Spanning Sets and Scattering Sets in Handcuffed Designs of order v and block size 3*

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ABSTRACT. For each admissible v we exhibit a $H(v,3,1)$ with a spanning set of minimum cardinality and a $H(v,3,1)$ with a scattering set of maximum cardinality.

1 Introduction

Let K_v be the complete undirected graph on v vertices. A *handcuffed design* $H(v,3,1)$ [3] is a pair $(V,\mathcal{B})$ where V is the vertex set of K_v and $\mathcal{B}$ is an edge disjoint decomposition of K_v into copies of P_2 (the simple path with 2 edges) such that each vertex belongs to exactly r copies of P_2. We call the elements of $\mathcal{B}$ *blocks*. Given a block $b \in \mathcal{B}$, we will use the same symbol b to denote its vertex set.

A handcuffed design $H(v,3,1)$ exists if and only if $v \equiv 1 \pmod 4$ [4]. It can be shown [3] that every element of an $H(v,3,1)$ must occur in an exterior (that is, the first or last) position of the same number of blocks, say u. Further, if $|\mathcal{B}| = t$, the following equalities can be shown: $u = \frac{v-1}{2}$, $r = \frac{3(v-1)}{4}$, $t = \frac{v(v-1)}{4}$.

A subset $S \subseteq V$ is *independent* if b is not a subset of S for every $b \in \mathcal{B}$. An independent set S is *maximal* if for all $y \in V \setminus S$, $S \cup \{y\}$ is not independent. An independent set $S \subset V$ is an *arc* if for all $b \in \mathcal{B}$, $|b \cap S| \leq 2$. An arc is *complete* when it is not contained in a larger arc. For $H(v,3,1)$ the notions of a (complete) arc and of a (maximal) independent set coincide.

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Let $X \subseteq V$. A block $b \in \mathcal{B}$ is called *secant* or *tangent* or *exterior* to X if $|b \cap X| = 2$ or 1 or 0 respectively.

The *spanned set* $C(X)$ is the set of $y \in V \backslash X$ such that there exists at least one secant meeting y. The subset X is a *spanning set* if for every $v \in V \backslash X$, $v \in C(X)$.

A *scattering set* $X \subseteq V$ is an arc for which every $y \in V \backslash X$ has the property that y appears in at most one secant.

It is straight-forward to verify that every complete arc is a spanning set; the converse need not hold.

Spanning and scattering sets in Steiner triple systems are studied in [1].

In this note we exhibit, for each $v \equiv 1 \pmod 4$, an $H(v,3,1)$ with a spanning set of minimum cardinality and an $H(v,3,1)$ with a scattering set of maximum cardinality.

2 Scattering and spanning sets

Consider an $H(v,3,1)$, $\mathcal{H} = (V, \mathcal{B})$. Using the terminology of [1] we define the *scattering number scat*($\mathcal{H}$) to be the size of a largest scattering set in $\mathcal{H}$, and we define the *spanning number span*($\mathcal{H}$) to be the size of a smallest spanning set in $\mathcal{H}$. Let *scat*(v) be the the maximum scattering number of a $H(v,3,1)$, and let *span*(v) be the minimum spanning number. Let X be a subset of V of cardinality x. It is easy to see that for X to be a scattering set, $x + \binom{x}{2} \leq v$. This implies $\text{scat}(v) \leq L(v) = \left\lfloor \frac{\sqrt{8v+1}-1}{2} \right\rfloor$.

A handcuffed design $\mathcal{H}$ is *scattered* if $\text{scat}(\mathcal{H}) = L(v)$.

The following result is proved in [4].

Lemma 1. ([4,Theorem 1]). *For every admissible $v \geq 5$ it is possible to embed an $H(v,3,1)$ in an $H(v+4,3,1)$.*

Proof: Let $v = 4m+1$, $m \geq 1$, and let $(V, \mathcal{B})$ be an $H(v,3,1)$ on $V = 1,2,\ldots,v$. Put $W = V \cup \{v+1, v+2, v+3, v+4\}$. Form the block set $\mathcal{C}$ by putting on it the following blocks:

(i) All the blocks of $\mathcal{B}$.

(ii) The blocks of an $H(5,3,1)$ on the vertex set $\{v, v+1, \ldots, v+4\}$.

(iii) For $h = 1,3,5,\ldots,2m-1$ the blocks $\{4m+2, 2h-1, 4m+3\}$, $\{4m+2, 2h, 4m+3\}$, $\{2h-1, 4m+4, 2h\}$, $\{2h-1, 4m+5, 2h\}$.

(iv) For $h = 2,4,6,\ldots,2m$ the blocks $\{4m+4, 2h-1, 4m+5\}$, $\{4m+4, 2h, 4m+5\}$, $\{2h-1, 4m+2, 2h\}$, $\{2h-1, 4m+3, 2h\}$.

It is straight-forward that $(W, \mathcal{C})$ is an $H(v4,3,1)$. □

Theorem 1. *For every $v \equiv 1 \pmod 4 \geq 17$, span$(v) = 4$. For $v = 5, 9$ and 13, span$(v) = 3$.*

Proof: Let X be a spanning set of minimum cardinality x in a handcuffed design $(V, \mathcal{B})$. Clearly $x \geq 2$. Since every element of V occurs in an exterior position of exactly $\frac{v-1}{2}$ blocks, we obtain $2(v - x - \binom{x}{2}) + \binom{x}{2} \leq \frac{(v-1)x}{2}$. Clearly this inequality implies that $x > 2$. The value $x = 3$ is possible only for $v \in \{5, 9, 13\}$, whereas $x = 4$ is possible for every admissible v. To prove that span$(v) = 3$ for $v = 5, 9, 13$ it is sufficient to see that the following $H(5,3,1)$, $H(9,3,1)$ and $H(13,3,1)$ contain the spanning set $X = \{a, b, c\}$: 1) Vertex set $V = X \cup \{0, 1\}$. Block set (all commas and brackets are omitted) $\mathcal{B} = \{ba0, 0bc, c01, 1ca, a1b\}$; 2) $V = X \cup \{0, 1, \ldots, 5\}$. $\mathcal{B} = \{ab1, ac2, bc3, a4b, a5c, b0c, 1a2, 3a0, 3b5, 21c, 314, 52b, 230, 54c, 051, 401, 024, 534\}$; 3) $V = X \cup \{0, 1, \ldots, 9\}$. $\mathcal{B} = \{1ab, 2bc, 3ca, a4b, b5c, c6a, a7b, b8c, c20, c9a, a0b, 2a3, 5a8, 1b3, 6b9, 4c7, 0c1, 123, 134, 142, 516, 718, 910, 250, 269, 275, 829, 397, 308, 536, 738, 460, 476, 485, 498, 540, 659, 786, 907\}$.

To prove that span$(v) = 4$ for every admissible $v \geq 17$, take the following $H(9,3,1)$ $V = \{1, 2, \ldots, 9\}$; $\mathcal{B} = \{126, 237, 348, 419, 135, 246, 617, 728, 863, 965, 674, 875, 185, 983, 493, 792, 154, 952\}$. By Lemma 1 embed $(V, \mathcal{B})$ in the $H(v+4, 3, 1)$ $(W, \mathcal{C})$, $W = \{1, 2, \ldots, v+4\}$. Since $X = \{1, 2, 3, 4\}$ is a spanning set in $(V, \mathcal{B})$, then for each $x \in V$ there is a secant X. Moreover $\{3, v+1, 4\}$, $\{3, v+2, 4\}$, $\{1, v+3, 2\}$ and $\{1, v+4, 2\}$ are secants X. This implies that X is still a spanning set in $(W, \mathcal{C})$. □

We observe that $X = \{a, b, c\}$ is not a scattering set in the $H(5,3,1)$ given in 1), X is a scattering set of maximum cardinality in the handcuffed design given in 2), and a scattering set, but not of maximum cardinality in 3). It follows from $x + \binom{x}{2} \leq v$ that scat$(5) = 2$ and scat$(9) = 3$.

Moreover it is easy to see that all the spanning sets constructed in the Theorem 1 are also complete arcs. This establishes the existence of handcuffed designs with complete arcs of minimum possible cardinality (the analogous problem for Steiner triple systems is posed in [2] and completely solved in [1]).

Corollary 1. *For every $v \equiv 1 \pmod 4$ there exists an $H(v,3,1)$ containing a complete arc of minimum cardinality. Which is 4 for $v \geq 17$ and 3 for $v = 5, 9$ and 13.*

For every integer $x \geq 2$ define $\alpha(x) \in \{0, 1, 2, 3\}$ by putting $\alpha(x) = 0$ for $x \equiv 1, 6 \pmod 8$, $\alpha(x) = 1$ for $x \equiv 0, 7 \pmod 8$, $\alpha(x) = 2$ for $x \equiv 2, 5 \pmod 8$ and $\alpha(x) = 3$ for $x \equiv 3, 4 \pmod 8$. The following theorem is an easy consequence of the inequality $x + \binom{x}{2} \leq v$.

Theorem 2. *Let X be a scattering set of maximum cardinality x in an $H(v,3,1)$, then $v \geq \binom{x+1}{2} + \alpha(x)$.*

Theorem 3. *For every integer $x \geq 2$ there exists a scattered $H(v,3,1)$ $\mathcal{H}$ on $v = \binom{x+1}{2} + \alpha(x)$ vertices such that scat$(\mathcal{H}) = x$.*

Proof: Let $X = \{1,2,\dots,x\}$ and $V = X \cup \{a_1, a_2, \dots, a_{v-x}\}$. We split the proof into two different cases: x even and x odd. Suppose x is even (see Example 1). Since scat(5) = 2, the theorem is proved for $x = 2$. Put in $\mathcal{B}_1$ the following blocks (where $h+j$ is reduced to the range $\{1,2,\dots,x\}$ modulo x):

(1) $\{a_{(h-1)x+j}, j, h+j\}$, for $h = 1,2,\dots,\frac{x-2}{2}$ and $j = 1,2,\dots,x$.

(2) $\{a_{\frac{x(x-2)+2j}{2}}, j, \frac{x}{2}+j\}$, for $j = 1,2,\dots,\frac{x}{2}$.

It is easy to verify that: $|\mathcal{B}_1| = \binom{x}{2} = v - x - \alpha(x)$; each element of $\{1,2,\dots,\frac{x}{2}\}$ is in the middle, resp. exterior, position of exactly $\frac{x}{2}$ resp. $\frac{x-2}{2}$, blocks of $\mathcal{B}_1$; each element of $\{\frac{x}{2}+1, \frac{x}{2}+2,\dots,x\}$ is in the middle, resp. exterior, position of exactly $\frac{x-2}{2}$, resp. $\frac{x}{2}$, blocks of $\mathcal{B}_1$.

Since $v-x$ is odd we can construct the difference sets $F_k = \{\{a_j, a_{k+j}\} : j = 1,2,\dots,v-x\}$ for every $k = 1,2,\dots,\frac{v-x-1}{2}$. (In the indices the sum is reduced to the range $\{1,2,\dots,v-x\}$ modulo $v-x$). Using the first $\frac{v-2x-1}{2}$ difference sets construct $\frac{(v-2x-1)(v-x)}{4}$ blocks such that any element of $V \backslash X$ appears $\frac{v-2x-1}{4}$ times in the middle position. Say $\mathcal{B}_2$ be the set of these blocks. With the last $\frac{x}{2}$ difference sets and the elements of X form the set $\mathcal{B}_3$ of $(v-x)\frac{x}{2}$ blocks such that: every block of $\mathcal{B}_3$ contains exactly one vertex $s \in X$; s is always in an exterior position; the number of blocks of $\mathcal{B}_3$ in which every s occurs is either $\frac{v-x+1}{2}$ if $s \in \{1,2,\dots,\frac{x}{2}\}$, or $\frac{v-x-1}{2}$ if $s \in \{\frac{x}{2}+1, \frac{x}{2}+2,\dots,x\}$; every block in $\mathcal{B}_1 \cup \mathcal{B}_3$ does not contain a repeated edge. At last using all the edges of K_v missing in the above blocks of $\mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$ construct the block set $\mathcal{B}_4$ in such a way that an element of X appears always in the middle position of any block of $\mathcal{B}_4$ (note that it is possible to form $\mathcal{B}_3$ or $\mathcal{B}_4$ in many different ways). It is easy to see that $\mathcal{H} = (V, \cup_{i=1}^4 \mathcal{B}_i)$ is an $H(v,3,1)$ and $X \subseteq V$ is a scattering set of maximum cardinality.

Let x be odd (see Example 2). Let $\mathcal{B}_1 = \{\{a_{(h-1)x+j}, j, h+j\} : h = 1,2,\dots,\frac{x-1}{2}, j = 1,2,\dots,x\}$ ($h+j$ is reduced to the range $\{1,2,\dots,x\}$ modulo x). It is easy to verify that: $|\mathcal{B}_1| = \binom{x}{2} = v - x - \alpha(x)$; each element of $\{1,2,\dots,x\}$ appears in an exterior position and in an interior position of exactly $\frac{x-1}{2}$ blocks of $\mathcal{B}_1$. Since $v-x$ is even we can construct the following difference sets, where we put $n = \frac{v-x}{2}$: $F = \{\{a_j, a_{j+k}\} : j = 1,2,\dots,v-x\}$ for $k = 1,2,\dots,n-1$, and $F = \{\{a_j, a_{j+n}\} : j = 1,2,\dots,n\}$ (in the indices the sum is reduced to the range $\{1,2,\dots,v-x\}$ modulo $v-x$). Let $\overline{F}_1 = F_1 \backslash \{\{a_n, a_{n+1}\}, \{a_{2n}, a_1\}\} \cup \{\{a_n, a_1\}, \{a_{2n}, a_{n+1}\}\}$ and $\overline{F}_{n-1} = F_{n-1}\{\{a_1, a_n\}, \{a_{n+1}, a_{2n}\}\} \cup \{\{a_{n+1}, a_n\}, \{a_1, a_{2n}\}\}$. Clearly $\overline{F}_1 \cup \overline{F}_{n-1} = F_1 \cup F_{n-1}$. With the edges of $\overline{F}_{n-1}$ we form the block set $\mathcal{B}_2 =$

$\{\{a_2, a_{n+1}, a_n\}, \{a_3, a_{n+2}, a_1\}, \{a_4, a_{n+3}, a_2\}, \ldots, \{a_n, a_{2n-1}, a_{n-2}\}, \{a_1, a_{2n}, a_{n-1}\}\}$. Take F_n and others $\frac{x-1}{2}$ difference sets F_h, $h \neq 1, n-1$. Form the set $\mathcal{B}_3$ of blocks such that any element of X appears in the exterior position exactly $\frac{v-x}{2}$ times, $|b \cap X| = 1$ for every $b \in \mathcal{B}_3$, every block in $\mathcal{B}_1 \cup \mathcal{B}_3$ does not contain a repeated edge. Using the remaining $\frac{v-2x-3}{2}$ difference sets (where we take $\overline{F}_1$ instead of F_1) construct $\frac{(v-2x-3)(v-x)}{4}$ blocks such that any element of $V \backslash X$ appears $\frac{v-2x-3}{4}$ times in the middle position. Say $\mathcal{B}_4$ be the set of these blocks. At last using all the edges of K_v missing in the above blocks of $\mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_4$, construct the block set $\mathcal{B}_5$ in such a way that every block of $\mathcal{B}_5$ has always in the middle position an element of X. It is easy to see that $\mathcal{H} = (V, \cup_{i=1}^{5} \mathcal{B}_i)$ is a $H(v,3,1)$ and $X \subseteq V$ is a scattering set of maximum cardinality. □

Example 1. Let $x = 4$. Then it is $\alpha(x) = 3$ and $v = 13$. The blocks (1) of $\mathcal{B}_1$ are (we omit all the commas and brackets) $a_1 12$, $a_2 23$, $a_3 34$, $a_4 41$; the blocks (2) of $\mathcal{B}_1$ are $a_5 13$, $a_6 24$. The blocks of $\mathcal{B}_2$ are $a_j a_{j+1} a_{j+3}$ for every $j = 1, 2, \ldots, 9$. It is easy to see that we can put $\mathcal{B}_3 = \{2a_1a_4, 1a_2a_5, 1a_3a_6, 1a_4a_7, 2a_5a_8, 1a_6a_9, 2a_7a_1, 2a_8a_2, 1a_9a_3, 3a_1a_5, 4a_2a_6, 2a_3a_7, 3a_4a_8, 3a_5a_9, 3a_6a_1, 4a_7a_2, 4a_8a_3, 4a_9a_4\}$ and $\mathcal{B}_4 = \{a_1 4 a_3, a_2 3 a_7, a_4 2 a_9, a_5 4 a_6, a_7 1 a_8, a_8 3 a_9\}$.

Example 2. Let $x = 5$. Then it is $\alpha(x) = 2$ and $v = 17$. The blocks of $\mathcal{B}_1$ are $a_1 12$, $a_2 23$, $a_3 34$, $a_4 45$, $a_5 51$, $a_6 13$, $a_7 24$, $a_8 35$, $a_9 41$, $a_{10} 52$. With the edges of $\overline{F}_5$ form the block set $\mathcal{B}_2 = \{a_2a_7a_6, a_3a_8a_1, a_4a_9a_2, a_5a_{10}a_3, a_6a_{11}a_4, a_1a_{12}a_5\}$. We use F_6, F_2 and F_3 to construct the block set $\mathcal{B}_3 = \{a_1a_7 1, a_2a_8 1, a_3a_9 1, a_4a_{10} 1, a_5a_{11} 1, a_6a_{12} 1, a_1a_3 2, a_2a_4 2, a_3a_5 2, a_4a_6 2, a_5a_7 3, a_6a_8 2, a_7a_9 3, a_8a_{10} 3, a_9a_{11} 3, a_{10}a_{12} 3, a_{11}a_1 5, a_{12}a_2 3, a_1a_4 5, a_2a_5 4, a_3a_6 4, a_4a_7 4, a_5a_8 5, a_6a_9 5, a_7a_{10} 2, a_8a_{11} 5, a_9a_{12} 5, a_{10}a_1 4, a_{11}a_2 4, a_{12}a_3 4\}$. With the edges of $\overline{F}_1$ and F_4 form the block set $\mathcal{B}_4 = \{a_1a_2a_6, a_2a_3a_7, a_3a_4a_8, a_4a_5a_9, a_5a_6a_{10}, a_6a_1a_5, a_7a_8a_{12}, a_8a_9a_1, a_9a_{10}a_2, a_{10}a_{11}a_3, a_{11}a_{12}a_4, a_{12}a_7a_{11}\}$. At last let $\mathcal{B}_5 = \{a_2 1 a_3, a_4 1 a_5, a_1 2 a_9, a_{11} 2 a_{12}, a_8 4 a_{10}, a_{11} 4 a_{12}, a_2 5 a_3, a_6 5 a_7\}$.

Theorem 4. *For every $v \equiv 1 \pmod 4$ it is scat$(v) = L(v)$.*

Proof: Since scat$(5) = 2$ and scat$(9) = 3$ we suppose $v \geq 13$. Obviously it is $\binom{L(v)+1}{2} \leq v$ for every admissible v. Put $w = \binom{L(v)+1}{2} + \alpha(L(v))$; it is easy to verify that $w \equiv 1 \pmod 4$. If it is $w > v$ we can put $w = v + 4\sigma$, $\sigma \geq 1$. Then $w = v + 4\sigma \geq \binom{L(v)+1}{2} + 4\sigma > \binom{L(v)+1}{2} + \alpha(L(v))$, this is impossible. Therefore $w \leq v$. By Theorem 3 there exists a scattered $H(w,3,1)$ $\mathcal{H}$ with scat$(\mathcal{H}) = L(v)$. Then we can consider only the v such that $w < v$. Let $X = \{1, 3, 5, \ldots, 2L(v) - 1\}$ be the scattering set of maximum cardinality in $\mathcal{H}$. Since $L(v) \leq \frac{v-5}{2}$ for $v \geq 17$, then by the embedding $w \to w + 4$ (Lemma 1) we construct a scattered $H(v,3,1)$ having X as scattering set of maximum cardinality $L(v)$ (note that no any block of (ii), (iii) and (iv) of Lemma 1 is secant to X).

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