# On Directed Incomplete Transversal Designs with Block Size Five

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ABSTRACT. Let $v$ and $u$ be positive integers. It is shown in this paper that the necessary condition for the existence of a directed TD$(5, v)$ - TD$(5, u)$, namely $v \geq 4u$, is also sufficient.

## 1 Introduction

Let $v$, $k$ and $\lambda$ be positive integers. A transversal design (TD) with parameters $v$, $k$ and $\lambda$, denote by TD$(k, \lambda; v)$, is a triple $(X, \mathbf{G}, \mathbf{A})$ where $X$ is $kv$-set (of points), $\mathbf{G}$ is a collection of $v$-subsets of $X$ (called groups) which partition $X$ and $\mathbf{A}$ is a collection of subsets of $X$ (called blocks), each meeting each group in exactly one point, such that every pairset of points from different groups occurs in exactly $\lambda$ blocks of $\mathbf{A}$. Thus it follows that each block contains $k$ points and there are $\lambda v^2$ blocks.

If we remove one or more subdesigns from a TD$(k, \lambda; v)$, we obtain a holey transversal design with index $\lambda$. In the case of one hole, it is called incomplete transversal design with index $\lambda$. This notion for $\lambda = 1$ is introduced by J. Horton [13] under the name of incomplete array. In the sequel, we write TD$(k, \lambda; v) - \Sigma_{1 \leq i \leq r}$ TD$(k, \lambda; u_i)$ for a structure $(X, (Y_i)_{i \leq r}, \mathbf{G}, \mathbf{A})$ where $X$ is a $kv$-set (of points), $\mathbf{G} = \{G_1, G_2, \ldots, G_k\}$ is a partition of $X$ into $k$ groups of $v$ points each, each $Y_i$ $(1 \leq i \leq r)$ is a set of $ku_i$ points (a hole of size $ku_i$) such that $|Y_i \cap G_j| = u_i$ for $1 \leq j \leq r$, $Y_i \cap Y_j = \emptyset$ for $1 \leq i < j \leq r$ and $\mathbf{A}$ is a collection of subsets of $X$ (called blocks), each meeting each group in exactly one point, such that no block contains two distinct points of any group or any hole, but any other pairset of points of $X$ is contained in exactly $\lambda$ blocks of A. When $\lambda = 1$, we drop the notation $\lambda$ and write TD$(k, v)$ and TD$(k, v) - \Sigma_{1 \le i \le r}$ TD$(k, i_i)$ for TD$(k, \lambda; v)$ and TD$(k, \lambda; v) - \Sigma_{1 \le i \le r}$ TD$(k, \lambda; u_i)$ respectively.

Now we define an analog of a transversal design TD$(k, v)$ in the directed case. A transitively ordered $k$-tuple $(a_1, a_2, \ldots, a_k)$ is defined to be the set $\{(a_i, a_j) \colon 1 \le i < j \le k\}$ consisting of $k(k-1)/2$ ordered pairs. A directed transversal design with block size $k$ and order $v$, denoted by DTD$(k, v)$, is a triple $(X, \mathbf{G}, \mathbf{A})$ where $X$ is $kv$-set (of points), $\mathbf{G}$ is a collection of $v$-subsets of $X$ (called groups) which partition $X$ and $\mathbf{A}$ is a collection of transitively ordered $k$-tuples of $X$ (called blocks), each meeting each group in exactly one point, such that every ordered pair of points from different groups occurs in exactly one block of A. The concept of transversal design with $r$ holes extends naturally to the directed case as well. We use the notation DTD$(k, v) - \Sigma_{1 \le i \le r}$ DTD$(k, u_i)$ for such a design. In the case of $r = 1$, we refer the design as a directed incomplete transversal designs and denote it by DTD$(k, v)$ - DTD$(k, u)$.

We are particularly interested in the existence of directed incomplete transversal designs, since they are very useful in the construction of other types of directed designs such as directed BIBDs, directed packings and coverings (see, for example, [3,9,15,16]). Simple counting arguments show DTD$(k, v)$ - DTD$(k, u)$ is $v \ge (k-1)u$.

The existence of a TD$(k, v)$ - TD$(k, u)$ implies the existence of a DTD$(k, v)$ - DTD$(k, u)$. The directed design is obtained by writing each block of the undirected one twice – once in some order and the other in the reverse order.

The following known results are taken from [12] and a preliminary version of [10] respectively.

**Lemma 1.1.** *For any integer $u \ge 2$, a TD$(4, v)$ - TD$(4, u)$ exists if and only if $v \ge 3u$.*

**Lemma 1.2.** *For any integer $u \ge 1$ a TD$(5, v)$ - TD$(5, u)$ exists if and only if $v \ge 4u$ except $(v, u) = (6, 1)$ and possibly the 65 values of $(v, u)$ shown in Table 1.*

With the above observation, we have the following two theorems from Lemmas 1.1 and 1.2.

**Theorem 1.1.** *For any integer $u \ge 2$, a DTD$(4, v)$ - DTD$(4, u)$ exists if and only if $v \ge 3u$.*

**Theorem 1.2.** *For any integer $u \ge 1$ a DTD$(5, v)$ - DTD$(5, u)$ exists if and only if $v \ge 4u$ except possibly $(v, u) = (6, 1)$ and the 65 values of $(v, u)$ shown in Table 1.*

| $u$ | $v$ |
|---|---|
| 1 | 10 |
| 2 | 13 14 15 16 17 |
| | 20 21 24 25 26 |
| | 27 28 31 |
| 3 | 20 21 25 26 29 |
| | 30 32 33 36 37 |
| | 41 42 44 |
| 4 | 19 23 25 38 42 |
| | 50 |
| 5 | 22 23 27 28 34 |
| | 38 |

| $u$ | $v$ |
|---|---|
| 6 | 26 27 28 32 33 |
| | 39 40 44 47 48 |
| | 52 53 |
| 7 | 30 34 41 45 |
| 9 | 38 42 |
| 10 | 43 |
| 11 | 50 |
| 13 | 54 |
| 15 | 66 |
| 17 | 74 |
| 21 | 90 |
| 29 | 122 |
| 30 | 123 |

**Table 1**

The purpose of this paper is to improve the result of Theorem 1.2 and prove that the necessary condition of the existence of a TD$(5, v)$ - TD$(5, u)$, namely $v \geq 4u$, is also sufficient.

## 2 Constructions

In order to establish our main result, we shall employ both direct and recursive methods of construction which we describe in this section.

Our first two constructions are the extension of the working corollaries of Theorems 1.1 and 1.2 in [6] to the directed case.

**Lemma 2.1.** *Suppose that a TD$(6,t)$ and a DTD$(5, m+m_j)$ - DTD$(5, m_j)$ (for $j = 1, 2, \ldots, t$) all exist. Then*

(1) *a DTD$(5, mt + \Sigma_{1\leq j\leq t} m_j)$ - DTD$(5, \Sigma_{1\leq j\leq t} m_j)$ exists;*

(2) *a DTD$(5, mt + \Sigma_{1\leq j\leq t} m_j)$ - DTD$(5, m_1 + m)$ exists if a DTD$(5, \Sigma_{1\leq j\leq t} m_j)$ - DTD$(5, m_1)$ exists' and*

(3) *a DTD$(5, mt + \Sigma_{1\leq j\leq t} m_j)$ - DTD$(5, t)$ exists if a DTD$(5, \Sigma_{1\leq j\leq t} m_j)$ and a DTD$(5, m+m_j)$ - DTD$(5, m_j)$ - DTD$(5, 1)$ (for $j = 1, 2, \ldots, t$) all exist.*

**Lemma 2.2.** *Suppose that the following designs exist:*

(1) *a TD$(5 + d, t)$;*

(2) *a DTD$(5, m)$; and*

(3) a $DTD(5, m+m_j)$ - $DTD(5, m_j)$ (for $j = 1, 2, \ldots, d$).

*Then there exists a* $DTD(5, mt + \Sigma_{1 \le j \le d} m_j)$ - $DTD(5, m + \Sigma_{1 \le j \le d} m_j)$.

The following is an extension of Wilson's Construction [17] to the directed case.

**Lemma 2.3.** *Suppose that a* $TD(7, t)$, *a* $DTD(5, m)$, *a* $DTD(5, m+1)$ - $DTD(5, 1)$, *and a* $DTD(5, m+2)$ - $2DTD(5, 1)$ *all exist. Then a* $DTD(5, mt + a + b)$ - $DTD(5, b)$ *exists if a* $DTD(5, a)$ *exists where* $0 \le a, b \le t$.

Now we consider the direct methods of construction. Most of our direct methods of construction are a variation of using difference sets in the construction of TDs (see [11]). We first make the observation that some directed incomplete transversal designs can be constructed as follows. If there exists a $\mathrm{TD}(k, 2; v)$ - $\mathrm{TD}(k, 2; u)$ which is generated by a set of base blocks and the base blocks can be rearranged so that each difference appears exactly once, taken from left to right, then the resultant design is a $\mathrm{DTD}(k, v)$ - $\mathrm{DTD}(k, u)$. In this fashion, we have the following results, in which the corresponding TDs are taken from [11] and [10] respectively.

**Lemma 2.4.** *There exists a* $DTD(7, 6)$ - $DTD(7, 1)$.

**Proof:** Let $X = (Z_5 \cup \{x\}) \times Z_7$, the group set $\mathbf{G} = \{(Z_5 \cup \{x\}) \times \{j\} : j \in Z_7\}$, the hole and $Y = \{x\} \times Z_7$ and $\mathbf{A}$ be the collection of blocks obtained by developing the following two base blocks under the action of $Z_5 \times Z_7$:

| | | | | | | |
|---|---|---|---|---|---|---|
| (0,1) | (1,2) | (4,3) | (4,4) | (1,5) | (0,6) | (x,0) |
| (x,0) | (0,6) | (2,5) | (3,4) | (3,3) | (2,2) | (0,1) |

**Lemma 2.5.** *There exists a* $DTD(5, 13)$ - $DTD(5, 2)$.

**Proof:** Let $X = (Z_{11} \cup \{x, y\}) \times Z_5$, the group set $\mathbf{G} = \{(Z_{11} \cup \{x, y\}) \times \{j\} : j \in Z_5\}$, the hole $Y = \{x, y\} \times Z_5$ and $\mathbf{A}$ be the collection of blocks obtained by developing the following base blocks under the action of $Z_{11} \times Z_5$:

| | | | | |
|---|---|---|---|---|
| (3.0) | (4,3) | (9,1) | (5,2) | (1,4) |
| (10,4) | (6,2) | (2,1) | (7,3) | (8,0) |
| (x,0) | (0,2) | (2,1) | (8,4) | (0,3) |
| (y,0) | (0,3) | (2,4) | (8,1) | (0,2) |
| (7,2) | (1,3) | (0,1) | (0,4) | (x,0) |
| (7,3) | (1,2) | (0,4) | (0,1) | (y,0) |

Finally we describe the following direct constructions.

**Lemma 2.6.** *There exists a* $DTD(5, v)$ - $DTD(5, u)$ *for each pair* $(v, u) \in \{(15, 2), (21, 2), (20, 3)\}$.

**Proof:** Take the point set $X = (Z_{v-u} \cup H) \times Z_5$, the group set $\mathbf{G} = \{(Z_{v-u} \cup H) \times \{j\}: j \in Z_5\}$ and the hole $Y = H \times Z_5$ where $H = \{x, y\}$ or $\{x, y, z\}$ depending on $u = 2$ or $u = 3$. The required blocks are listed below.

A DTD(5, 2; 15) - DTD(5, 2; 2):

| | | | | | |
|---|---|---|---|---|---|
| (x,0) | (0,1) | (3,2) | (7,3) | (2,4) | (mod 13, −) |
| (x,1) | (2,0) | (0,2) | (3,3) | (7,4) | (mod 13, −) |
| (x,2) | (7,0) | (2,1) | (0,3) | (3,4) | (mod 13, −) |
| (x,3) | (3.0) | (7.1) | (2.2) | (0.4) | (mod 13, −) |
| (x,4) | (0,0) | (3.1) | (7,2) | (2,3) | (mod 13, −) |
| (01,) | (10,2) | (6,3) | (11,4) | (x,0) | (mod 13, −) |
| (11,0) | (0,2) | (10,3) | (6,4) | (x,1) | (mod 13, −) |
| (6,0) | (11,1) | (0,3) | (10,4) | (x,2) | (mod 13, −) |
| (10,0) | (6.1) | (11.2) | (0,4) | (x,3) | (mod 13, −) |
| (0,0) | (10,1) | (6,2) | (11,3) | (x,4) | (mod 13, −) |
| (y,0) | (0,1) | (11,2) | (4,3) | (3,4) | (mod 13, −) |
| (y,1) | (3,0) | (0,2) | (11,3) | (4,4) | (mod 13, −) |
| (y,2) | (4,0) | (3,1) | (0,3) | (11,4) | (mod 13, −) |
| (y,3) | (11,0) | (4,1) | (3,2) | (0,4) | (mod 13, −) |
| (y,4) | (0,0) | (11,1) | (4,2) | (3,3) | (mod 13, −) |
| (0,1) | (2,2) | (9,3) | (10,4) | (y,0) | (mod 13, −) |
| (10,0) | (0,2) | (2,3) | (9,4) | (y,1) | (mod 13, −) |
| (9,0) | (10,1) | (0,3) | (2,4) | (y,2) | (mod 13, −) |
| (2.0) | (9,1) | (10,2) | (0,4) | (y,3) | (mod 13, −) |
| (0,0) | (2,1) | (9,2) | (10,3) | (y,4) | (mod 13, −) |
| (0,0) | (0,1) | (0,2) | (0,3) | (0,4) | (mod 13, −) |
| (0,4) | (0,3) | (0,2) | (0,1) | (0,0) | (mod 13, −) |
| (0,4) | $(2^{j_1},3)$ | $(2^{j+2},2)$ | $(2^{j+3},1)$ | $(2^{j+4},0)$ | (mod 13, −) |

(j = 0,1, ... ,11)

A DTD(5, 21) - DTD(5, 2):

| | | | | | |
|---|---|---|---|---|---|
| (x,0) | (0,1) | (7,3) | (3,2) | (8,4) | (mod 19, 5) |
| (11,4) | (16,2) | (12,3) | (0,1) | (x,0 | (mod 19, 5) |
| (y,0) | (0,1) | (9,2) | (10,4) | (16,3) | (mod 19, 5) |
| (3,3) | (9,4) | (10,2) | (0,1) | (y,0) | (mod 19, 5) |
| (13,2) | (7,4) | (15,3) | (0,0) | (18,1) | (mod 19, 5) |
| (1,1) | (0,0) | (4,3) | (12,4) | (6,2) | (mod 19, 5) |
| (17,3) | (0,0) | (14,4) | (9,2) | (13,1) | (mod 19, 5) |
| (6,1) | (10,2) | (5,4) | (0,0) | (2,3) | (mod 19, 5) |
| (9,1) | (2,3) | (2,4) | (0,0) | (0,2) | (mod 19, −) |
| (9,2) | (2,0) | (2,4) | (0,1) | (0,3) | (mod 19, −) |
| (9,3) | (2,0) | (2,1) | (0,2) | (0,4) | (mod 19, −) |
| (9,4) | (2,1) | (2,2) | (0,0) | (0,3) | (mod 19, −) |
| (9,0) | (2,2) | (2,3) | (0,1) | (0,4) | (mod 19, −) |
| (0,4) | (0,3) | (0,2) | (0,1) | (0,0) | (mod 19, −) |

A DTD(5, 20) - DTD(5, 3):

| | | | | | |
|---|---|---|---|---|---|
| (x,0) | (14,4) | (11,3) | (9,2) | (3,1) | (mod 17, –) |
| (x,1) | (11,4) | (9,3) | (3,2) | (14,0) | (mod 17, –) |
| (x,2) | (9,4) | (3,3) | (14,1) | (11,0) | (mod 17, –) |
| (x,3) | (3,4) | (14,2) | (11,1) | (9,0) | (mod 17, –) |
| (x,4) | (14,3) | (11,2) | (9,1) | (3,0) | (mod 17, –) |
| (y,0) | (12,4) | (7,3) | (15,2) | (5,1) | (mod 17, –) |
| (y,1) | (7,4) | (15,3) | (5,2) | (12,0) | (mod 17, –) |
| (y,2) | (15,4) | (5,3) | (12,1) | (7,0) | (mod 17, –) |
| (y,3) | (5,4) | (12,2) | (7,1) | (15,0) | (mod 17, –) |
| (y,4) | (12,3) | (7,2) | (15,1) | (5,0) | (mod 17, –) |
| (z,0) | (3,4) | (6,3) | (8,2) | (14,1) | (mod 17, –) |
| (z,1) | (6,4) | (8,3) | (14,2) | (3,0) | (mod 17, –) |
| (z,2) | (8,4) | (14,3) | (3,1) | (6,0) | (mod 17, –) |
| (z,3) | (14,4) | (3,2) | (6,1) | (8,0) | (mod 17, –) |
| (z,4) | (3,3) | (6,2) | (8,1) | (14,0) | (mod 17, –) |
| (5,4) | (10,3) | (2,2) | (12,1) | (x,0) | (mod 17, –) |
| (10,4) | (2,3) | (12,2) | (5,0) | (x,1) | (mod 17, –) |
| (2,4) | (12,3) | (5,1) | (10,0) | (x,2) | (mod 17, –) |
| (12,4) | (5,2) | (10,1) | (2,0) | (x,3) | (mod 17, –) |
| (5,3) | (10,2) | (2,1) | (12,0) | (x,4) | (mod 17, –) |
| (0,4) | (1,2) | (1,3) | (0,1) | (y,0) | (mod 17, –) |
| (1,3) | (1,4) | (0,0) | (0,2) | (y,1) | (mod 17, –) |
| (1,4) | (0,1) | (0,3) | (1,0) | (y,2) | (mod 17, –) |
| (0,2) | (0,4) | (1,0) | (1,1) | (y,3) | (mod 17, –) |
| (0,3) | (1,1) | (1,2) | (0,0) | (y,4) | (mod 17, –) |
| (4,3) | (0,1) | (0,4) | (4,2) | (z,0) | (mod 17, –) |
| (4,4) | (4,3) | (0,2) | (0,0) | (z,1) | (mod 17, –) |
| (0,3) | (4,0) | (4,4) | (0,1) | (z,2) | (mod 17, –) |
| (0,4) | (0,2) | (4,1) | (4,0) | (z,3) | (mod 17, –) |
| (4,2) | (0,0) | (0,3) | (4,1) | (z,4) | (mod 17, –) |
| (4,1) | (0,0) | (6,2) | (11,4) | (7,3) | (mod 17, –) |
| (13,1) | (0,0) | (11,2) | (6,4) | (10,3) | (mod 17, –) |
| (0,0) | (12,2) | (8,1) | (14,3) | (5,4) | (mod 17, –) |
| (0,0) | (1,1) | (6,3) | (10,2) | (7,4) | (mod 17, –) |
| (0,0) | $(4j, 1)$ | $(6j, 2)$ | $(7j, 3)$ | $(11j, 4)$ | (mod 17, –) |

$(j \in Z_{17}\{0, 1, 2, 13, 16\})$

□

**Lemma 2.7.** *There exists a DTD*(5, 3).

**Proof:** Take the point set $X = Z_{15}$, the group set $\mathbf{G} = \{\{j, 5+j, 10+j\} : j = 0, 1, 2, 3, 4\}$. Then the required blocks are

| | | | | |
|---|---|---|---|---|
| 12 | 0 | 1 | 14 | 8 |
| 13 | 1 | 2 | 9 | 0 |
| 14 | 2 | 3 | 10 | 1 |
| 3 | 0 | 4 | 11 | 2 |
| 4 | 1 | 5 | 12 | 3 |
| 5 | 2 | 6 | 13 | 4 |
| 6 | 3 | 7 | 14 | 5 |
| i | i+12 | i+1 | i+14 | i+8 |
| j | j+3 | j+6 | j+9 | j+12 |

where $i = 7, 8, \ldots, 14$, $j = 0, 1, 2$ and all sums are calculated in $Z_{15}$. □

## 3 Results

In this section we apply the previous constructions to establish our results. For this purpose, we make extensive use of the obvious fact that the existence of TD$(5, v)$ - TD$(5, 1)$ implies the existence of a DTD$(5, v)$, a DTD$(5, v)$ - DTD$(5, 1)$ and a DTD$(5, v)$ - 2DTD$(5, 1)$ (the latter only if $v > 4$). We also require the following two known results. The first one can be found in [1, 2, 4, 6, 7, 8, 14] and the second is taken from [16].

**Lemma 3.1.** *For any integer $v \geq 5$ and $v \neq 6, 10, 14, 18, 22, 34$ or 42, there exists a TD$(6, v)$.*

**Lemma 3.2.** *There exists a DTD$(5, 2)$.*

Now we work towards establishing our results.

**Lemma 3.3.** *There exists a DTD$(5, v)$ - DTD$(5, 1)$ when $v = 6$ or 10.*

**Proof:** A DTD$(5, 6)$ - DTD$(5, 1)$ follows from Lemma 2.4. From Brouwer [5] we have a TD$(5, 10)$ - TD$(5, 2)$ - TD$(5, 1)$. Filling the hole of size 10 by a DTD$(5, 2)$ creates a DTD$(5, 10)$ - DTD$(5, 1)$. □

**Lemma 3.4.** *A DTD$(5, v)$ - DTD$(5, 2)$ exists for each $v \in \{14, 16, 17, 20, 24, 25, 26, 27, 28, 31\}$.*

**Proof:** For $v \in \{14, 16, 24, 28\}$, there is a TD$(5, v/2)$ - TD$(5, 1)$ by Lemma.2. We then apply Wilson's Fundamental Construction [18] with weight 2 to obtain the result, since a DTD$(5, 2)$ exists by Lemma 3.2.

For $v = 20$, we give weight 2 to a TD$(5, 10)$ - TD$(5, 2)$ - TD$(5, 1)$. Wilson's Fundamental Construction then produces a TD$(5, 20)$ - TD$(5, 4)$ - TD$(5, 2)$. Filling the hole of size 20 by a DTD$(5, 4)$ establishes the result.

For the remaining values of $v$, we apply Lemma 2.1 (1) and Lemma 2.3 with the equations:

$$17 = 3.5 + (1 + 1)$$
$$25 = 3.7 + 2 + 2$$
$$27 = 3.7 + 2 + 4$$
$$31 = 3.9 + 2 + 2$$

where we used the results in Lemmas 1.2, 3.1 and 3.2. □

**Lemma 3.5.** *A DTD*$(5, v)$ *- DTD*$(5, 3)$ *exists for each* $v \in \{21, 25, 26, 29, 30, 32, 33, 36, 37, 41, 42, 44\}$.

**Proof:** For each of these values of $v$, simple calculations show that it can be written in the form $v = 3t + w$ where $t \in \{7, 9, 11, 13\}$ and $0 \le w < t$. Therefore the result follows from Lemma 2.1 (2) with $m = 3$, $m_1 = 0$ and $m_j = 0$ or $1$ $(2 \le j \le t)$ so that $\Sigma_{1 \le j \le t} m_j = w$. The required DTD(5, 3) is constructed in Lemma 2.7. □

**Lemma 3.6.** *A DTD*$(5, v)$ *- DTD*$(5, 4)$ *exists for each* $v \in \{19, 23, 25, 38, 42, 50\}$.

**Proof:** For the cases $v = 19$ and $v = 25$, the result follows from Lemma 2.1 (1) with $m = 3$, $t \in \{5, 7\}$ and $\Sigma_{1 \le j \le t} m_j = 1 + 1 + 1 + 1$. For $v \in \{23, 38, 42, 50\}$, the result is taken care of by Lemma 3.2 and Lemma 2.1 (2) with $m = 4$, $m_1 = 0$ and $m_j = 0$ or $1$ $(2 \le j \le t)$ where $t \in \{5, 9, 11\}$. □

**Lemma 3.7.** *A DTD*$(5, v)$ *- DTD*$(5, 5)$ *exists for each* $v \in \{22, 23, 27, 28, 34, 38\}$.

**Proof:** Taking $(m, t, a, b) = (3, 7, 5, 1), (3, 7, 5, 2), (3, 9, 5, 2)$ and $(3, 9, 5, 6)$ in Lemma 2.3 gives the result for $v \in \{27, 28, 34, 38\}$. Lemma 2.1 (3) works for the cases $v = 22 = 4.5 + (1 + 1)$ and $v = 23 = 4.5 + (1 + 1 + 1)$. □

**Lemma 3.8.** *A DTD*$(5, v)$ *- DTD*$(5, 6)$ *exists for each* $v \in \{26, 27, 28, 32, 33, 39, 40, 44, 47, 48, 52, 53\}$.

**Proof:** For $v \in \{32, 33, 44, 47, 48, 52, 53\}$, we can write $v = 6t + w$ such that $t \in \{5, 7, 8\}$ and $0 < w < t$. The result then follows from Lemma 2.1 (2) with $m_1 = 0$, $m = 6$ and $m_j = 0$ or $1$ $(2 \le j \le t)$, since a DTD(5, 6) exists by Lemma 3.3. Lemma 2.2 takes care of the cases $v = 26 = 5.5 + 1$ and $v = 27 = 3.8 + (1 + 1 + 1 + 1)$. The remaining values of $v$ are covered by Lemmas 2.1 (1) and 2.3, since $28 = 3.7 + 1 + 6$, $39 = 3.11 + 6$ and $40 = 3.9 + 7 + 6$. □

**Lemma 3.9.** *There exists a DTD*$(5, v)$ *- DTD*$(5, u)$ *for any pair* $(v, u) \in \{(30, 7), (34, 7), (41, 7), (45, 7), (38, 9), (42, 9), (43, 10), (50, 11), (54, 13), (66, 15), (74, 17), (90, 21), (122, 29), (123, 30)\}$.

**Proof:** Because of Lemmas 2.7, 3.1 - 3.3 and Theorem 1.2, Lemma 2.1 (3) with $t = u$, $m_j = 0$ or 1 works for all cases except for $(v, u) \in \{(41, 7), (43, 10)\}$. For the case $(v, u) = (41, 7)$, the result follows from Lemma 2.1 (1) with $t = 5$, $m = 7$, $m_1 = m_2 = 0$ and $m_3 = m_4 = m_5 = 2$. The auxiliary design DTD(5,9) - DTD(5,2) comes from Theorem 1.2. Since $43 = 3.11 + 10$, a DTD(5,43) - DTD(5,10) exists by taking $t = 11$, $m = 3$, $m_+1 = 0$ and $m_j = 1$ $(2 \leq j \leq 10)$ in Lemma 2.1 (1). This completes the proof. □

The foregoing can be summarized as follows.

**Theorem 3.10.** *Let $v$, and $u$ be positive integers. Then a DTD$(5, v)$ - DTD$(5, u)$ exists if and only if $v \geq 4u$.*

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