The sizes k of the complete k-caps in $PG(n, q)$, for small q and $3 \leq n \leq 5$

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Abstract. *It is known that there exists a one-to-one correspondence between the classes of equivalent* $[n, n - k, 4]$*-codes over* $GF(q)$ *and the classes of projectively equivalent complete n-caps in* $PG(k - 1, q)$ *(see* [20], [40]). *Hence all results on caps can be translated in terms of such codes. This fact stimulated many researches on the fundamental problem of determining the spectrum of the values of k for which there exist complete k-caps in* $PG(n, q)$. *This paper reports the result of a computer search for the spectrum of k's that occur as a size of a complete k-cap in some finite projective spaces. The full catalog of such sizes k is given in the following projective spaces:* $PG(3, q)$, *for* $q \leq 5$, $PG(4, 2)$, $PG(4, 3)$, $PG(5, 2)$. *Concrete examples of such caps are presented for each possible k.* *

1 Introduction

A k-cap K in $PG(n, q)$, the n-dimensional projective space over the finite field $GF(q)$, is a set of k points, no three of which are collinear. If $n = 2$, then a k-cap is also called a *k-arc*. A k-cap K is called complete if it is not contained in a $(k + 1)$-cap of the same projective space.

It is known that there exists a one-to-one correspondence between the classes of equivalent $[n, n - k, 4]$-codes over $GF(q)$ and the classes of projectively equivalent n-caps in $PG(k - 1, q)$ (see [20], [40]). Hence all results on caps can be translated in terms of such codes. This fact stimulated many researches on the fundamental problem of determining the spectrum of the values of k for which there exist complete k-caps in $PG(n, q)$.

This paper reports the result of a computer search for the spectrum of k's that occur as a size of a complete k-cap in the following finite projective spaces:

$PG(3, q)$, for $q \leq 5$, $PG(4, 2)$, $PG(4, 3)$, $PG(5, 2)$. Concrete examples of such caps are presented for each possible k.

Looking back at the progress made in this area, in [8] the spectrum of k's that occur as a size of a complete k-arc in $PG(2, q)$, for $q \leq 23$, has been investigated.

2 Preliminaries

The subject of complete caps in $PG(n, q)$ is vast and we will introduce only the concepts and the results that we need in this paper. For more details, one can refer to the excellent books [16], [18], [20].

The cardinality of the largest complete k-cap in $PG(n, q)$ is denoted by $m_2(n, q)$, while the cardinality of the smallest complete k-cap is denoted by $n_2(n, q)$. In our context, also the cardinality of the second largest complete k-cap in $PG(n, q)$ is very important. Let us denote this by $m'_2(n, q)$. The definition of $m'_2(n, q)$ can be formulated as an embedding theorem, namely any complete k-cap having more than $m'_2(n, q)$ points can be embedded in a k-cap of size $m_2(n, q)$, i.e. an *ovaloid*. Some lemmas will be helpful in the following section.

Lemma 2. 1 (*a*) *For* $q > 2$, $m_2(3, q) = q^2 + 1$ (Bose [3], Qvist [30]).
(*b*) $m_2(n, 2) = 2^n$ (Bose [3]).
(*c*) $m_2(4, 3) = 20$ (Pellegrino [27]).
(*d*) $m_2(5, 3) = 56$ (Hill [14]).

Lemma 2.2 (*a*) *For q odd or* $q = 4$, *an ovaloid in* $PG(3, q)$, *is an elliptic quadric* (Barlotti [2], Panella [26]).
(*b*) *In* $PG(n, 2)$, *a* 2^n*-cap is the complement of a hyperplane* (Segre [33]).
(*c*) *There are nine projectively distinct 20-caps in* $PG(4, 3)$ (Hill [15]).
(*d*) *The 56-cap in* $PG(5, 3)$ *is projectively unique* (Hill [14]).

Lemma 2.3 (*a*) $m'_2(3, 4) = 14$ (Hirschfeld and Thas [21]).
(*b*) $m'_2(3, 5) = 20$ (Abatangelo et al. [1]).
(*c*) $m'_2(4, 3) = 19$ (Tallini [39], Penttila and Royle [29]).
(*d*) $m'_2(n, 2) = 2^{n-1} + 2^{n-3}$ (Davydov and Tombak [5]).

Lemma 2.4 (*a*) $n_2(3, 2) = 5$ (Hirschfeld [16]).
(*b*) $n_2(4, 2)$ (Gabidulin et al. [11]).
(*c*) $n_2(5, 2) = 13$ (Gabidulin et al. [11]).

Lemma 2.5 *In* $PG(n, q)$, *if K is a complete k-cap, then*

$$\binom{k}{2}(q+1) - k(k-2) \geq |PG(n,q)|.$$

Proof. Let $K = \{P_1, P_2, ..., P_k\}$ and $K_i = \{P_{i+1}, P_{i+2}, ..., P_k\}$, $i = 1, 2, ..., k - 1$. Let Γ be the set of all the pairs $\{(P, b) \mid P \in PG(n, q), b \in B, \text{and } P \in b\}$, where B is the set of all bisecants of K in $PG(n, q)$. We recall that $|B| = \binom{k}{2}$, and $|b| = q + 1$, for all $b \in B$. Thus we have that $|\Gamma| = \binom{k}{2}(q + 1)$. Furthermore, since K is a complete cap, it follows that every point of $PG(n, q)$ belongs to at least a bisecant of K, and that $|PG(n, q)| \leq |\Gamma|$. Since there are $k - 1$ bisecants of K through P_1, it follows that $|PG(n, q)| \leq |\Gamma| - (k - 2)$. Also, there are $k - 2$ bisecants of K_1 through P_2, but P_2 and every point of K_2 belong to a bisecant of K through P_1. Since $|K_2| = k - 2$, it follows that $|PG(n, q)| \leq |\Gamma| - (k-2) - 2(k - 2)$. Furthermore, there are $k - 3$ bisecants of K_2 through P_3, but P_3 and every point of K_3 belong to a bisecant of K through P_1. Since $|K_3| = k - 3$, it follows that $|PG(n, q)| \leq |\Gamma| - (k - 2) - 2(k - 2) - 2(k - 3)$, and so on for $P_j \in K$. Finally, we have that

$$|PG(n, q)| \leq \binom{k}{2}(q + 1) - (k - 2) - 2\sum_{j=1}^{k-2} j = \binom{k}{2}(q + 1) - (k - 2)\, k.$$

Regarding the spectrum of the values k for which there exists a complete k-cap, general methods were developed for constructing such structures in $PG(n, q)$. In our context, the following list shows the most significant among them.

Lemma 2.6 (*a*) *In* $PG(n, 3)$, $n \geq 3$, *there exist complete* (2^n)*-caps.* (Segre [33]).

(*b*) *Suppose* $n \geq 3$. *In* $PG(n, 4)$, *there exist complete k-caps with* $k = 2^{n+1} - 2$ (Segre [33]).

(*c*) *In* $PG(4, q)$, $q > 2$ *even, there exist complete k-caps with* $k = 2q^2 + q + 5$ (Tallini [39]).

(*d*) *In* $PG(4, q)$, $q \geq 3$ *odd, there exist complete k-caps with* $k = 2q^2 + 1$ (Tallini [39]).

(*e*) *For each* $g = 0, 2, 3, ..., r - 1$, *there exists a complete* $(2^{r-1} + 2^{r-1-g})$*-cap in* $PG(r, 2)$ (Davydov-Tombak[5]).

(*f*) *There exist complete* $(3q + 2)$*-caps in* $PG(3, q)$, $q = 2^h$ (Segre [32]).

(*g*) *In* $PG(3, 4)$, *there exist:* (*i*) *a complete* 10*-cap* (Faina-Pambianco [10]).
(*ii*) *a complete* 12*-cap* (Segre [33]).

(*h*) *Let r be the remainder of the division* $[((q - 3)/2) : 3]$. *In* $PG(3, q)$, $q = 9$ *or q prime with* $q \geq 5$, *there exist complete k-caps with* $k = (q^2 + rq + 6)/3$ (Faina-Pambianco [9]).

(*i*) *Let* $q = 4t \pm 1$ *and let m be the maximum integer such that* $\binom{m}{2} < t$. *In* $PG(3, q)$, *there exist complete k-caps with* $k = (m + 1)(q + 1) + 2$ (Pellegrino [28]).

(*l*) *Let* $q = 4t \pm 1$. *In* $PG(3, q)$, *there exist complete k-caps with* $k = 4 + [(q - 1)^2/2]$ (Pellegrino [28]).

(*m*) *In* $PG(3, 5)$, *there exist complete* 16-*caps* (Faina [7]).

Lemma 2. 7 *In* $PG(3, 2)$, *a complete k-cap K is one of the following:*
(*i*) $k = 8$ *and K is the complement of a plane;*
(*ii*) $k = 5$ *and K is an elliptic quadric* (Hirschfeld [18]).

Two k-caps in $PG(n, q)$ will be considered *equivalent* if there is a projectivity which maps one onto the other. Since the group of projectivities is transitive on the coordinate systems of $PG(n, q)$, any two $(n + 2)$-cap in $PG(n, q)$ are equivalent. It follows that if H is an $(n + 2)$-cap and H' is a k-cap $(k \geq n + 3)$, then H can be extended to a k-cap which is equivalent to H'. Let $U_0, ..., U_n$ be the points $(1, 0, ..., 0), ..., (0, ..., 0, 1)$ respectively and let U_{n+1} be $(1, 1, ..., 1)$ (i.e. the $n+2$ points of the canonical coordinate system of $PG(n, q)$). A k-cap which contains $U_0, U_1, ..., U_{n+1}$ will be said *standard.* Every k-cap $(k \geq n + 2)$ is equivalent to a standard k-cap. So, an exhaustive computer search to produce a complete k-cap for each admissible k in every fixed $PG(n, q)$, can be performed by trying to complete in all possible ways the standard $(n + 2)$-cap $S = \{U_0, U_1, ..., U_{n+1}\}$. In the following sections we describe the results of our computer search for the values k for which there exists a complete k-cap in PG(3, q), for $q \leq 5$, $PG(4, 2)$, $PG(4, 3)$, $PG(5, 2)$.

Our computer program tried to find the complete k-caps by an exhaustive backtrack search. Anybody who interested can get a copy of our programs from the authors.

3 The spectrum of the values *k* for which there exists a complete *k*-cap in some *PG*(3, *q*)

Let $s(n, q)$ be the set $\{n_2(n, q), ..., m'_2(n, q), m_2(n, q)\}$ of all the integer k for which there exists a complete k-cap in $PG(n, q)$. We can now state the main result of this paper.

Theorem (*i*) $s(3, 2) = \{5, 8\}$.
(*ii*) $s(3, 3) = \{8, 10\}$.
(*iii*) $s(3, 4) = \{10, 12, 13, 14, 17\}$.
(*iv*) $s(3, 5) = \{12, 13, 14, 15, 16, 17, 18, 20, 26\}$.
(*v*) $s(4, 2) = \{9, 10, 16\}$.
(*vi*) $s(4, 3) = \{11, 16, 17, 18, 19, 20\}$.
(*vii*) $s(5, 2) = \{13, 17, 18, 20, 32\}$.

Proof. (*i*) We have our result as a direct consequence of Lemma 2.7.

(*ii*) In $PG(3, 3)$, from Lemmas 2.1 (*a*) and 2.6 (*a*) it follows that $m_2(3, 3) = 10$ and that complete 8-caps there exist. After trying out all the cases, we have not found complete k-caps with $k \notin \{8, 10\}$. Therefore $n_2(3, 3) = m'_2(3, 3)$, and $s(3, 3)$ is as desired.

(*iii*) In $PG(3, 4)$, from Lemmas 2.1 (*a*), 2.3 (*a*) and 2.5 it follows that $m_2(3, 4) = 17$ (see also [35]), $m'_2(3, 4) = 14$ and $n_2(3, 4) \geq 8$, respectively. From Lemma 2.6 (g) it follows that 10-caps and 12-caps there exist in $PG(3, 4)$. Our computer search finished without finding complete k-caps with $k \leq 9$ and $k = 11$. Finally, we have found a unique (up to projectivities) complete 13-cap. Therefore $n_2(3, 4) = 10$, and $s(3, 4)$ is as above desired. Below we list the points for the complete 13-cap $K_{13}(3, 4)$ found under our computer search. We recall that S denotes the point set of the canonical coordinate system of $PG(n, q)$. In our search has been proved that, every complete k-cap, for $k = 10, 13, 14, 17$, is unique (up to projectivities) and that there are five non equivalent complete 12-caps.

$$K_{13}(3, 4) = S \cup \{(0, 1, 1, 2), (0, 1, 2, 1), (0, 1, 3, 3), (1, 0, 1, 2), (1, 0, 2, 1), (1, 0, 3, 3), (1, 1, 0, 2), (1, 1, 2, 0)\}.$$

(*iv*) In $PG(3, 5)$, from Lemmas 2.1 (*a*), 2.3 (*b*) and 2.5, it follows that $m_2(3, 5) = 26$, $m'_2(3, 5) = 20$ and $n_2(3, 5) \geq 10$, respectively. From Lemma 2.6 (*h*), and (*m*), it follows that complete 12-caps and 16-caps there exist. Our exhaustive computer search finished without finding complete k-caps with $k \leq 11$ and $k = 19$, but we have found complete k-caps with $k \in \{13, 14, 15, 17, 18\}$. So $n_2(3, 5) = 12$, and $s(3, 5)$ is as above desired. Examples of complete k-caps, with $k \in \{13, 14, 15, 17, 18\}$, in $PG(3, 5)$ are given below. In [1], it is shown that there are two non equivalent 20-caps.

$$K_{13}(3, 5) = S \cup \{(0, 1, 1, 2), (0, 1, 2, 1), (0, 1, 3, 3), (1, 0, 1, 2), (1, 0, 2, 1), (1, 1, 0, 2), (1, 4, 0, 1), (1, 4, 2, 0)\}.$$

$$K_{14}(3, 5) = S \cup \{(0, 1, 1, 2), (0, 1, 2, 1), (0, 1, 3, 3), (1, 0, 1, 2), (1, 0, 2, 1), (1, 1, 0, 2), (1, 1, 2, 3), (1, 2, 0, 3), (1, 4, 2, 0)\}.$$

$K_{15}(3, 5) = S \cup \{(0, 1, 1, 2), (0, 1, 2, 1), (0, 1, 3, 3), (1, 0, 1, 2), (1, 0, 2, 1), (1, 1, 0, 2),$
$(1, 1, 4, 4), (1, 3, 0, 4), (1, 3, 4, 2), (1, 4, 0, 1)\}.$
$K_{17}(3, 5) = S \cup \{(0, 1, 1, 2), (0, 1, 2, 1), (0, 1, 3, 3), (1, 0, 1, 2), (1, 0, 2, 1), (1, 1, 0, 2),$
$(1, 1, 3, 0), (1, 1, 4, 4), (1, 2, 2, 0), (1, 2, 4, 3), (1, 3, 2, 4), (1, 4, 0, 1)\}.$
$K_{18}(3, 5) = S \cup \{(0, 1, 1, 2), (0, 1, 2, 1), (0, 1, 3, 3), (1, 0, 1, 2), (1, 0, 2, 1), (1, 1, 0, 2),$
$(1, 1, 3, 0), (1, 1, 4, 4), (1, 2, 2, 0), (1, 2, 3, 4), (1, 2, 4, 1), (1, 3, 2, 3),$
$(1, 3, 4, 2)\}.$

(*v*) From Lemmas 2.1 (*b*), 2.3 (*d*), and 2.4 (*b*), it follows that $m_2(4,2) = 16$, $m'_2(4, 2) = 10$, and $n_2(4, 2) = 9$, respectively. So $s(4, 2)$ is as desired.

(*vi*) In $PG(4, 3)$, from Lemmas 2.1 (*c*), 2.3 (*c*) and 2.5, it follows that $m_2(4, 3) = 20$, $m'_2(4, 3) = 19$ and $n_2(4, 3) \geq 11$, respectively. From Lemma 2.6 (*a*) it follows that complete 16-caps there exist. Our exhaustive computer search finished without finding complete k-caps with $k \leq 10$ and $k \in \{12, 13, 14, 15\}$, but we found complete k-caps with $k \in \{11, 16, 17, 18\}$. So $n_2(4, 3) = 11$, and $s(4, 3)$ is as desired. Examples of complete k-caps, with $k \in \{11, 16, 17, 18\}$, in $PG(4, 3)$ are given below. Finally, it is also shown that there is a unique (up to projectivities) 11-cap.
$K_{11}(4, 3) = S \cup \{(0, 1, 1, 2, 2), (1, 0, 2, 1, 2), (1, 1, 2, 2, 0), (1, 2, 0, 2, 1), (1, 2, 1, 0, 2)\}.$
$K_{16}(4, 3) = S \cup \{(0, 0, 1, 1, 1), (0, 1, 0, 1, 1), (0, 1, 1, 0, 1), (0, 1, 1, 1, 0), (1, 0, 0, 1, 1),$
$(1, 0, 1, 0, 1), (1, 0, 1, 1, 0), (1, 1, 0, 0, 2), (1, 1, 0, 2, 0), (1, 1, 1, 0, 0)\}.$
$K_{17}(4, 3) = S \cup \{(0, 0, 1, 1, 1), (0, 1, 0, 1, 1), (0, 1, 1, 0, 1), (0, 1, 1, 1, 0), (1, 0, 0, 1, 1),$
$(1, 0, 1, 0, 1), (1, 0, 1, 1, 2), (1, 1, 0, 0, 2), (1, 1, 0, 1, 0), (1, 1, 2, 0, 0),$
$(1, 2, 1, 0, 2)\}.$
$K_{18}(4, 3) = S \cup \{(0, 0, 1, 1, 1), (0, 1, 0, 1, 1), (0, 1, 1, 0, 1), (0, 1, 1, 1, 0), (1, 0, 0, 1, 1),$
$(1, 0, 1, 0, 1), (1, 0, 1, 1, 0), (1, 1, 0, 0, 2), (1, 1, 0, 2, 0), (1, 1, 1, 2, 2),$
$(1, 1, 2, 0, 1), (1, 1, 2, 1, 0)\}.$

(*vii*) In $PG(5, 2)$, from Lemmas 2.1 (*b*), 2.3 (*d*) and 2.4 (*c*), it follows that $m_2(5, 2) = 32$, $m'_2(5, 2) = 20$ and $n_2(5, 2) = 13$, respectively. From Lemma 2.6 (*e*), it follows that complete 17-caps and 18-caps there exist. Our exhaustive computer search finished without finding complete k-caps with $k \in \{14, 15, 16, 19\}$. So $s(5, 2)$ is as above desired.

4 Concluding Remarks

The problem of compiling the full catalog of the possible sizes of the complete k-caps in $PG(n, q)$ seems to be extremely hard. Practically nothing is known about exact values of the cardinality of the second largest complete cap and the size of the smallest complete cap in $PG(3, q)$ for $q \geq 7$ or in $PG(n, q)$

for $n \geq 6$. However, many people have put a lot of work also in this area, and the main results obtained are given in [24].

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