# On interval colourings of bi-regular bipartite graphs

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ABSTRACT. In this paper we consider interval colourings – edge colourings of bipartite graphs in which the colours represented at each vertex form an interval of integers. These colourings, corresponding to certain types of timetables, are not always possible. In the present paper it is shown that if a bipartite graph with bipartition $(X, Y)$ has all vertices of $X$ of the same degree $d_X = 2$ and all vertices of $Y$ of the same degree $d_y$, then an interval colouring can always be established.

## 1 Introduction

Call a colouring of the edges of a graph with colours $1, 2, 3, \ldots$ an 'interval colouring' if the colours received by the edges incident to each vertex are distinct and form an interval of integers. Let $G$ be a bipartite graph with vertex-partition $(X, Y)$. If all vertices of $X$ have the same degree $d_X$ and all vertices of $Y$ have degree $d_Y$, does $G$ allow an interval colouring?

This is an instance, due to Hansen [4], of a general problem, which is discussed in the book *Graph Coloring Problems* by T.R. Jensen and B. Toft [5] under the name consecutive colouring. It arose from a practical scheduling problem: At the Saint Canute High School in Odense, Denmark, parent consultations are arranged by letting each parent, or couple of parents, decide beforehand on a list of teachers that he/she/they would like to consult. Each meeting between parent(s) and a teacher lasts for the same fixed amount of time. The problem is to create a schedule without waiting periods for either the parents or the teachers. Letting $X$ represent the set of teachers and $Y$ the set of parents/couples of parents, and letting an edge $xy$ represent a meeting of teachers $x$ with parent(s) $y$, we obtain an instance of the above-mentioned colouring-problem (without the degree constraints), in which each colour corresponds to an assigned time slot.

This particular variation of edge-colouring for multigraphs in general seems first to have been studied by Asratian and Kamalian [1] and by Sevast'janov [8]. There are further results due to Asratian and Kamalian in [2], in particular if a triangle-free simple graph $G$ has an interval colouring using $t$ colours then $t \leq |V(G)| - 1$. Independently of these, with the timetablelling problem as a starting point, Hansen [4] obtained a number of fundamental results.

Consider the problem for bipartite graphs without degree constraints. In this case it is not always possible to colour the edges such that the graph will have an interval colouring. The first example showing this seems due to Sevast'janov [8]. In 1991 examples were found independently by P. Erdös and by A. Hertz and D. de Werra. To obtain Erdös' example, take a finite projective plane $P$ of order $p$ and let $X$ represent the set of points of $P$ and $Y$ the set of lines, with $xy$ an edge if and only if the point $x$ belongs to the line $y$. Finally join one new vertex $z$ to all the vertices of $Y$. For $p \geq 3$, this bipartite graph $G$ cannot be given an interval colouring. The vertices of the part $X \cup \{z\}$ of the bipartition contains vertices of two different degrees $p+1$ and $p^2+p+1$ whereas the vertices in $Y$ all have degree $p+2$. For $p = 3$, $G$ has 27 vertices and $\Delta = 13$. The example by Sevast'janov has 28 vertices and $\Delta = 21$. A smaller example, due to Hertz and de Werra has 21 vertices and $\Delta = 14$. We do not know any examples with $\Delta \leq 12$. For $\Delta \leq 3$, such examples do not exist as proved by Hansen [4].

It is noteworthy that the above-mentioned examples have the property that vertices of both low and high degree appear on the same side of the bipartition. We consider graphs where this is not the case. If $G$ is $\Delta$-regular, then by the 1-factor theorem of D. König [6], it is always possible to $\Delta$-edge-colour $G$, and naturally such a colouring is also an interval colouring. For bi-regular bipartite graphs, where all vertices in $X$ have degree $d_X$ and all vertices in $Y$ have degree $d_Y$, Hansen [4] noted that the answer to the question initially asked is affirmative in the where case $d_X = 2$ and $d_Y$ is any even number, and that this statement is in fact equivalent to the 2-factor theorem of Petersen [7]: Every $2r$-regular graph has a decomposition into 2-regular edge-disjoint subgraphs.

In this paper we shall give an affirmative answer also in the case in which $d_X = 2$ and $d_Y$ is any odd number. It is interesting to note that if there is a meeting between two people which must take place at a certain time then it is possible to arrange this and still keep all the meetings consecutive for everybody. The simplest unsolved cases of the problem stated above are thus $(d_X, d_Y) = (3,4)$. A.V. Kostochka, Novosibirsk, has informed us that he also obtained a solution in the case $d_X = 2$ and $d_Y$ odd.

## 2 The case $d_X = 2$ and $d_Y = 3$

**Lemma 1.** *Let $H$ be a bipartite graph with bipartition $(X, Y)$, and with $d_X = 2$ and $d_Y = 3$, then $H$ can be given an interval colouring using 4 colours. Moreover, in such a colouring, we can choose any edge $e$ of $H$ and give it a specific colour.*

**Proof:** $H$ being bipartite implies that the edge-chromatic number satisfies $\chi' = \Delta = 3$, [6] (this is not an interval colouring however). Consider the colours represented at the vertices of $X$ and $Y$ in a proper colouring of $H$. The colours 1, 2 and 3 are represented at each $y \in Y$. Each vertex $x \in X$ has one of the following pairs of colours represented there: 1 and 2, 2 and 3, 1 and 3. The only vertices which cause a problem for an interval colouring are those in $X$ with colours 1 and 3 represented there. At these vertices, change colour 1 to colour 4. These vertices now have the colours 3 and 4 represented at them and the colouring forms an interval on $X$. The colouring remains an interval colouring on $Y$ as any $y \in Y$ now has either the colours 1, 2 and 3 or 2, 3 and 4 represented there. Thus $H$ has been given an interval colouring using 4 colours.

Now suppose that we wish to give a particular edge $e$ a specific colour. Again, we begin with a proper colouring of $H$ using the colours 1, 2 and 3. If we want $e$ to have colour 2 or 3, we simply permute the proper colouring to give the desired colour to $e$ and then make the colouring an interval colouring as above. Forcing $e$ to have the colours 1 or 4 takes a bit more manipulation. First, permute the proper colouring of $H$ so that $e$ has the colour 1. Let $x$ be the endpoint of $e$ in $X$. If we want $e$ to keep the colour 1 when we make the proper colouring an interval colouring, we first insure that $f$, the other edge with $x$ as an endpoint, has colour 2. If $f$ has colour 3, exchange the colours 2 and 3 in the proper colouring. If we want $e$ to change to the colour 4, we first insure that $f$ has colour 3. If $f$ has colour 2, exchange the colours 2 and 3 as before. Now make the colouring an interval colouring as above and we obtain the required colouring. □

## 3 Factors in regular graphs

In the cases where $d_Y \geq 5$ we make use of the following theorem of Tutte [9] on factors in a regular graph.

**Theorem.** (Tutte [9]). *A multigraph $G^\circ$ contains a $k$-factor if and only if $q(S,T) + \sum_{v \in T}(k - d_{G^\circ - S}(v)) \leq k|S|$ for all choices of disjoint sets $S$, $T \subseteq V(G^\circ)$, where $q(S,T)$ is the number of components $Q$ of $G^\circ - S - T$*

*for which $|[V(Q),T]| + k|V(Q)|$ is odd ($[R,L]$ denotes the set of edges with endpoints in both the sets $R$ and $L$).*

**Theorem 1.** *Let $G^\circ$ be a $(2n+1)$-regular graph. If $n > (3k-1)/2$ and $G^\circ$ has at most $[(1/k)(2n-k+1)]$ bridges, then $G^\circ$ has a $2k$-factor, $k \geq 1$.*

**Proof:** From Tutte's Theorem, $G^\circ$ does not have a $2k$-factor if and only if

$$q(S,T) - \sum_{v \in T} d_{G^\circ - S}(v) \geq 2k(|S| - |T|) + 1. \tag{1}$$

In this instance, $q(S,T)$ counts the components $Q$ of $G^\circ - S - T$ for which $|[V(Q),T]|$ is odd since $2k|V(Q)|$ is always even. Note that $\sum_{v\in T} d_{G^\circ - S}(v) = \sum_{v \in T} d(v) - h = (2n+1)|T| - h$, where $h = |[S,T]|$. Define $q_1$, $q_2$ and $q_3$ as follows (see Figure 1):

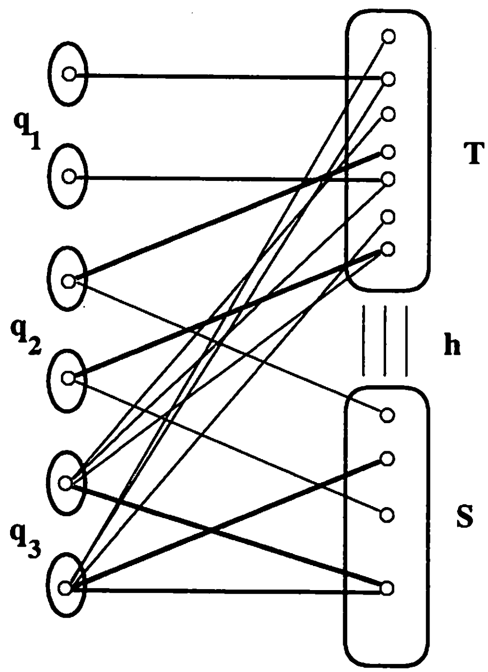


**Figure 1**

$q_1 =$ the number of components $Q$ of $G^\circ - S - T$ for which $|[V(Q),T]| = 1$ and $|[V(Q),S]| = 0$
$q_2 =$ the number of components $Q$ of $G^\circ - S - T$ for which $|[V(Q),T]| = 1$ and $|[V(Q)S]| \geq 1$
$q_3 =$ the number of components $Q$ of $G^\circ - S - T$ for which $|[V(Q),T]| = (2p+1)$, $p \geq 1$.

Let $q = q(S,T)$. Notice that $q = q_1 + q_2 + q_3$ and that $q_1$ is bounded above by the number of bridges in $G^\circ$. Inequality (1) is equivalent to the following:

$$q + h \geq 2k|S| + (2n - 2k + 1)|T| + 1. \tag{2}$$

Since $G^\circ$ is $(2n+1)$-regular, the number of edges originating in $T$ is bounded above by $(2n+1)|T|$, thus $(2n+1)|T| \geq q_1 + q_2 + 3q_3 + h = q+h+2q_3 \geq 2k|S|+(2n-2k+1)|T|+1+2q_3$ giving $2k|T| \geq 2k|S|+1+2q_3$ and by parity arguments, $2k|T| \geq 2k|S| + 2 + 2q_3$, i.e.

$$|T| \geq |S| + 1/k + (1/k)q_3 \tag{3}$$

Substituting (3) into (2) we obtain the inequality $q + h \geq (2n+1)|S| + (1/k)(2n-2k+1) + (1/k)(2n-2k+1)q_3 + 1$. The number of edges that originate in $S$ is bounded above by $(2n+1)|S|$ thus $(2n+1)|S| + q_1 + q_3 \geq q + h$. Now we can write

$$\begin{aligned}(2n+1)|S| + q_1 + q_3 \geq q + h \geq (2n+1)|S| + (1/k)(2n-k+1)\\ + (1/k)(2n-2k+1)q_3,\end{aligned} \tag{4}$$

i.e. $q_1 \geq (1/k)(2n-k+1) + (1/k)(2n-3k+1)q_3$. If $G^\circ$ has at most $[(1/k)(2n-k+1)]$ bridges, then $q_1 \leq (1/k)(2n-k+1)$. Thus inequality (4) implies $0 \geq (1/k)(2n-3k+1)q_3$, which then implies that either $(3k-1)/2 \geq n$ or $q_3 = 0$. We must have $q_3 = 0$ since we have assumed that $(3k-1)/2 < n$.

Substituting $q_1 \leq (1/k)(2n-k+1)$ and $q_3 = 0$ into (4) we have $(2n+1)|S| + q_1 \geq q_1 + q_2 + h \geq (2n+1)|S| + q_1$, i.e.

$$(2n+1)|S| = q_2 + h \tag{5}$$

and, by (3),

$$|T| \geq |S| + 1/k. \tag{6}$$

From (5) along with $q = q_1 + q_2$ and (2) we obtain $q_1 + q_2 + h \geq 2k|S| + (2n-2k+1)|T| + 1$ giving $(1/k)(2n-k+1) + (2n+1)|S| \geq 2k|S| + (2n-2k+1)|T| + 1$, which implies $1/k + |S| \geq |T|$. By (6) we have

$$1/k + |S| = |T|. \tag{7}$$

Since $|S|$ and $|T|$ are both natural numbers, we have a contradiction for $k > 1$.

Now we must consider the case when $k = 1$, i.e. when we are looking for a 2-factor. In this instance, we obtain the following from inequality (4):

$$q_1 \geq 2n + (2n-2)q_3. \tag{8}$$

Thus we know that $G^\circ$ has a 2-factor whenever $q_1 < 2n$. We must, therefore, investigate the case when $q_1 = 2n$. Assume in this case that $G^\circ$ does not have a 2-factor, implying $q_3 = 0$ by inequality (8). Equation (7) becomes $1 + |S| = |T|$. This along with equation (5), $q_1 = 2n$ and $q_3 = 0$ leads to

$$(2n+1)|T| = (2n+1)|S| + (2n+1) = q_2 + h + 2n + 1 = q + h + 1.$$

We conclude that there is one edge originating in $T$ that has not been accounted for. Let this edge be edge $f$. Both endpoints of $f$ cannot be in $T$ since $(2n+1)|T| = q+h+1$. Also, the other endpoint of $f$ cannot be in $S$ as $f$ was not counted by $h$, hence it is in some component $Q$ of $G^\circ - S - T$. Therefore $[V(Q), T] = f$ and $Q$ would have been counted by $q = q_1 + q_2$, a contradiction.

We conclude that $G^\circ$ has a $2k$-factor whenever $(3k-1)/2 < n$ and $G^\circ$ has at most $[(1/k)(2n-k+1)]$ bridges. □

**Corollary 1.** *Let $G^\circ$ be a $(2n+1)$-regular graph, $n \geq 2$, with at most $2n$ bridges, then $G^\circ$ has a 2-factor .*

For $n = 1$ Corollary 1 is a famous theorem of Petersen [7]. For general $n$ and graphs with at most one bridge it is due to Bäbler [3].

**Corollary 2.** *Let $G^\circ$ be a $(2n+1)$-regular graph, $n \geq 3$, with at most $n-1$ bridges, then $G^\circ$ has a 4-factor .*

## 4 The case $d_X = 2$ and $d_Y$ odd and greater than one

**Theorem 2.** *A bipartite graph $G$ with bipartition $(X, Y)$, and $d_X = 2$ and $d_Y = 2n+1$, $n \geq 1$, can be given an interval colouring using $2n+2$ colours; further, we can choose any edge of $G$ and specify its colour.*

**Proof:** By Lemma 1, the theorem is true for $n = 1$ ($d_Y = 3$). We induct on $n$. Assume that the theorem is true for $n = m$; let $n = m+1$. In this case $G$ is a bipartite graph with $d_X = 2$ and $d_Y = 2m+3$. Let $G^\circ$ be the condensed version of $G$ on the vertices of $Y$, that is, if each edge of $G^\circ$ is subdivided once, we obtain $G$. The vertices used to subdivide the edges of $G^\circ$ are the vertices of the set $X$ of $G$. Note that $G^\circ$ is $(2m+3)$-regular. There are two cases which we must consider:

1) Assume that $G^\circ$ has at most one bridge. By Corollary 1, $G^\circ$ has a 2-factor, $J^\circ$, and a $(2m+1)$-factor, $H^\circ$. Let $J$ and $H$ be the bipartite subdivisions of $J^\circ$ and $H^\circ$ (see Figure 2). Note that $J$ and $H$ are edge-disjoint subgraphs of $G$ such that $J \cup H = G$. In $H$, $d_W = 2$ and $d_Y = 2m+1$, while in $J$, $d_U = 2$ and $d_Y = 2$, where $W = X(H)$ and $U = X(J)$. By assumption, $H$ can be given an interval colouring using $2m+2$ colours. We must now colour the edges of $J$, so that colouring $J$ finishes the interval colouring on $G$. $J$ is the union of even cycles, thus, $J$ has two edge disjoint

perfect matchings. Give the edges in one of the perfect matchings the colour $2m+3$. Now consider the vertices of $Y$ (in $G$). Either the colours 1 to $2m+1$ and $2m+3$ or 2 to $2m+2$ and $2m+3$ are represented at these vertices. Also, each $y$ in $Y$ has one edge which as yet, has no colour. If $y$ has the colours 1 to $2m+1$ and $2m+3$, give the uncoloured edge the colour $2m+2$. If $y$ has the colours 2 to $2m+3$, give the uncoloured edge the colour $2m+4$. We now have a colouring of $G$ which forms intervals on $W$ and $Y$. At each vertex $x$ of $U$, either the colours $2m+2$ and $2m+3$ or the colours $2m+3$ and $2m+4$ are represented. Therefore, the colouring also forms intervals on $U$ implying $G$ has an interval colouring using $2m+4$ colours.

We will now prove that we can specify the colour for a particular edge $e$ in the interval colouring of $G$. We consider the cases where $m = 1$ and where $m \geq 2$ separately.

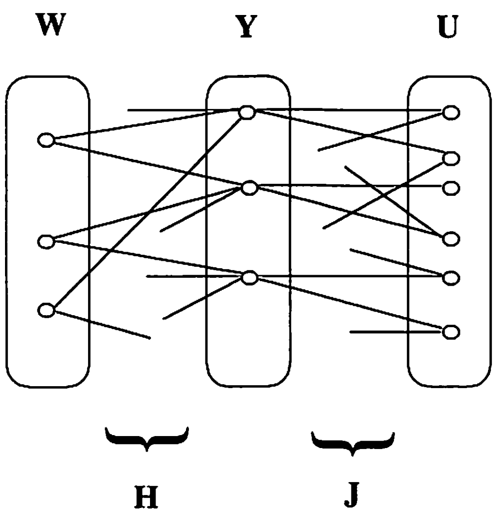


**Figure 2**

Assume that $m = 1$ (i.e. $d_Y = 5$). Split $G$ into $J$ and $H$ as above. If $e$ is in $H$, we can give it any colour from 1 to 4 by Lemma 1 and then we complete the colouring of $G$. If we want $e$ to have colour 5 (6 respectively), first give $e$ the colour 2 (1 respectively), then complete the colouring of $G$. Once this has been done, permute the colouring of $G$ in the following manner: exchange 1 and 6, exchange 2 and 5, and exchange 3 and 4. Edge $e$ now has colour 5 (6 respectively). If $e$ is in $J$, we can give it any colour from 4 to 6 when we complete the colouring of $G$. To give $e$ the colour 5, we simply insure that $e$ is in the perfect matching of $J$ that receives the colour 5. On the other hand, to give the colour 4 (6 respectively) to $e$, we must first insure that $e$ is in the perfect matching which is not coloured 5.

Then $y$, the endpoint of $e$ in $Y$, must have the colours 1, 2, and 3 (2, 3 and 4 respectively) represented at it in $H$. This is done by choosing one of the edges coming out of $y$ in $H$ and giving it the colour 1 (4 respectively) in the interval colouring of $H$. Thus, when we finish the interval colouring of $G$, the colour 4 (6 respectively) will be given to $e$. To give $e$ the colour 1, 2 or 3, first give $e$ the colour 6, 5 or 4, then permute the colours as described previously.

Assume that $m \geq 2$. By Corollary 2, $G^\circ$ has a 4-factor which implies that $G^\circ$ has two edge-disjoint 2-factors, $J_1^\circ$ and $J_2^\circ$. Let $H_1^\circ$ and $H_2^\circ$, respectively, be the $(2m+1)$-factors that complement these 2-factors. Let $J_1$, $H_1$ and $J_2$, $H_2$ be the bipartite subdivisions of the factors. Split $G$ into $J_1$ and $H_1$. If $e$ is in $H_1$, then by assumption we can give e any colour we wish from 1 to $2m+2$. To give $e$ the colour $2m+3$ ($2m+4$ respectively), first give $e$ the colour 2 (1 respectively), and then, in $G$, permute the colours. If $e$ is in $J_1$, we will use the other subgraphs of $G$, $J_2$ and $H_2$. Thus $e$ is in $H_2$, and we can give $e$ any colour we wish in the interval colouring of $G$.

2) Assume that $G^\circ$ has more than one bridge. This implies that $G^\circ$ has more than two bridgeless subgraphs. Take a bridgeless subgraph of $G^\circ$, along with the bridge(s) that connects (connect) it to the rest of $G^\circ$, and call this $B^\circ$. Now construct $D^\circ$, a $(2m+3)$-regular supergraph of $B^\circ$ as follows: If $B^\circ$ has an even number of bridges, group the bridges into pairs and connect their endpoints of degree 1 (in $B^\circ$) with $2m+2$ edges. If $B^\circ$ has an odd number of bridges, group all but one them into pairs; these pairs, we treat as above. We attach a triangle, $F^\circ$, to the remaining bridge, and add $m$ extra edges between each pair of vertices of $F^\circ$. Finally, add one extra edge between the two vertices of $F^\circ$ which are not incident with the bridge. In the Figure 3 we have demonstrated the case when $m = 1$.

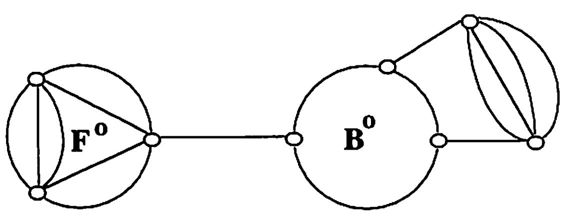


**Figure 3**

Thus $D^\circ$ is a $(2m+3)$-regular supergraph of $B^\circ$ with at most 1 bridge. Let $D$ be the bipartite subdivision of $D^\circ$. We can use the method of case 1) to give an interval colouring to $D$ using $2m+4$ colours. However, $D$ isn't quite what we're interested in. Define $B$ as the bipartite subdivision of $B^\circ$ excluding the edge(s) of the subdivided bridge(s) which doesn't (don't) have an endpoint in the bridgeless subgraph (such a constructed subgraph $B$ of $G$ will be called a section of $G$). $B$ has been given an interval colouring using $2m + 4$ colours via the interval colouring of $D$. Thus we can give an interval colouring to each section of $G$ using this method.

We represent a section of $G$ by a circle along with the edge(s) attached to it (see Figure 4, the square vertices are in $Y$ and the round vertices are in $X$). Colouring each section of $G$ colours $G$ and this colouring will be an interval colouring, except possibly at certain vertices. Such vertices belong to two sections and will be called joining vertices. Since case 1) applies to the constructed supergraphs of the sections of $G$, we can use this to make the colours represented at these joining vertices an interval in the following manner. Pick a section of $G$. Manipulate the interval colouring of the neighbouring sections so that the colours represented at the joining vertices form an interval. Repeat, moving outward along the subdivided bridges. Eventually $G$ has an interval colouring with $2m + 4$ colours. A particular edge $e$ of $G$ can be given a specific colour as follows: Identify the section $e$ is in and arrange it so that $e$ has the desired colour. Now we manipulate the colours of the other sections as described above to insure that the colours at the joining vertices form an interval. □

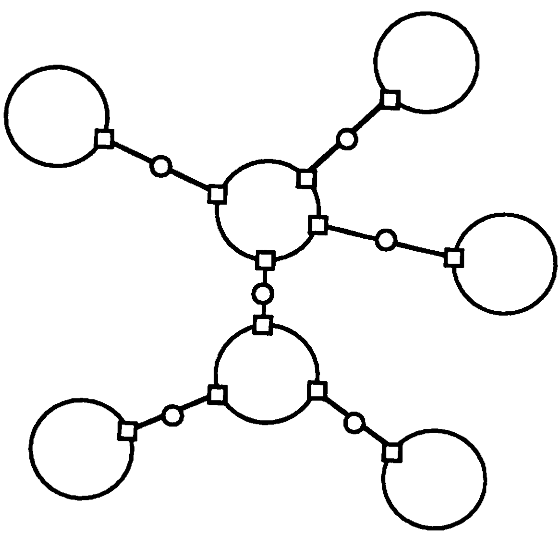

**Figure 4**

## 5 Concluding Remarks

In [4] Hansen proved that the complete bipartite graph $K_{m,n}$ has an interval colouring using $m + n - \gcd(m, n)$ colours, and he claimed that in general $d_X + d_Y - \gcd(d_X, d_Y)$ colours are necessary for a bi-regular bipartite graph with bipartition $(X, Y)$. He also noted, however, that this number of colours is not always sufficient.

Asratian and Kamalian [1] proved that for a given bipartite graph with bipartition $(X, Y)$ it is an NP-complete problem to decide if there exists a proper edge-colouring such that edges incident with each vertex $x$ in $X$ are coloured with colours $1, 2, \ldots, d(x)$. Sevast'janov [8] proved that in general it is an NP-complete problem to decide for a given bipartite graph if it has an interval colouring (allowing any number of colours).

For non-bipartite graphs, interval colourings were studied by Asratian and Kamalian [1]. Jensen and Toft [5] noted that a $k$-regular graph can be given an interval colouring if and only if it can be $k$-edge-coloured.

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