Minus k-subdomination in graphs

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Abstract

Let $G = (V, E)$ be a graph and $k \in \mathbf{Z}^+$ such that $1 \leq k \leq |V|$. A k-subdominating function (kSF) to $\{-1, 0, 1\}$ is a function $f : V \rightarrow \{-1, 0, 1\}$ such that the closed neighborhood sum $f(N[v]) \geq 1$ for at least k vertices of G. The weight of a kSF f is $f(V) = \sum_{v \in V} f(v)$. The k-subdomination number to $\{-1, 0, 1\}$ of a graph G, denoted by $\gamma_{ks}^{-101}(G)$, equals the minimum weight of a kSF of G. In this paper we characterize minimal kSF's, calculate γ_{ks}^{-101} for an arbitrary path and determine the least order of a connected graph G for which $\gamma_{ks}^{-101}(G) = -m$ for an arbitrary positive integer m.

1 Introduction

Let $G = (V, E)$ be a graph and let v be a vertex in V. The *open neighborhood* of v is defined as the set of vertices adjacent to v, i.e., $N(v) = \{u | uv \in E\}$. The *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$. For a set S of vertices, we define the open neighborhood $N(S)$ as $\cup_{v \in S} N(v)$, and the

closed neighborhood $N[S]$ as $N(S) \cup S$. A set S of vertices is a *dominating set* if $N[S] = V$. The *domination number* of a graph G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set in G.
For any real valued function $f : V \rightarrow \mathbf{R}$ and $S \subseteq V$, let $f(S) = \sum_{u \in S} f(u)$. The *weight* of f is defined as $f(V)$. We will also denote $f(N[v])$ by $f[v]$, where $v \in V$.
A *minus dominating function* is defined in [2] as a function $f : V \rightarrow \{-1, 0, 1\}$ such that $f[v] \geq 1$ for every $v \in V$. The *minus domination number* of a graph G is $\gamma^-(G) = \min\{f(V) \mid f$ is a minus dominating function on $G\}$.
A *signed dominating function* is defined in [3] as a function $f : V \rightarrow \{-1, 1\}$ such that $f[v] \geq 1$ for every $v \in V$. The *signed domination number* of a graph G is $\gamma_s(G) = \min\{f(V) \mid f$ is a signed dominating function on $G\}$.
A *majority dominating function* is defined in [4] as a function $f : V \rightarrow \{-1, 1\}$ such that $f[v] \geq 1$ for at least half the vertices $v \in V$. The *majority domination number* of a graph G is $\gamma_{maj}(G) = \min\{f(V) \mid f$ is a majority dominating function on $G\}$.
Let $G = (V, E)$ be a graph and $k \in \mathbf{Z}^+$ such that $1 \leq k \leq |V|$. A *k-subdominating function* (kSF) *to* $\{-1, 1\}$ for G is defined in [1] as a function $f : V \rightarrow \{-1, 1\}$ such that $f[v] \geq 1$ for at least k vertices of G. The *k-subdomination number to* $\{-1, 1\}$ of a graph G, denoted by $\gamma_{ks}^{-11}(G)$, is equal to $\min\{f(V) \mid f$ is a kSF to $\{-1, 1\}$ of $G\}$. In the special cases where $k = |V|$ and $k = \lceil \frac{|V|}{2} \rceil$, $\gamma_{ks}^{-11}(G)$ is respectively the signed domination number and the majority domination number.
We now generalize the concept of minus domination. Let $G = (V, E)$ be a graph and $k \in \mathbf{Z}^+$ such that $1 \leq k \leq |V|$. A *k-subdominating function* (kSF) *to* $\{-1, 0, 1\}$ for G is defined as a function $f : V \rightarrow \{-1, 0, 1\}$ such that $f[v] \geq 1$ for at least k vertices of G. The *k-subdomination number to* $\{-1, 0, 1\}$ of a graph G, denoted by $\gamma_{ks}^{-101}(G)$, is equal to $\min\{f(V) \mid f$ is a kSF to $\{-1, 0, 1\}$ of $G\}$. Since functions to $\{-1, 1\}$ play no further role in the remainder of this paper, we will omit the phrase *to* $\{-1, 0, 1\}$ throughout when dealing with a kSF to $\{-1, 0, 1\}$ and with the k-subdomination number to $\{-1, 0, 1\}$.
Alon (see [4]) proved that $\gamma_{maj}(G) \leq 2$ for a connected graph G. Let k be an integer such that $1 \leq k \leq \lceil \frac{1}{2}|V| \rceil$. Since every majority domination function is a kSF to $\{-1, 0, 1\}$, it follows that $\gamma_{ks}^{-11}(G) \leq \gamma_{maj}(G)$. Hence, if G is connected, then $\gamma_{ks}^{-101}(G) \leq 2$.
There is a wide variety of possible applications for this variation of domination. By assigning the values $+1$, 0 and -1 to the vertices of a graph we can model such things as networks of people or organizations in which global decisions must be made (e.g. yes-abstain-no, agree-neutral-disagree, like-neutral-dislike, etc.). In such a context, for example, the k-subdomination

number represents the minimum number of people whose positive votes can assure that at least k of the local groups of voters (represented by closed neighborhoods) have more positive than negative voters, even though the entire network may have a large majority of negative voters.
In this paper we characterize minimal k-subdominating functions, calculate γ_{ks}^{-101} for an arbitrary path and determine the least order of a connected graph G for which $\gamma_{ks}^{-101}(G) = -m$ for an arbitrary positive integer m.

2 Minimal k-subdominating functions

Let, throughout this section, $G = (V, E)$ be a graph. The kSF f is called *minimal* if no $g < f$ is a kSF. Following [1], we now characterize minimal k-subdominating functions. Let f be a kSF for the graph G. We use three sets for such an f:

$$\begin{aligned} B_f &= \{v \in V | f[v] = 1\}, \\ P_f &= \{v \in V | f(v) \geq 0\} \\ \text{and } C_f &= \{v \in V | f[v] \geq 1\}. \end{aligned}$$

A vertex $v \in C_f$ is *covered* by f, otherwise it is *uncovered* by f. Note that $B_f \subseteq C_f$.
For $A, B \subseteq V$ we say that A *dominates* B (denoted by $A \succ B$) if for each $b \in B$ we have $N[b] \cap A \neq \emptyset$. We are now in a position to characterize minimal kSF's.

Theorem 1 *A kSF f is minimal iff for each k-subset K of C_f we have $B_f \cap K \succ P_f$.*

Proof. If f is a kSF satisfying the above condition which is not minimal, then there is a kSF g with $g < f$. Consider a k-subset $K' \subseteq C_g \subseteq C_f$ and a vertex v with $g(v) < f(v)$. Then $g(v) \leq 0$ and $f(v) \geq 0$, t.i. $v \in P_f$. By the assumption, $B_f \cap K' \succ \{v\}$, i.e., there exists a $w \in B_f \cap K' \cap N[v]$. But then $f[w] = 1$ and $v \in N[w]$, hence $g[w] \leq 0$, contradicting $w \in C_g$.
Conversely, suppose f is a minimal kSF but there is a k-subset $K \subseteq C_f$ with $B_f \cap K \not\succ \{v\}$ for some $v \in P_f$. Define $h : V \to \{-1, 0, 1\}$ by $h(v) = f(v) - 1$ and $h(w) = f(w)$ for $w \in V - \{v\}$. We prove that $h[w] \geq 1$ for each $w \in K$ by considering two cases:

- If $w \in K \cap B_f$, then $w \notin N[v]$ so that $v \notin N[w]$ and hence $h[w] = f[w] = 1$.
- If $w \in K - B_f$, then $f[w] \geq 2$ so that $h[w] \geq f[w] - 1 \geq 1$.

Hence the set K shows that h is a kSF. This contradicts the minimality of f. ∎

3 The value of $\gamma_{ks}^{-101}(P_n)$

In this section we calculate $\gamma_{ks}^{-101}(P_n)$. We start with following result.

Lemma 1 *Let the vertex sequence of P_n, $n \geq 3$ be $1, \ldots, n$ and let k be an integer such that $1 \leq k \leq n-1$. There exists a minimum kSF f such that either $\{1, \ldots, k\} \subseteq C_f$ or $\{1, n\} \nsubseteq C_f$.*

Proof. Let $V = \{1, \ldots, n\}$ and let $f : V \to \{-1, 0, 1\}$ be a kSF of weight $\gamma_{ks}^{-101}(P_n)$. If $1 \notin C_f$ or $n \notin C_f$ or $\{1, \ldots, k\} \subseteq C_f$ we are done. Hence, we assume that $1 \in C_f$ and $n \in C_f$ and $\{i | i \notin C_f\} \neq \emptyset$. Let $nc_f = \max\{i | i \notin C_f\}$. Note that nc_f is not an endvertex of P_n, that $\{f(1), f(2)\} = \{0, 1\}$ or $\{f(1), f(2)\} = \{1, 1\}$ and that $\{f(n-1), f(n)\} = \{0, 1\}$ or $\{f(n-1), f(n) = \{1, 1\}$. Let $c_f = nc_f + 1$. When the function f is clear from context, we will find it convenient to use c for c_f and nc for nc_f.

Case 1 $f(c) = -1$.

Since $c \in C_f$, we must have that $f(nc) = f(c+1) = 1$. Note that $c+1$ is not an endvertex and that $f(c+2) = 1$. Before proceeding, we prove the following claim.

Claim *If $c+2$ is not an endvertex, there exists a minimum kSF h such that $h(x) = f(x)$ for all $x \in \{1, \ldots, c+2\}$ and $\{h(c+3), \ldots, h(n)\} \subseteq \{0, 1\}$.*

Proof. If $\{f(c+3), \ldots, f(n)\} \subseteq \{0, 1\}$, let $h = f$. If this is not the case, there exists an $i \in \{c+3, \ldots, n\}$ such that $f(i) = -1$. Let $m = \max\{i | f(i) = -1\}$. Then, since $m \in C_f$, it follows that $f(m-1) = f(m+1) = 1$. Furthermore, since $m+1 \in C_f$, it follows that $m+1$ is not an endvertex and $f(m+2) = 1$. Define $g : V \to \{-1, 0, 1\}$ by $g(x) = f(x)$ for all $x \in V - \{m, m+1\}$ and $g(m) = g(m+1) = 0$. Then g is a minimum kSF such that $g(x) = f(x)$ for all $x \in \{1, \ldots, c+2\}$ and with fewer vertices in $\{c+3, \ldots, n\}$ having the value -1 under g. The required function h will be found by iterating this procedure. □

By the Claim we may assume that $\{f(c+3), \ldots, f(n) \subseteq \{0, 1\}$. We define a new function $h : V \to \{-1, 0, 1\}$ as follows: $(h(1), \ldots, h(n)) = (f(c+1) - 1, f(c+2), \ldots, f(n), f(1), f(2), \ldots, f(nc), f(c) + 1)$. The function h still covers at least k vertices and $h(V) = f(V)$. Since $nc \notin C_f$ and $f(nc) = 1$, we have that $f(nc-1) \leq 0$. If $f(nc_f - 1) = -1$, then $nc_h = n - 1$ and the function h is an example of Case 2.1, below. If $f(nc_f - 1) = 0$, then in the function $|C_h| \geq |C_f| + 1$. If $\{i | i \notin C_h\} = \emptyset$, we are done. If not we proceed to find nc_h. If $h(c_h) \geq 0$, then the function h is an example of Case 2 below. If $h(c_h) = -1$, we repeat the above procedure which will lead to a function in which the vertices $1, \ldots, k$ are covered.

Case 2 $f(c) \geq 0$.

Since $c \in C_f$, it follows that $f(nc) + f(c) \geq 0$. Furthermore, since $nc \notin C_f$, $f(nc) + f(c) \leq 1$. Thus we have two cases.

Case 2.1 $f(nc) + f(c) = 1$.
Since $nc \notin C_f$, we have that $f(nc-1) = -1$. Define $h : V \to \{-1,0,1\}$ by $(h(1),\dots,h(n)) = (f(nc), f(c), f(c+1),\dots,f(n), f(1), f(2),\dots,f(nc-1))$. Then $|C_h| \geq k$ and $h(V) = f(V)$. Since $h[n] \leq 0$, h is the required function
Case 2.2 $f(nc) + f(c) = 0$.
Since $c \in C_f$, we have that $f(nc-1) + f(nc) \leq 0$. Define $h : V \to \{-1,0,1\}$ by $(h(1),\dots,h(n)) = (f(c), f(c+1),\dots,f(n), f(1),\dots,f(nc))$. Then $|C_h| \geq k$ and $h(V) = f(V)$. Since $h[n] \leq 0$, h is the required function.
■

Proposition 1 *Let $n \geq 3$ be an integer. Then, for an integer $1 \leq k \leq n-1$, there exists a minimum kSF f of P_n such that $\{1,2,\dots,k\} \subseteq C_f$.*

Proof. The proof is by induction on n, the number of vertices of the path. The result is trivial for paths of order 3, so suppose $n \geq 4$ and assume the result is true for all P_m, $3 \leq m \leq n-1$. Let k be an integer such that $1 \leq k \leq n-1$ and let $V = V(P_n)$.
Suppose $k = n-1$. By Lemma 1, there exists a minimum kSF f such that $\{1,\dots,n-1\} \subseteq C_f$ or $\{1,n\} \not\subseteq C_f$. Since $k = n-1$, there is at most one vertex not covered by f. So by reversing the path if necessary, we obtain a minimum kSF such that $\{1,\dots,n-1\} \subseteq C_f$.
Now suppose that $k \leq n-2$. By the induction hypothesis, there is a minimum kSF f on P_{n-1} such that $\{1,\dots,k\} \subseteq C_f$. Define $g : V \to \{-1,0,1\}$ by $g(i) = f(i)$ for $1 \leq i \leq n-1$ and $g(n) = -1$. Now g is a kSF for P_n. It remains to be shown that g is the smallest such function.
By the Lemma 1, there exists a kSF h of P_n such that $\{1,\dots,k\} \subseteq C_h$ or $\{1,n\} \not\subseteq C_h$. In the first case we are done. So assume that $\{1,n\} \not\subseteq C_h$. By reversing the path, if necessary, we may assume that $n \notin C_h$. Assume that $h(V) < g(V)$ and let $U' = V - \{n\}$. Then $h(U) + h(n) < g(U) + g(n)$, so that $h(U) < f(U) - 1 - h(n)$. If $h(n) \leq 0$, then h' defined as h restricted to U covers k vertices and $h'(U) < f(U) = \gamma_{ks}^{-101}(P_{n-1})$, which is a contradiction.
If $h(n) = 1$, then, since h is minimal and $n \notin C_h$, we must have that $h[n-1] = 1$. This implies that $h(n-1) = -1$ and $h(n-2) = 1$. Define $h' : U \to \{-1,0,1\}$ by $h'(i) = h(i)$ for $1 \leq i \leq n-2$ and $h'(n-1) = 0$. Then h' covers k vertices and $h'(U) = h(U) + 1 < f(U) - h(n) < f(U) = \gamma_{ks}^{-101}(P_{n-1})$, which is a contradiction. ■
The following lemma was proved in [2] and is given without proof here.

Theorem 2 $\gamma_{ns}^{-101}(P_n) = \lceil \frac{n}{3} \rceil$. ■

Theorem 3 *If $n \geq 2$ is an integer and $1 \leq k \leq n-1$, then $\gamma_{ks}^{-101}(P_n) = \lceil \frac{k}{3} \rceil + k - n + 1$.*

Proof. We start by showing that $\gamma_{ks}^{-101}(P_n) \leq \lceil \frac{k}{3} \rceil + k - n + 1$ for $1 \leq k \leq n-1$, using induction on n.
Suppose $n = 2$. Then $k = 1$ and it is clear that $\gamma_{1s}^{-101}(P_2) = 1 = \lceil \frac{k}{3} \rceil + k - n + 1$ and the result holds. Let $n > 2$ be given and assume for all paths of order $j(< n)$ that $\gamma_{ks}^{-101}(P_j) \leq \lceil \frac{k}{3} \rceil + k - j + 1$ holds for $1 \leq k \leq j-1$.
Suppose $1 \leq k \leq n-1$. If $k = n-1$, then the path P_{n-1} with vertex set $\{1, \ldots, n-1\}$ has minus domination number $\lceil \frac{n-1}{3} \rceil$. Assigning the value 0 to n allows an $(n-1)SF$ with weight $\lceil \frac{n-1}{3} \rceil$, and the result holds in this case.
If $k \leq n-2$, then by the inductive hypothesis and Proposition 1 applied to P_{n-1}, the first k vertices can be covered by a kSF with weight at most $\lceil \frac{k}{3} \rceil + k - (n-1) + 1$. If the remaining vertex is given the value -1, none of the covered vertices will be affected, and so $\gamma_{ks}^{-101}(P_n) \leq \gamma_{ks}^{-101}(P_{n-1}) - 1 \leq \lceil \frac{k}{3} \rceil + k - (n-1) + 1 - 1$ and the inequality follows.
To show that $\gamma_{ks}^{-101}(P_n) \geq \lceil \frac{k}{3} \rceil + k - n + 1$, assume g is a minimum kSF and suppose, by Proposition 1, that g covers the first k vertices of P_n. If $k + 1 < n$, let P' be the subgraph spanned by the vertices $\{k+2, \ldots, n\}$. If $k + 1 = n$, let P' be the empty graph. We obtain the required lower bound in cases by calculating $g(V)$ in each case: the result then follows since $\gamma_{ks}^{-101}(P_n) = g(V)$.
Case 1 If $k \equiv 0$ (mod3), then $g(V) = g(1) + g[3] + g[6] + \ldots + g[k] + g(V(P'))$. Because $g[1] \geq 1$, we must have $g(1) \geq 0$. Thus $g(V) \geq 0 + \lceil \frac{k}{3} \rceil - 1(n-k-1) = \lceil \frac{k}{3} \rceil - n + k + 1$.
Case 2 If $k \equiv 1$ (mod3), then $g(V) = g[1] + g[4] + \ldots + g[k] + g(V(P'))$. So in this case $g(V) \geq \lceil \frac{k}{3} \rceil - 1(n-k-1) = \lceil \frac{k}{3} \rceil - n + k + 1$.
Case 3 If $k \equiv 2$ (mod3), then $g(V) = g[2] + g[5] + \ldots + g[k] + g(V(P'))$. So in this case $g(V) \geq \lceil \frac{k}{3} \rceil - 1(n-k-1) = \lceil \frac{k}{3} \rceil - n + k + 1$. ■

4 The least order of a connected graph G for which $\gamma_{ks}^{-101}(G) = -m$

In this section we determine the least order of a connected graph G for which $\gamma_{ks}^{-101}(G) = -m$, where m is a positive integer m.

Theorem 4 *Let m be a positive integer and let $G = (V, E)$ be a connected graph such that $\gamma_{ks}^{-101}(G) = -m$ with k an integer such that $1 \leq k \leq p = p(G)$. Then*
(a) if $k = 1$, then $p \geq m + 3$,
(b) if $2 \leq k \leq p - 2$, then $p \geq m + 4$,
(c) if $k = p - 1$, then $p \geq 2\ell + m$, where $\ell = \min\{\ell \in \mathbf{Z}^+ | m \leq (\ell - 1)^2 - \ell\}$,
(d) if $k = p$, then $p \geq 2\ell + m$, where $\ell = \min\{\ell \in \mathbf{Z}^+ | m \leq \frac{\ell^2 - 3\ell}{2}\}$.
All these bounds are best possible.

Proof. Let, throughout this proof, f be a kSF of weight $\gamma_{k_s}^{-101}(G) = -m$. For such an f, let P, Z and M denote the subsets of V that are assigned the values $+1$, 0 and -1 respectively by f. Then $|P| - |M| = -m$, so that $|M| = |P| + m$. Also, since $k \geq 1$, at least one vertex, say v, is assigned the value $+1$. It follows that $p = |P| + |M| + |Z| \geq 2|P| + m \geq 2 + m$.
(a) We now prove that $p \geq m + 3$ if $k = 1$. Suppose, to the contrary, that $p = m + 2$. Then, since G is connected, v is adjacent to some vertex of M. But then $f[u] \leq 0$ for all $u \in V(G)$, which is a contradiction. This result is best possible, since, by Theorem 3, $\gamma_{1_s}^{-101}(P_{m+3}) = -m$.
(b) Before proceeding further, we prove the following for f: If $|P| = 1$, then $|Z| \geq 2$: Assume, to the contrary, that $|P| = 1$ and $|Z| < 2$. Clearly, $f[w] \leq 0$ for all $w \in M$. Hence the only vertices that can have $f[w] \geq 1$ are the vertex in P and vertices of Z (if any). Since $k \geq 2$ we must have $|Z| = 1$. But then there can be no edges between $P \cup Z$ and M, contradicting the connectedness of G.
Note that $p = |P| + |M| + |Z| = |P| + |P| + m + |Z| = 2|P| + |Z| + m$. If $|P| = 1$, then our claim implies that $2|P| + |Z| + m \geq 2 + 2 + m = 4 + m$. If $|P| \geq 2$, then $2|P| + |Z| + m \geq 2|P| + m \geq 4 + m$.
To see that this result is best possible, consider the graph $G = K_{2,2+m}$ with partite sets $U = \{u_1, u_2\}$ and $V = \{v_1, \ldots, v_{m+2}\}$. We will show that $\gamma_{k_s}^{-101}(G) = -m$:
To prove the inequality $\gamma_{k_s}^{-101}(G) \geq -m$, suppose there is a kSF g of weight $\gamma_{k_s}^{-101}(G) \leq -m - 1$. Let P', Z' and M' denote the subsets of V that are assigned the values $+1$, 0 and -1 respectively by g. Note that the weight of g is $g(u_1) + g(u_2) + |P'| - |M'| \leq -m - 1$, therefore we have $|M'| \geq g(u_1) + g(u_2) + m + 1 + |P'|$.
If a vertex $u_i \in U$ has $g[u_i] \geq 0$, then $g[u_i] = g(u_i) + |P'| - |M'| \geq 0$ implying that $-1 \leq g(u_1) + g(u_2) + |P'| - |M'| = \gamma_{k_s}^{-101}(G) \leq -m - 1 \leq -2$, a contradiction. Hence neither vertex of U is covered.
The covered vertices are therefore all in V. However, a vertex $w \in V$ can only be covered if $w \in P'$ and $g(u_1) + g(u_2) \geq 0$ or $w \in M' \cup Z'$ and $g(u_1) + g(u_2) \geq 1$. In both cases we have $|P'| + g(u_1) + g(u_2) \geq 1$ and hence that $|M'| \geq m + 2$ and therefore that $M' = V$ and $P' = Z' = \emptyset$. But then, to cover a vertex, we must have $g(u_1) = g(u_2) = 1$, causing g to have weight $-m$, a contradiction.
To show that $\gamma_{k_s}^{-101}(G) \leq -m$ we exhibit a kSF of G of weight $-m$. Define g by $g(v) = 1$ if $v \in U$ and $g(v) = -1$ if $v \in V$. It is easily seen that at least k, $k = 1, \ldots, m + 2 = p - 2$, of the vertices of G are covered by g and that g has weight $-m$. This proves assertion (b).
(c) and (d) Let f be as before.
Suppose first that x is the only vertex not covered by f. We distinguish two cases:

Case 1 $x \in P$
Each $u \in M$ is adjacent to at least two vertices of P. Hence there exists at least $2|M|$ edges between M and P. To ensure that $w \in P - \{x\}$ is covered, it is adjacent to at most $|P| - 1$ vertices in M. Hence there are at most $|M| + (|P| - 1)^2$ edges between P and M. But then $2|M| \leq |M| + (|P| - 1)^2$ which implies that $|P| + m = |M| \leq (|P| - 1)^2$, whence $m \leq (|P| - 1)^2 - |P|$. Let $\ell_1 = \min\{\ell \in \mathbf{Z}^+ | m \leq (\ell - 1)^2 - \ell\}$. Then $\ell_1 \leq |P|$, so that $p \geq |P| + |M| \geq \ell_1 + \ell_1 + m = 2\ell_1 + m$.
Case 2 $x \in M \cup Z$.
As before, there exists at least $2(|M| - 1)$ edges between M and P. To ensure that $w \in P$ is covered, it is adjacent to at most $|P| - 1$ vertices in M. Hence there exists at most $|P|(|P| - 1)$ edges between P and M, so that $2|M| - 2 \leq |P|^2 - |P|$, which implies that $2|P| + 2 \leq |P|^2 - |P| + 2$, whence $m \leq \frac{|P|^2 - 3|P| + 2}{2}$. Let $\ell_2 = \min\{\ell \in \mathbf{Z}^+ | m \leq \frac{\ell^2 - 3\ell + 2}{2}\}$. Then $|P| \geq \ell_2$, so that $p \geq |P| + |M| \geq \ell_2 + (\ell_2 + m) = 2\ell_2 + m$.
Now suppose that all the vertices of G are covered by f. Then each vertex of M is adjacent to at least two vertices of P. Again, there are at most $|P|(|P| - 1)$ edges between P and M, so that $2|P| + 2m = 2|M| \leq |P|^2 - |P|$, whence $m \leq \frac{|P|^2 - 3|P|}{2}$. Let $\ell_3 = \min\{\ell \in \mathbf{Z}^+ | m \leq \frac{\ell^2 - 3\ell}{2}\}$. Then $\ell_3 \leq |P|$, so that $p \geq 2\ell_3 + m$.
The inequalities $\ell_1 \leq \ell_2$ and $\ell_1 \leq \ell_3$ now follow readily. This proves (c) and (d).
We now show that these results are best possible by constructing a connected graph G of order $p = 2\ell_1 + m$ such that $\gamma_{ks}^{-101}(G) = -m$. Let $\ell = \ell_1$.
Construct the graph G as follows: take a complete graph K_ℓ with vertex set $U = \{u_1, \ldots, u_\ell\}$ and a set of isolated vertices $V = \{v_1, \ldots, v_{\ell+m}\}$. Join u_ℓ to every vertex in the set V, join u_i to every vertex in the set $\{v_{1+(i-1)(\ell-1)}, \ldots, v_{i(\ell-1)}\}$ for $i = 1, \ldots, \lfloor \frac{m+\ell}{\ell-1} \rfloor$ and join $u_{\lfloor \frac{m+\ell}{\ell-1} \rfloor + 1}$ to every vertex in the set $\{v_{1+\lfloor \frac{m+\ell}{\ell-1} \rfloor (\ell-1)}, \ldots, v_{m+\ell}\}$. Note that this is possible since $(\ell - 1)^2 \geq m + \ell$. Note also that every u_i, $i \neq \ell$, has at most $\ell - 1$ neighbors in V and that every v_i has two neighbors in U. We now prove that $\gamma_{(p-1)s}^{101}(G) = -m$: Define g by $g(v) = 1$ if $v \in U$ and $g(v) = -1$ if $v \in V$. It is easy to check that every vertex, except u_ℓ, is covered by g, so that g is $(p-1)SF$ of G. Since the weight of g is equal to $-m$, it follows that $\gamma_{(p-1)s}^{-101}(G) \leq -m$.
To prove the inequality $\gamma_{(p-1)s}^{-101}(G) \geq -m$, let f be a $(p-1)SF$ of weight $\gamma_{(p-1)s}^{-101}(G)$. Since the vertex u_ℓ dominates the graph G, it follows that $f[u_\ell] = \gamma_{(p-1)s}^{-101}(G) \leq -m$. However, since only one vertex is not covered, every other vertex must be covered.

We now prove that $u_\ell \in P$. Suppose, to the contrary, that $u_\ell \in M \cup Z$. Let $j \in \{1, \ldots, m+\ell\}$. Then, since $f[v_j] = f(\{v_j, u_i, u_\ell\}) \geq 1$, we must have that $f(v_j) + f(u_i) \geq 1$, which implies that $f(v_j) \geq 0$ and $f(u_i) \geq 0$. Hence $f(v_j) \geq 0$ for all $j \in \{1, \ldots, m+\ell\}$ and $f(u_i) \geq 0$ for all $i = 1, \ldots, \lfloor \frac{m+\ell}{\ell-1} \rfloor + 1$. If $\lfloor \frac{m+\ell}{\ell-1} \rfloor + 1 \geq \ell - 1$, then, since there is at least one vertex that has been assigned the value $+1$ under f, it follows that $f(V(G)) \geq 0$, which is a contradiction. We may, therefore, assume that $\lfloor \frac{m+\ell}{\ell-1} \rfloor + 1 < \ell - 1$. Then $N[u_{\ell-1}] = U$, so that $f[u_{\ell-1}] = f[U] \geq 1$, whence $f[u_\ell] = f(V(G)) = f[U] + f[V] \geq 1$, which is a contradiction. We conclude that $u_\ell \in P$ and that Case 1 can be applied.

Suppose $\gamma^{-101}_{(p-1)s}(G) = -m' < -m$. Let $\ell' = \min\{\ell \in \mathbf{Z}^+ | m' \leq (\ell-1)^2 - \ell\}$. Then $m' \leq (\ell' - 1)^2 - \ell'$, so that $m \leq (\ell' - 1)^2 - \ell'$ (since $m < m'$). We conclude that $\ell \leq \ell'$. The proof of Case 1 then implies that $p \geq 2\ell' + m'$. But $2\ell + m = p \geq 2\ell' + m' \geq 2\ell + m'$, so that $m \geq m'$, which is a contradiction. The proof that (c) is best possible is now complete. Assertion (d), albeit in a different guise, first appeared in [2], where it is also shown that this result is best possible. ■

For each integer $q \geq 1$, let $I_q = \{q(q+1)/2, q(q+1)/2+1, \ldots, q(q+1)/2+q\}$. Then the smallest integer in I_q is one larger than the largest integer in I_{q-1} (if $q \geq 2$), while the largest integer in I_q is one smaller than the smallest integer in I_{q+1}. Hence each positive integer is contained in a unique interval I_q for some $q \geq 1$. The following theorem appears in [2].

Theorem 5 *Let $q \geq 1$ be an integer and $m \in I_q$. Let G be a connected graph of order p with $\gamma^{-101}_{ps}(G) = -m$. Then $p \geq 2(q+3) + m$.* ■

The statement of Theorem 5 and statement (d) of Theorem 4 are equivalent, as may be seen from the following: If $q = \ell - 3$, then $m \leq \frac{\ell^2 - 3\ell}{2}$ if and only if $m \leq \frac{q(q+1)}{2} + q$.

For each integer $q \geq 1$, let $J_q = \{(q-1)q, (q-1)q+1, \ldots, (q-1)q+2q-1\}$. Then the smallest integer in J_q is one larger than the largest integer in J_{q-1} (if $q \geq 2$), while the largest integer in J_q is one smaller than the smallest integer in J_{q+1}. Hence, each positive integer is contained in a unique interval J_q for some $q \geq 1$. In this way we obtain an equivalent statement for statement (c) of Theorem 4 by letting $q = \ell - 2$.

Theorem 6 *Let $q \geq 1$ be an integer and $m \in J_q$. Let G be a connected graph of order p with $\gamma^{-101}_{(p-1)s}(G) = -m$. Then $p \geq 2(q+2) + m$.* ■

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