's homomorphism, *Demonstratio Math.* **21** (1988), 441–455.

[9] U.M. Maurer, A asymptotically-tight bounds on the number of cycles in generalized de Bruijn graphs, *Discrete Applied Mathemathics* **37/38** (1992), 421–436.

[10] J. Mykkeltveit, M. Siu, P. Tong, On the cycle structure of some nonlinear shift register sequences, *Inform. Contr.* **43** (1979), 202–215.

[11] R. Ruppel, *Analysis and Design of Stream Ciphers*, Springer Verlag, Berlin, Heidelberg 1986.

[12] P. Wlaź, J. Zurawiecki, An algorithm for generating M-sequences using universal circuit matrix, *Ars Combinatoria* 41 (1995), 203–216.

[13] J. Zurawiecki, The strong similitude of Hamiltonian circuits of a de Bruijn graph, *Journal of Inf. Process. Cybern. EIK* **28** (1992) 385–399.