Minimum Order of a Graph with Given Deficiency and Either Minimum or Maximum Degree

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ABSTRACT. Let G be a simple graph of order n having a maximum matching M. The deficiency def(G) of G is the number of vertices unsaturated by M. In this paper we find lower bounds for n when def(G) and the minimum degree (or maximum degree) of vertices are given. Further, for every n not less than the bound and of the same parity as def(G), there exists a graph G with the given deficiency and minimum (maximum) degree.

1 Introduction

In this paper all graphs are finite and have neither loops nor multiple edges. For most of our notation and terminology we follow that of Bondy and Murty [3]. Let $G = (V, E)$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The *order* and *size* of G are $|V(G)|$ and $|E(G)|$ respectively. The *minimum* and *maximum* degrees are denoted by $\delta(G)$ and $\Delta(G)$ respectively.

A *matching* M in G is a subset of $E(G)$ in which no two edges have a vertex in common. M is a *maximum matching* if $|M| \geq |M'|$ for any other matching M' of G. A vertex v is *saturated* by M if an edge of M is incident

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with v, otherwise v is said to be *unsaturated*. A matching M is called a *perfect matching* (or one-factor) if it saturates every vertex of the graph.

The *deficiency* $\operatorname{def}(G)$ of a graph G is the number of vertices unsaturated by a maximum matching. Thus, if $\operatorname{def}(G) = 0$, then G has a perfect matching. If M is a maximum matching, then $\operatorname{def}(G) = |V(G)| - 2|M|$, so $\operatorname{def}(G)$ and $|V(G)|$ have the same parity.

Many problems concerning matchings in graphs have been studied; [6] is a very good reference. Bollobás and Eldridge [2] have studied the greatest lower bound of the size of a matching in a graph of given order, minimal degree and maximal degree of vertices. The deficiencies of regular graphs have been studied in [4] and [5].

In this paper we study the lower bound of the order of a connected graph G when $\operatorname{def}(G)$ and $\delta(G)$ are given. We obtain a lower bound and show that for every n not less than the bound and of the same parity as $\operatorname{def}(G)$, there exists a connected graph of order n having deficiency $\operatorname{def}(G)$ and minimum degree δ. We also obtain the corresponding result when minimum degree is replaced by maximum degree.

2 The Bounds

Let G be a graph. If S is a subset of $V(G)$, $G-S$ denotes the graph formed from G by deleting all the vertices in S together with their incident edges. A component of G is called *odd* or *even* according as its order is odd or even. The number of odd components of a graph G is denoted by $o(G)$. We need Berge's formula ([1], p159) to establish our results.

Berge's Formula:

$$\operatorname{def}(G) = \max_{S \subset V(G)} \{o(G-S) - |S|\}$$

□

Our first result is on the lower bound of the order of a connected graph G when $\operatorname{def}(G)$ and $\delta(G)$ are given.

Theorem 1. *Let G be a connected graph of order n with $\delta(G) = \delta$, $\delta \geq 1$. If $\operatorname{def}(G) = d$, then*

(a) $n \geq \delta + 1$, *for odd δ and $d = 0$ or even δ and $d = 1$*

(b) $n \geq \delta + 2$, *for even δ and $d = 0$ or odd δ and $d = 1$*

(c) $n \geq 2\delta + d$, *otherwise.*

Proof: Parts (a) and (b) are obvious. So suppose that $d \geq 2$. By Berge's formula, there exists a vertex set $S \subset V(G)$ such that

$$o(G-S) = |S| + d.$$

Since $d \geq 2$, therefore $|S| \geq 1$.

Let n_0 be the minimum order of odd components of $G - S$. Since the minimum degree is δ,

$$|S| \geq \delta + 1 - n_0.$$

Counting the number of vertices, we have

$$\begin{aligned} n &\geq |S| + o(G - S)n_0 \\ &= |S| + (|S| + d)n_0. \end{aligned}$$

If $n_0 \geq \delta$, then

$$\begin{aligned} n &\geq 1 + (1 + d)\delta \\ &= 1 + \delta + d\delta \\ &\geq 2\delta + d \end{aligned}$$

If $n_0 \leq \delta - 1$, then we use $|S| \geq \delta + 1 - n_0$, and

$$\begin{aligned} n &\geq |S| + (|S| + d)n_0 \\ &\geq (\delta + 1 - n_0) + (\delta + 1 - n_0 + d)n_0 \\ &= -n_0^2 + (\delta + d)n_0 + \delta + 1 \end{aligned}$$

Since $1 \leq n_0 \leq \delta - 1$, we have

$$\begin{aligned} n &\geq \min\{-(1)^2 + (\delta + d)(1) + \delta + 1, \\ &\quad - (\delta - 1)^2 + (\delta + d)(\delta - 1) + \delta + 1\} \\ &= \min\{2\delta + d, 2\delta + (\delta - 1)d\} \\ &= 2\delta + d. \end{aligned}$$

□

When a graph G of even order has no perfect matching, obviously $\mathrm{def}(G) \geq 2$. Theorem 1 has the following corollary.

Corollary 1. *Let G be a connected graph of order n, n even, with minimum degree δ. If G has no perfect matching, then $n \geq 2\delta + 2$.* □

Let G be a connected graph with $\Delta(G) = \Delta$. If $\Delta = 0$, then $\mathrm{def}(G) = 1$; if $\Delta = 1$, then $\mathrm{def}(G) = 0$; and if $\Delta = 2$, then $\mathrm{def}(G)$ is 0 or 1 according as the order of G is even or odd. The next result is on the lower bound of the order of G when $\mathrm{def}(G)$ and the maximum degree Δ are given, and $\Delta > 2$.

Theorem 2. *Let G be a connected graph of order n with $\Delta(G) = \Delta, \Delta \geq 3$.*

If $\mathrm{def}(G) = d$, then

(a) $n \geq \Delta + 1$, *when* $d < \Delta$ *and* d *has a different parity from* Δ,

(b) $n \geq \Delta + 2$, *when* $d < \Delta$ *and* d *has the same parity as* Δ,

(c) $n \geq 2\lceil \frac{d-1}{\Delta-2} \rceil + d$, *otherwise.*

Proof: Parts (a) and (b) are obvious, since $n \geq \Delta + 1$ and n has the same parity as d. Suppose $d \geq \Delta$. By Berge's formula, there exists a vertex set $S \subset V(G)$ such that

$$o(G - S) = |S| + d.$$

Since G is connected, we must have

$$\begin{aligned}\Delta|S| &\geq o(G - S) + |S| - 1 \\ &= 2|S| + d - 1,\end{aligned}$$

or

$$|S| \geq \left\lceil \frac{d-1}{\Delta - 2} \right\rceil.$$

Hence

$$\begin{aligned} n &\geq |S| + o(G - S) \\ &= 2|S| + d \\ &\geq 2\left\lceil \frac{d-1}{\Delta-2} \right\rceil + d \end{aligned}$$

□

3 The Constructions

In this section we will show that, for every n not less than the bounds in Theorem 1 or Theorem 2 such that n has the same parity as def(G), there exists a connected graph of order n with deficiency def(G) and with given minimum or maximum degree. This implies that the bounds are sharp.

Theorem 3. *Let δ and d be non-negative integers and let*

$$n_1 = \begin{cases} \delta + 1, & \text{if } \delta \text{ is odd and } d = 0 \text{ or } \delta \text{ is even and } d = 1 \\ \delta + 2, & \text{iff } \delta \text{ is even and } d = 0 \text{ or } \delta \text{ is odd and } d = 1 \\ 2\delta + d, & \text{otherwise.} \end{cases}$$

Then for every integer $n \geq n_1$, there exists a connected graph G of order n with $\delta(G) = \delta$ and def$(G) = d$.

Proof: Choose $d = 0$ or $d = 1$ and $n \geq n_1$. A graph G_1 of order n with $\delta(G_1) = \delta$ and $\mathrm{def}(G_1) = d$ can be formed from K_n by deleting $n - 1 - \delta$ edges which have one vertex in common.

Choose $d \geq 1$ and $n \geq 2\delta + d$. A graph G_2 of order n with $\delta(G_2) = \delta$ and $\mathrm{def}(G_2) = d$ can be formed as follows. Take an empty graph $\bar{K}_n$ with vertices $v_1, v_2, \ldots, u_a, u_1, u_2, \ldots, u_b$, where $a = \frac{n+d}{2}$ and $b = \frac{n-d}{2}$. Then for every i, $1 \leq i \leq a$, join v_i to every u_j where $j \equiv i + t \pmod{b}$, $1 \leq t \leq \delta$. □

Theorem 4. *Let Δ and d be non-negative integers, $\Delta \geq 3$, and let*

$$n_1 = \begin{cases} \Delta + 1, & \text{if } d < \Delta \text{ and } d \text{ has a different parity from } \Delta, \\ \Delta + 2, & \text{if } d < \Delta \text{ and } d \text{ has the same parity as } \Delta, \\ 2\lceil \frac{d-1}{\Delta-2} \rceil + d, & \text{otherwise.} \end{cases}$$

Then for every integer $n \geq n_1$, there exists a connected graph of order n with$\Delta(G) = \Delta$ and $\mathrm{def}(G) = d$.

Proof: Suppose $d < \Delta$ and $n \geq n_1$. We form a graph G_3 as follows. Take a star $K_{1,\Delta}$ with center v_1 and other vertices $v_2, v_3, \ldots, v_{\Delta+1}$. If $d \leq \Delta - 3$, then join v_{2i}, to v_{2i+1} for every i such that, $1 \leq i \leq \frac{\Delta-1-d}{2}$. Also, if $n > \Delta + 1$, then take a path $u_1, u_2, \ldots, u_{n-\Delta-1}$ and join u_1 to $v_{\Delta+1}$. The resulting graph G_3 is a connected graph of order n with $\Delta(G_3) = \Delta$ and $\mathrm{def}(G_3) = d$.

Suppose $d \geq \Delta$ and $n \geq n_1$. Let $s = \lceil \frac{d-1}{\Delta-2} \rceil$. We form a graph G_4 as follows. Take an empty graph $\bar{K}_{2s+d}$ with vertices $v_1, v_2, \ldots, v_s, u_1, u_2, \ldots, u_{s+d}$. For every v_i, $1 \leq i \leq s$, join v_i to u_j, for every $j \equiv (\Delta-1)(i-1)+t (\mathrm{mod}\, s+d)$, $1 \leq t \leq \Delta$. If $n > 2s + d$, which implies $n \geq 2s + d + 2$, then also take a path $w_1, w_2, \ldots, w_{n-2s-d}$ and join w_1 to u_1. The resulting graph G_4 is connected, of order n, with $\Delta(G_4) = \Delta$ and $\mathrm{def}(G_4) = d$. □

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