Some New Partition Identities

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July 29, 1998

ABSTRACT: Using the Jacobi triple product identity and the quintuple product identity, we obtain identities involving several partition functions.

KEYWORDS: partition, Jacobi triple product identity, quintuple product identity

AMS CLASSIFICATION: 11P81, 11P83

INTRODUCTION: If n is a natural number, let $p(n)$, $q(n)$, $q_0(n)$, $g(n)$ denote respectively the number of partitions of n without restriction, into distinct parts, into distinct odd parts, into distinct parts such that if $2m$ is a part, then m is odd and m is not a part. If $f(n)$ is any of these functions, define $f(0) = 1$, and $f(n) = 0$ if n is not a non-negative integer. Using the Jacobi triple product identity and the quintuple product identity, we obtain identities concerning the four above-mentioned partition functions.

PRELIMINARIES

Definition 1 (Pentagonal numbers): If $n \in \mathbf{Z}$, let $\omega(n) = \frac{1}{2}n(3n-1)$.

Identities

Let $|x| < 1$, $z \neq 0$. Then

(1) (Jacobi triple product identity)

$$\prod_{n=1}^{\infty}(1-x^{2n})(1+x^{2n-1}z)(1+x^{2n-1}z^{-1}) = 1 + \sum_{n=1}^{\infty} x^{n^2}(z^n + z^{-n})$$

(2) (quintuple product identity)

$$(1-z^{-1})\prod_{n=1}^{\infty}(1-x^n)(1-x^n z)(1-x^n z^{-1})(1-x^{2n-1}z^2)(1-x^{2n-1}z^{-2})$$

$$= \sum_{n=-\infty}^{\infty} x^{\omega(-n)}(z^{3n} - z^{-3n-1})$$

(3) $$\prod_{n=1}^{\infty}(1-x^n) = 1 + \sum_{n=1}^{\infty}(-1)^n(x^{\omega(n)} + x^{\omega(-n)})$$

(4) $$\prod_{n=1}^{\infty}(1-x^n)^{-1} = \sum_{n=0}^{\infty} p(n)x^n$$

(5) $$\prod_{n=1}^{\infty}(1+x^n) = \prod_{n=1}^{\infty}(1-x^{2n-1})^{-1} = \sum_{n=0}^{\infty} q(n)x^n$$

(6) $$\prod_{n=1}^{\infty}(1+x^{2n-1}) = \sum_{n=0}^{\infty} q_0(n)x^n$$

(7) $$\prod_{n=1}^{\infty}(1+x^n)^{-1} = \sum_{n=0}^{\infty}(-1)^n q_0(n)x^n$$

(8) $$\prod_{n=1}^{\infty}(1+x^{2n-1}+x^{4n-2})=\sum_{n=0}^{\infty}g(n)x^n$$

(9) $$\prod_{n=1}^{\infty}(1-x^{dn})=1+\sum_{n=1}^{\infty}(-1)^n(x^{d\omega(n)}+x^{d\omega(-n)})$$

(10) $$\prod_{n=1}^{\infty}(1-x^{dn})^{-1}=\sum_{n=0}^{\infty}p(\tfrac{n}{d})x^n$$

(11) $$\prod_{n=1}^{\infty}(1+x^{dn})=\prod_{n=1}^{\infty}(1-x^{d(2n-1)})^{-1}=\sum_{n=0}^{\infty}q(\tfrac{n}{d})x^n$$

(12) $$\prod_{n=1}^{\infty}(1+x^{dn})^{-1}=\sum_{n=0}^{\infty}(-1)^n q_0(\tfrac{n}{d})x^n$$

(13) $$\prod_{n=1}^{\infty}(1-x^n)^3=\sum_{n=0}^{\infty}(-1)^n(2n+1)x^{\frac{1}{2}n(n+1)} \quad \text{(Jacobi)}$$

(14) $$(\sum_{n=0}^{\infty}a(n)x^n)(\sum_{n=0}^{\infty}b(n)x^n)=\sum_{n=0}^{\infty}(\sum_{k=0}^{n}a(n-k)b(k))x^n$$

Remarks: (1) is Theorem 352, p. 282 in [4]. For a proof of (2), see [2] or [3]. (3), (4), (13) are Theorems 14.3, 14.2, 14.7 respectively in [1]. As for (5) and (6), see Table 14.1 in [1], as well as Theorem 344 in [4]. (7) follows from (5) and (6). (8) is mentioned in [6]. (9) through (12) follow from (3), (4), (5), and (7) respectively. (14) is the Cauchy product (see [5]).

THE MAIN RESULTS

Theorem 1

$$q_0(n)^2 + 2\sum_{k=0}^{n} q_0(k)q_0(2n-k) = p(n) + 2\sum_{j\geq 1} p(n-2j^2) \tag{15}$$

$$\sum_{k=0}^{n} q_0(k)q_0(2n+1-k) = \sum_{j\geq 0} p(n-2j^2-2j) \tag{16}$$

$$(-1)^n q_0(n) = p(n) + 2\sum_{j\geq 1}(-1)^j p(n-j^2) \tag{17}$$

$$\sum_{j\geq 0}(-1)^{n+j-\omega(\pm j)} q_0(n-\omega(\pm j)) = \begin{cases} 1 & \text{if } n=0 \\ 2(-1)^r & \text{if } n=r^2>0 \\ 0 & \text{otherwise} \end{cases} \tag{18}$$

Proof: Set $z=1$ in (1) to obtain:

$$\prod_{n=1}^{\infty}(1-x^{2n})(1+x^{2n-1})^2 = 1+2\sum_{n=1}^{\infty} x^{n^2}.$$

Let

$$b(n) = \begin{cases} 1 & \text{if } n=0 \\ 2 & \text{if } n=r^2>0 \\ 0 & \text{otherwise} \end{cases}$$

so that

$$\sum_{n=0}^{\infty} b(n)x^n = 1+2\sum_{n=1}^{\infty} x^{n^2}.$$

Then we have

$$\prod_{n=1}^{\infty}(1-x^{2n})(1+x^{2n-1})^2 = \sum_{n=0}^{\infty} b(n)x^n. \tag{19}$$

Thus

$$\prod_{n=1}^{\infty}(1+x^{2n-1})^2 = \left(\prod_{n=1}^{\infty}(1-x^{2n})^{-1}\right)\left(\sum_{n=0}^{\infty} b(n)x^n\right).$$

Now (6) and (10) imply

$$(\sum_{n=0}^{\infty} q_0(n)x^n)^2 = (\sum_{n=0}^{\infty} p(\tfrac{n}{2})x^n)(\sum_{n=0}^{\infty} b(n)x^n);$$

(13) implies

$$\sum_{t=0}^{\infty}(\sum_{k=0}^{t} q_0(k)q_0(t-k))x^t = \sum_{t=0}^{\infty}(\sum_{k=0}^{t} p(\tfrac{t-k}{2})b(k))x^t.$$

Matching coefficients of like powers of x, we have

$$\sum_{k=0}^{t} q_0(k)q_0(t-k) = \sum_{k=0}^{t} p(\tfrac{t-k}{2})b(k).$$

But

$$\sum_{k=0}^{t} p(\tfrac{t-k}{2})b(k) = p(\tfrac{t}{2}) = 2\sum_{m\geq 1} p(\tfrac{t-m^2}{2}),$$

so that

$$\sum_{k=0}^{t} q_0(k)q_0(t-k) = p(\tfrac{t}{2}) + 2\sum_{m\geq 1} p(\tfrac{t-m^2}{2}).$$

If $t = 2n$, we have

$$\sum_{k=0}^{2n} q_0(k)q_0(2n-k) = p(n) + 2\sum_{m\geq 1} p(n - \tfrac{1}{2}m^2),$$

that is, we have (15).

If $t = 2n+1$, then

$$\sum_{k=0}^{2n} q_0(k)q_0(2n+1-k) = p\left(\frac{2n+1}{2}\right) + 2\sum_{m\geq 1} p\left(\frac{2n+1-m^2}{2}\right),$$

that is,

$$2\sum_{k=0}^{n} q_0(k)q_0(2n+1-k) = 2\sum_{j\geq 0} p\left(\frac{2n+1-(2j+1)^2}{2}\right),$$

which yields (16).

If we replace x by $-x$ in (19), we get

$$\prod_{n=1}^{\infty}(1-x^{2n})(1-x^{2n-1})^2 = \sum_{n=0}^{\infty}(-1)^n b(n)x^n.$$

Simplifying, we have

$$\prod_{n=1}^{\infty}(1-x^{n})(1-x^{2n-1}) = \sum_{n=0}^{\infty}(-1)^n b(n)x^n, \tag{20}$$

so that

$$\prod_{n=1}^{\infty}(1-x^{2n-1}) = (\prod_{n=1}^{\infty}(1-x^{n}))^{-1})(\sum_{n=0}^{\infty}(-1)^n b(n)x^n).$$

Now (4) and (6) imply

$$\sum_{n=0}^{\infty}(-1)^n q_0(n)x^n = (\sum_{n=0}^{\infty}p(n)x^n)(\sum_{n=0}^{\infty}(-1)^n b(n)x^n).$$

Invoking (14), we get

$$\sum_{n=0}^{\infty}(-1)^n q_0(n)x^n = \sum_{n=0}^{\infty}(\sum_{k=0}^{n}p(n-k)(-1)^k b(k))x^n.$$

Matching coefficients of like powers of x, we have

$$(-1)^n q_0(n) = \sum_{k=0}^{n}p(n-k)(-1)^k b(k),$$

from which Equation (17) follows.

Write (20) as

$$\prod_{n=1}^{\infty}(1-x^{2n-1})\prod_{n=1}^{\infty}(1-x^{n}) = \sum_{n=0}^{\infty}(-1)^n b(n)x^n.$$

Now (3) implies

$$\prod_{n=1}^{\infty}(1-x^{n}) = \sum_{n=0}^{\infty}c(n)x^n \text{ where } c(n) = \begin{cases} (-1)^j \text{ if } n=\omega(\pm j) \\ 0 \text{ otherwise.} \end{cases}$$

By virtue of (6), we have

$$(\sum_{n=0}^{\infty}(-1)^n q_0(n)x^n)(\sum_{n=0}^{\infty} c(n)x^n) = \sum_{n=0}^{\infty}(-1)^n b(n)x^n.$$

Invoking (14) and matching coefficients of like powers of x, we get

$$\sum_{n=0}^{\infty}(-1)^{n-k}q_0(n-k)c(k) = \begin{cases} 1 & \text{if } n=0 \\ 2(-1)^r & \text{if } n=r^2>0 \\ 0 & \text{otherwise,} \end{cases}$$

from which (18) follows. QED

Remarks: In [6], we set $z = i$ in (1) to obtain a result similar to that of Equation (18), namely

Theorem A:

$$q_0(n) + \sum_{k\geq 1}(-1)^k(q_0(n-\omega(k)) + q_0(n-\omega(-k))) = \begin{cases} 2(-1)^m \text{ if } n = 2m^2 \\ 0 \text{ otherwise.} \end{cases}$$

(Theorem A is Theorem 2 in [6]).

Theorem 2:

(21)
$$q_0(n) + \sum_{m\geq 1}(-1)^m(q_0(n-2\omega(m)) + q_0(n-2\omega(-m)))$$
$$= \begin{cases} (-1)^{[(1\mp r)/2]} \text{ if } n = \omega(\pm r) \\ 0 \text{ otherwise} \end{cases}$$

(22)
$$q_0(n) = \sum_{m\geq 0}(-1)^{[(1\mp m)/2]}p\left(\frac{n-\omega(\pm m)}{2}\right).$$

Proof: Setting $z = i$ in (2), we get

$$(1+i)\prod_{n=1}^{\infty}(1-x^n)(1-ix^n)(1+ix^n)(1+x^{2n-1})^2$$

$$= \sum_{n=-\infty}^{\infty} x^{\omega(-n)}(i^{3n} + i^{-3n-1}),$$

that is,

$$(1+i)\prod_{n=1}^{\infty}(1-x^n)(1+x^{2n})(1+x^{2n-1})^2 = \sum_{n=-\infty}^{\infty} x^{\omega(-n)}((-i)^n + i(-i)^n).$$

Thus

$$(1+i)\prod_{n=1}^{\infty}(1-x^n)(1+x^n)(1+x^{2n-1}) = (1+i)\sum_{n=0}^{\infty} x^{\omega(-n)}(-1)^{[(n+1)/2]},$$

which yields

$$\prod_{n=1}^{\infty}(1-x^{2n})(1+x^{2n-1}) = \sum_{n=0}^{\infty}(-1)^{[(n+1)/2]}x^{\omega(-n)},$$

that is,

$$\prod_{n=1}^{\infty}(1-x^{2n})(1+x^{2n-1}) = 1 + \sum_{n=1}^{\infty}((-1)^{[(1-n)/2]}x^{\omega(n)} + (-1)^{[(1+n)/2]}x^{\omega(-n)}).$$

If we let

$$h(n) = \begin{cases} (-1)^{[(1\mp m)/2]} \text{ if } n = \omega(\pm m) \\ 0 \text{ otherwise} \end{cases}$$

then we have

$$\prod_{n=1}^{\infty}(1-x^{2n})(1+x^{2n-1}) = \sum_{n=0}^{\infty} h(n)x^n. \tag{23}$$

Now (6) and (9) imply

$$(\prod_{n=0}^{\infty} q(n)x^n)(\prod_{n=0}^{\infty} f(n)x^n) = \sum_{n=0}^{\infty} h(n)x^n,$$

where

$$f(n) = \begin{cases} (-1)^j \text{ if } n = 2\omega(\pm j) \\ 0 \text{ otherwise} \end{cases}$$

Therefore, invoking (14) and matching coefficients of like powers of x, we get

$$\sum_{k=0}^{n} q_0(n-k)f(k) = h(n),$$

that is, we get (21).

Moreover, (23) implies that

$$\prod_{n=1}^{\infty}(1+x^{2n-1}) = (\prod_{n=1}^{\infty}(1-x^{2n})^{-1})(\sum_{n=0}^{\infty} h(n)x^n).$$

Now (6) and (10) imply

$$\sum_{n=0}^{\infty} q_0(n)x^n = (\sum_{n=0}^{\infty} p(\tfrac{n}{2})x^n)(\sum_{n=0}^{\infty} h(n)x^n).$$

Therefore, again invoking (14) and matching coefficients of like powers of x, we get $q_0(n) = \sum_{k=0}^{n} p(\frac{n-k}{2})h(k)$, from which (22) follows. QED

Theorem 3:

(24)
$$q(n) + \sum_{j\geq 1}(-1)^j(q(n-3\omega(j)) + q(n-3\omega(-j)))$$
$$= q\left(\frac{n}{3}\right) + \sum_{j\geq 1}\left(q\left(\frac{n-\omega(j)}{3}\right) + q\left(\frac{n-\omega(-j)}{3}\right)\right).$$

(25)
$$g(n) = p\left(\frac{n}{3}\right) + \sum_{j\geq 1}\left(p\left(\frac{n-\omega(j)}{3}\right) + p\left(\frac{n-\omega(-j)}{3}\right)\right).$$

Proof: Write (2) as:

$$(1-z^{-1})\prod_{n=1}^{\infty}(1-x^n)(1-(z+z^{-1})x^n + x^{2n})(1-(z^2+z^{-2})x^{2n-1} + x^{4n-2})$$
$$= \sum_{n=-\infty}^{\infty} x^{\omega(-n)}(z^{3n} - z^{-3n-1}).$$

Now we let $z = e^{2\pi i/3}$ to obtain

$$(1 - e^{-2\pi i/3}) \prod_{n=1}^{\infty} (1 - x^n)(1 + x^n + x^{2n})(1 + x^{2n-1} + x^{4n-2})$$

$$= \sum_{n=-\infty}^{\infty} x^{\omega(-n)}(1 - e^{-2\pi i/3}),$$

so that

$$\prod_{n=1}^{\infty} (1 - x^{3n})(1 + x^{2n-1} + x^{4n-2}) = \sum_{n=-\infty}^{\infty} x^{\omega(-n)}.$$

If we let

$$d(n) = \begin{cases} 1 \text{ if } n = \omega(\pm k) \\ 0 \text{ otherwise} \end{cases}$$

then we have

$$\prod_{n=1}^{\infty} (1 + x^{2n-1} + x^{4n-2})(1 - x^{3n}) = \sum_{n=0}^{\infty} d(n) x^n, \tag{26}$$

that is,

$$\prod_{n=1}^{\infty} (1 - x^{2n-1})^{-1}(1 - x^{6n-3})(1 - x^{3n}) = \sum_{n=0}^{\infty} d(n) x^n.$$

Thus

$$\prod_{n=1}^{\infty} (1 - x^{2n-1})^{-1}(1 - x^{6n-3})(1 - x^{3n})(1 + x^{3n}) = (\prod_{n=1}^{\infty} (1 + x^{3n}))(\sum_{n=0}^{\infty} d(n) x^n),$$

which simplifies to

$$\prod_{n=1}^{\infty} (1 - x^{2n-1})^{-1}(1 - x^{3n}) = (\prod_{n=1}^{\infty} (1 + x^{3n}))(\sum_{n=0}^{\infty} d(n) x^n).$$

Now (5), (9) and (11) imply

$$(\sum_{n=0}^{\infty} q(n) x^n)(1 + \sum_{n=1}^{\infty} (-1)^n (x^{3\omega(n)} + x^{3\omega(-n)})) = (\sum_{n=0}^{\infty} q(\tfrac{n}{3}) x^n)(\sum_{n=0}^{\infty} d(n) x^n).$$

The conclusion, namely (24), follows from invoking (14) and matching coefficients of like powers of x. Note also that (26) implies

$$\prod_{n=1}^{\infty}(1+x^{2n-1}+x^{4n-2}) = (\prod_{n=1}^{\infty}(1-x^{3n})^{-1})(\sum_{n=0}^{\infty}d(n)x^n).$$

Therefore (8) and (10) imply

$$\sum_{n=0}^{\infty}g(n)x^n = (\sum_{n=0}^{\infty}p(\tfrac{n}{3})x^n)(\sum_{n=0}^{\infty}d(n)x^n).$$

Invoking (14) and matching coefficients of like powers of x, we obtain (25).QED

Theorem 4:

(27) $$\sum_{k=0}^{n}q(\tfrac{n-k}{3})g(k) = q(n).$$

(28) $$g(n) = \sum_{k=0}^{n}(-1)^{n-k}q_0(\tfrac{n-k}{3})q(k).$$

Proof: Set $z = e^{\pi i/3}$ in (2) to obtain:

$$(1-e^{-\pi i/3})\prod_{n=1}^{\infty}(1-x^n)(1-x^n+x^{2n})(1+x^{2n-1}+x^{4n-2})$$

$$= \sum_{n=-\infty}^{\infty} x^{\omega(-n)}((-1)^n - (-1)^n e^{-\pi i/3})$$

$$= (1-e^{-\pi i/3})(1+\sum_{n=1}^{\infty}(-1)^n(x^{\omega(n)}+x^{\omega(-n)})).$$

Now (3) implies

$$\prod_{n=1}^{\infty}(1-x^n)(1-x^n+x^{2n})(1+x^{2n-1}+x^{4n-2}) = \prod_{n=1}^{\infty}(1-x^n),$$

hence

$$\prod_{n=1}^{\infty}(1-x^n+x^{2n})(1+x^{2n-1}+x^{4n-2}) = 1,$$

so that

$$\prod_{n=1}^{\infty}(1+x^n)(1-x^n+x^{2n})(1+x^{2n-1}+x^{4n-2}) = \prod_{n=1}^{\infty}(1+x^n),$$

that is,

$$\prod_{n=1}^{\infty}(1+x^{3n})(1+x^{2n-1}+x^{4n-2}) = \prod_{n=1}^{\infty}(1+x^n). \tag{29}$$

Now (5), (8), and (11) imply

$$(\sum_{n=0}^{\infty} q(\tfrac{n}{3})x^n)(\sum_{n=0}^{\infty} g(n)x^n) = \sum_{n=0}^{\infty} q(n)x^n.$$

Invoking (14) and matching coefficients of like powers of x, we obtain (27). Note also that (29) implies

$$\prod_{n=1}^{\infty}(1+x^{2n-1}+x^{4n-2}) = (\prod_{n=1}^{\infty}(1+x^{3n})^{-1})(\prod_{n=1}^{\infty}(1+x^n)).$$

Now (5), (8), and (12) imply

$$\sum_{n=0}^{\infty} g(n)x^n = (\sum_{n=0}^{\infty}(-1)^n q(\tfrac{n}{3})x^n)(\sum_{n=0}^{\infty} q(n)x^n).$$

Invoking (14) and matching coefficients of like powers of x, we obtain (27).QED

Theorem 5:

$$\sum_{k\geq 0}(-1)^{n-\frac{1}{2}k(k+3)}(2k+1)q_0(n-\tfrac{1}{2}k(k+1))$$

$$= q(n) + \sum_{k\geq 1}((1-6k)q(n-\omega(k)) + (1+6k)q(n-\omega(-k))).$$

Proof: Write (2) as

$$(1-z^{-1})\prod_{n=1}^{\infty}(1-x^n)(1-x^n z)(1-x^n z^{-1})(1-x^{2n-1}z^2)(1-x^{2n-1}z^{-2})$$

$$= \sum_{n=-\infty}^{\infty} x^{\omega(-n)} \left(\frac{z^{6n+1}-1}{z^{3n+1}} \right).$$

Thus we have

(30) $$\prod_{n=1}^{\infty}(1-x^n)(1-x^n z)(1-x^n z^{-1})(1-x^{2n-1}z^2)(1-x^{2n-1}z^{-2})$$

$$= \sum_{n=-\infty}^{\infty} x^{\omega(-n)} \left(\frac{z^{6n+1}-1}{z^{3n}(z-1)} \right).$$

The right member of (30) may be written as $\sum_{n=-\infty}^{\infty} x^{\omega(-n)}(\sum_{j=-3n}^{3n} z^j)$. Setting $z=1$, we obtain

(31) $$\prod_{n=1}^{\infty}(1-x^n)^3(1-x^{2n-1})^2 = \sum_{n=-\infty}^{\infty}(1+6n)x^{\omega(-n)}.$$

Let

$$h(n) = \begin{cases} 1 \mp 6k \text{ if } n = \omega(\pm k) \\ 0 \text{ otherwise} \end{cases}$$

so that $\sum_{n=0}^{\infty} h(n)x^n$ is the right member of (31). This yields

$$(\prod_{n=1}^{\infty}(1-x^{2n-1}))(\prod_{n=1}^{\infty}(1-x^n)^3) = (\prod_{n=1}^{\infty}(1-x^{2n-1})^{-1})(\sum_{n=0}^{\infty}h(n)x^n).$$

Now (6), (13), and (5) imply

$$(\sum_{n=0}^{\infty}(-1)^n q_0(n)x^n)(\sum_{n=0}^{\infty}(-1)^n(2n+1)x^{\frac{1}{2}n(n+1)}) = (\sum_{n=0}^{\infty}q(n)x^n)(\sum_{n=0}^{\infty}h(n)x^n).$$

Let

$$r(n) = \begin{cases} (-1)^k(2k+1) \text{ if } n = \frac{1}{2}k(k+1) \\ 0 \text{ otherwise} \end{cases}$$

so that

$$\sum_{n=0}^{\infty} r(n)x^n = \sum_{n=0}^{\infty}(-1)^n(2n+1)x^{\frac{1}{2}n(n+1)}.$$

Then we have

$$(\sum_{n=0}^{\infty}(-1)^n q_0(n)x^n)(\sum_{n=0}^{\infty} r(n)x^n) = (\sum_{n=0}^{\infty} q(n)x^n)(\sum_{n=0}^{\infty} h(n)x^n).$$

Invoking (14) and matching like powers of x, we get

$$\sum_{j=0}^{n}(-1)^{n-j}q(n-j)r(j) = \sum_{j=0}^{n} q(n-j)h(j),$$

from which the conclusion follows. QED

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