A series of quasi-multiple BIB designs from Hadamard matrices

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ABSTRACT. A method of construction of quasi-multiple balanced incomplete block (BIB) designs from certain group divisible designs is described. This leads to a series of quasi-multiple designs of symmetric BIB designs and new non-isomorphic solutions of designs listed as unknown in the tables of Mathon and Rosa [3,4]. In the process a series of semi-regular group divisible designs is also obtained.

1 Introduction

We assume that the reader is familiar with some basic concepts in design theory. A balanced incomplete block (BIB) design is an arrangement of a set of v elements, called treatments, into b subsets, called blocks, of size k each such that every treatment occurs in r blocks and any unordered pair of treatments is contained in λ blocks, denoted by BIB(v, b, r, k, λ). Here we have the relations $vr = bk$ and $r(k-1) = \lambda(v-1)$.

By taking m copies of the BIB design we get another BIB(v, mb, mr, k, $m\lambda$), which is called the m-multiple design of the BIB(v, b, r, k, λ). A BIB(v, mb, mr, k, $m\lambda$) which is not obtained by taking m copies of the

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BIB(v, b, r, k, λ) is called an m-quasi-multiple design of the BIB(v, b, r, k, λ). A BIB design may or may not exist but its quasi-multiple BIB design may exist.

Mathon and Rosa [3,4] tabulated BIB designs with $r \leq 41$ including quasi-multiple designs. This is a good source of references on quasi-multiple BIB designs. Constructions of m-quasi-multiple designs were also dealt with in Sinha [5,6], Sinha and Singh [9], and Logan, Singh and Sinha [2].

A group divisible (GD) design is an arrangement of $v(= mn)$ treatments in b blocks such that each block contains $k(< v)$ distinct treatments, each treatment is replicated r times, and the set of v treatments can be partitioned into $m(\geq 2)$ groups of $n(\geq 2)$ treatments each, any two distinct treatments occurring together in λ_1 blocks if they belong to the same group, and in λ_2 blocks if they belong to different groups. Furthermore, if $r - \lambda_1 = 0$, the GD design is said to be singular (S). If $r - \lambda_1 > 0$ and $rk - v\lambda_2 = 0$, it is said to be semi-regular (SR), while it is called regular if $r - \lambda_1 > 0$ and $rk - v\lambda_2 > 0$. GD designs with $r, k \leq 10$ were tabulated by Clatworthy [1] and Sinha [7].

In this note, a method of construction of BIB designs from certain GD designs is described. This yields a series of BIB designs and new 2-quasi-multiple BIB designs whose solutions are listed as unknown in Mathon and Rosa [3,4], i.e., in the column "References" of their Table, there are no references given for designs. In the process a series of semi-regular GD design is also obtained.

2 Constructions

Consider two GD designs with parameters

$$v_1 = m_1 n, b_1, r_1, k, \lambda_{11} = 0, \lambda_{21}, m_1 = m_2 + 1, n \tag{1}$$

$$v_2 = m_2 n, b_2, r_2, k, \lambda_{12}, \lambda_{22}, \lambda_{12} > \lambda_{22}, m_2, n \tag{2}$$

such that $\alpha_1 r_1 + \alpha_2 r_2 = \alpha_1 n r_1$ and $\alpha_2 \lambda_{12} = \alpha_1 \lambda_{21} + \alpha_2 \lambda_{22} = n \alpha_1 \lambda_{21}$ for positive integers α_1 and α_2. Then by juxtaposing α_1 and α_2 copies of GD designs with (1) and (2) respectively and considering the n treatments of the last group in the design with (1) as one treatment and suitably renumbering them, we can get a BIB design with parameters

$$v = m_2 n + 1, b = \alpha_1 b_1 + \alpha_2 b_2, r = n\alpha_1 r_1, k, \lambda = n\alpha_1 \lambda_{21}/p$$

where $\alpha_i = |\lambda_{1j} - \lambda_{2j}|/p, i \neq j, i, j = 1, 2$, and p is the greatest common factor of λ_{21} and $|\lambda_{12} - \lambda_{22}|$.

As a special case of the above construction method we can obtain the following.

For a positive integer s the existence of a SRGD design with parameters

$$v = 4s = b, r = 2s = k, \lambda_1 = 0, \lambda_2 = s, m = 2s, n = 2 \quad (3)$$

and a SGD design with parameters

$$v = 2(2s-1) = b, r = 2s = k, \lambda_1 = 2s, \lambda_2 = s, m = 2s-1, n = 2 \quad (4)$$

implies the existence of a BIB design with parameters

$$v = 4s-1, b = 2(4s-1), r = 4s, k = 2s, \lambda = 2s \quad (5)$$

which can also be shown to be a 2-quasi-multiple of a symmetric BIB($v' = b' = 4s-1, r' = k' = 2s, \lambda' = s$), whose complement is a symmetric BIB($4s-1, 2s-1, s-1$).

Case I: $s = 2t$

A solution of a SRGD design with (3) when $s = 2t$ is reported in Sinha [8] as

$$\frac{1}{2}(J+H) \otimes I + \frac{1}{2}(J-H) \otimes (J-I),$$

where H is a Hadamard matrix of order $4t$, J is a matrix with all elements unity and I is the identity matrix, while a solution of the SGD design is obtained from the Hadamard matrix of order $4t$ as follows: the existence of a Hadamard matrix of order $4t$ implies the existence of a symmetric BIB($4t-1, 2t, t$) which in turn implies the existence of a SGD design (see Clatworthy [1]) which on duplication yields a SGD design with (4) when $s = 2t$. It also follows from the construction of the SRGD design with (3) whose block intersection numbers are 0 and $2t$, and the block intersection numbers for the $8t$ blocks, obtained from the SRGD design with (3) by merging the last two treatments in the last group as one, are $1, 2t$ and $2t+1$ and each of the other $2(4t-1)$ blocks of the SGD design occurs twice, whereas in the 2-multiple of a symmetric BIB($8t-1, 4t, 2t$) each of the blocks will be duplicated. Hence the BIB design with (5) for $s = 2t$ is non-isomorphic to the 2-multiple of a symmetric BIB($8t-1, 4t, 2t$). Thus, this fact always holds under the existence of a Hadamard matrix of order $4t$.

The complement of (5) when $s = 2t$ is a BIB design with parameters

$$v = 8t-1, b = 2(8t-1), r = 2(4t-1), k = 4t-1, \lambda = 2(2t-1)$$

which is clearly non-isomorphic to a 2-multiple of a symmetric BIB($8t-1, 4t-1, 2t-1$). In particular, when $t = 2, 3, 4, 5$, the corresponding designs are of numbers MR109 (known), MR327 (new), MR642 (possibly new), MR1021 (possibly new), in the table of Mathon and Rosa [3,4], as 2-quasi-multiple BIB designs.

Remark. Terms "new" and "possibly new" regarding the status of solutions of BIB designs are here used as follows.

(i) "new" means that the present solution is non-isomorphic to known solutions in tables by Mathon and Rosa [3,4]. This gives the increase of the number of non-isomorphic solutions by one.

(ii) "possibly new" means that it may be impossible to show the non-isomorphism of the present solution to the available ones in their tables, by using the information given in their tables and the present block structure of our designs, i.e., there is still a possibility of the present solution being non-isomorphic to the known solutions. Therefore, it cannot be also said as "known" solutions. Namely, a problem in this regard is: "Is there *another* symmetric BIB design whose derived design has the same parameters as the present design? Then, how to show the non-isomorphism of the present designs to a derived design of another symmetric BIB design?" This is not easy, because a well-known property that block intersection numbers of any two blocks are at most λ, cannot be even utilized here. Thus, we do not say "known", but we can say at least "possibly new" only.

Case II: general s (any positive integer ≥ 2)

The existence of a Hadamard matrix of order $4s$ provides a solution of the BIB design with parameters $v = 4s-1 = b, r = 2s-1 = k, \lambda = s-1$, whose residual design is a BIB design with parameters $v = 2s, b = 2(2s-1), r = 2s-1, k = s, \lambda = s-1$, and whose derived design is a BIB design with parameters $v = 2s-1, b = 2(2s-1), r = 2(s-1), k = s-1, \lambda = s-2$, whose complement is a BIB design with parameters $v = 2s-1, b = 2(2s-1), r = 2s, k = s = \lambda$.

Now the existence of a Hadamard matrix of order $4s$ implies the existence of a BIB design with parameters

$$v = 2s, b = 2(2s-1), r = 2s-1, k = s, \lambda = s-1 \tag{6}$$

which in turn implies the existence of a SRGD design, by the application of Remark 1 in Sinha et al. [10], with parameters

$$v = 4s = b, r = 2s = k, \lambda_1 = 0, \lambda_2 = s, m = 2s, n = 2, \tag{7}$$

which may be a new series. On the other hand, the existence of a Hadamard matrix of order $4s$ implies the existence of a BIB design with parameters

$$v = 2s-1, b = 2(2s-1), r = 2s, k = s = \lambda \tag{8}$$

which in turn implies the existence of a SGD design with parameters

$$v = 2(2s-1) = b, r = 2s = k, \lambda_1 = 2s, \lambda_2 = s, m = 2s-1, n = 2. \tag{9}$$

By juxtaposing designs with (7) and (9) and considering two treatments in the last group in (7), as one, we can obtain a BIB design with parameters

$$v = 4s - 1, b = 2(4s - 1), r = 4s, k = 2s = \lambda,$$

whose complement is given by

$$v = 4s - 1, b = 2(4s - 1), r = 2(2s - 1), k = 2s - 1, \lambda = 2(s - 1). \quad (10)$$

By the construction the BIB designs with (6) and (8) are not duplicate designs, and thus the derived design with (10) cannot be a duplicate design. In particular, when $s = 3, 5, 7, 9$, by designs: (*i*) MR4 and a BIB(5,3,3), (*ii*) MR33 and a complement of MR24, (*iii*) MR89 and a complement of MR77, (*iv*) MR179 and a complement of MR158, we can obtain the designs of numbers (*i*) MR47 (known), (*ii*) MR208 (possibly new), (*iii*) MR464 (possibly new), (*iv*) MR802 (possibly new), respectively, as 2-quasi-multiple BIB designs.

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