Short Proofs of Theorems of Nash-Williams and Tutte *

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ABSTRACT. We give short proofs of theorems of Nash-Williams (on edge-partitioning a graph into acyclic subgraphs) and of Tutte (on edge-partitioning a graph into connected subgraphs). We also show that each theorem can be easily derived from the other.

1 Introduction

Throughout this paper, the word graph will refer to an undirected graph with multiple edges but no loops. Other terminology and notation will be standard except as indicated. A good reference for any undefined terms is [1].

One of the most fundamental problems in graph theory is to determine whether a graph can be vertex- (or edge-) partitioned into a given number of subgraphs, all having a specified property. In this paper, we will be concerned with two classical problems of this type.

Problem 1. Given a graph G and a positive integer k, can G be edge-partitioned into k *acyclic* subgraphs? (The smallest integer k for which such an edge-partition exists is called the *arboricity* of G, denoted $a(G)$.)

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Problem 2. Given a graph G and a positive integer k, can G be edge-partitioned into k *connected spanning* subgraphs (equivalently, does G contain k edge-disjoint spanning trees)?

Elegant solutions for these two problems were given in the early 1960's in the following theorems of Nash-Williams and Tutte, respectively.

Nash-Williams' Theorem [9]. *A graph G can be edge-partitioned into k acyclic subgraphs if and only if for all nonempty $X \subseteq V(G), |E\langle X\rangle| \leq k(|X|-1)$.*

Before stating Tutte's theorem, we introduce the following notation. Given a partition P of $V(G)$, let $|P|$ denote the number of sets in the partition P and let $E(P)$ denote the set of edges in G which join different sets in P.

Tutte's Theorem [8], [11]. *A graph G can be edge-partitioned into k connected spanning subgraphs if and only if for every partition P of $V(G)$, $|E(P)| \geq k(|P|-1)$.*

Nash-Williams' theorem is certainly one of the best known results in graph theory (see for example [6]), though it is more often expressed in the equivalent form below.

Nash-Williams' Theorem (Alternate Form). *For any graph G, $a(G) = \max_{X \subseteq V(G)} \left\lceil \frac{|E\langle X\rangle|}{|X|-1} \right\rceil$.*

Tutte's theorem, on the other hand, does not appear to be as well known as it deserves to be. Our goal in this paper is to present a short proof for each of these theorems, and also to show that either theorem can be easily derived from the other. We hope thereby to make both theorems more accessible, and to help give Tutte's theorem the recognition it deserves. In Section 2 of this paper, we give a short proof of Nash- Williams' theorem, and then show that Tutte's theorem can be easily derived from Nash-Williams' theorem as a corollary. In Section 3, we reverse this order—we begin by giving a new proof of Tutte's theorem, and then derive Nash-Williams' theorem as a corollary.

Several other proofs of these theorems, besides the originals cited above, have appeared in the literature. Edmonds gave a constructive generalization of both theorems via his matroid partitioning algorithm (see [4] and [7, Section 8.7]). Nash-Williams even suggested in 1982 [10] that the application of Edmonds' algorithm to the special case of graphs provided 'probably the neatest proof' of the two theorems up to that time. In 1976, Bollobas [2] gave an alternate proof of Tutte's theorem, which is easier than the original proofs in [8] and [11] and uses only ideas from graph theory. More recently, Enomoto [5] and Chen et. al. [3] have given new proofs of Nash-Williams' theorem, and we consider the proof of Chen et. al. to be especially elegant. Both of these recent proofs are quite different from our proof in Section 2.

2 A Short Proof of Nash-Williams' Theorem

We now give a short proof of Nash-Williams' theorem which does not assume Tutte's theorem, and then show how Tutte's theorem can be easily derived as a corollary.

2.1 Proof of Nash-Williams' Theorem

The necessity is clear and thus we establish just the sufficiency.

Suppose G cannot be edge-partitioned into k forests and that G is edge-minimal in this regard. Let e_o be any edge of G. Then $G - e_o$ can be edge-partitioned into k forests, say $F_1, F_2, \ldots, F_k$.

Label as many edges of G as possible using the following simple labelling algorithm.

algorithm {edge-labelling}
Initially all edges in G are unlabelled. Label e_o with 0. If the end vertices of an edge labelled r are joined by a path P entirely contained in one of the forests $F_1, \ldots, F_k$, give any unlabelled edge on P label $r + 1$. □

Let us call an edge *forest-complete* if its end vertices are joined by a path in each of the forests $F_1, F_2, \ldots, F_k$. Clearly we may assume e_o is forest-complete. We now establish the following lemma.

Lemma. *Every labelled edge is forest-complete.*

Proof of the Lemma: Suppose some labelled edge e_r with label $r \geq 1$ is not forest-complete. In particular, suppose the endvertices of e_r are not joined by a path in the forest F_s. Since e_r is labelled r, there is a sequence of edges $e_r, e_{r-1}, \ldots, e_1, e_o$ such that for $j = 1, 2, \ldots, r$, e_j is labelled j and belongs to a path entirely contained in one of the forests joining the endvertices of e_{j-1}.

For any edge $e \neq e_o$, let ind(e) permanently denote the index of the forest in $F_1 \cup \ldots \cup F_k$ to which e belongs. Define the set of edges X_j initially to be $E(F_j)$, for $j = 1, 2, \ldots, k$, and then perform the edge relocations below in the order indicated (here $e \to X$ means to move the edge e from the set—if any—to which it currently belongs into the set X):

$$e_r \to X_s, e_{r-1} \to X_{ind(e_r)}, \ldots, e_1 \to X_{ind(e_2)}, e_o \to X_{ind(e_1)}$$

Note that $X_1 \cup \ldots \cup X_k$ is now a partition of $E(G)$, and thus if $\langle X_i \rangle$ were acyclic for every i, we would have the contradiction that G could be edge-partitioned into k forests, and the proof of the Lemma would be complete. Thus, we now complete the proof of the Lemma by showing that every $\langle X_i \rangle$ remains acyclic after each of the edge relocations above.

Certainly this is true after the first relocation $e_r \to X_s$. Suppose it is true after the relocations of $e_r, e_{r-1}, \ldots, e_{j+1}$, and consider the relocation

$e_j \to X_{ind(e_{j+1})}$. The path P_{j+1} in $F_{ind(e_{j+1})}$ joining the endvertices of e_j has largest edge label $j+1$, and thus P_{j+1} remains intact in the forest $\langle X_{ind(e_{j+1})}\rangle$ after relocating each of the edges $e_r, e_{r-1}, \ldots, e_{j+2}$. So when we finally do the relocation $e_{j+1} \to X_{ind(e_{j+2})}$, there will no longer be a path in the forest $\langle X_{ind(e_{j+1})}\rangle$ joining the end vertices of e_j, and thus the relocation $e_j \to X_{ind(e_{j+1})}$ would leave $\langle X_{ind(e_{j+1})}\rangle$ acyclic. This proves the Lemma.

Let $G_L = (V_L, E_L)$ denote the subgraph of G induced by the labelled edges. It is trivial that G_L is connected. Let F_j' denote the forest on V_L with edge set $E(F_j) \cap E_L$, for $j = 1, 2, \ldots, k$.

We now prove F_j' is a connected forest (i.e., a spanning tree) on V_L. Let $v, w \in V_L$. Since G_L is connected, v and w are joined by a path of labelled edges $e_1' \, e_2' \ldots e_s'$. Since each e_i' is forest- complete by the Claim, the end vertices of e_i' are joined by a path P_{ij} in F_j. Since e_i' is labelled, every edge in P_{ij} must also be labelled, and thus P_{ij} belongs to F_j'. Thus v, w are joined by the walk $P_{1j} \, P_{2j} \ldots P_{sj}$ in F_j', and so F_j' is a connected forest as asserted.

By the above paragraph, we have $|E(F_j')| = |V_L| - 1$, for every j. Thus,

$$|E\langle V_L\rangle| \geq |E_L| = 1 + \sum_{j=1}^{k} |E(F_j')| = 1 + k\,(|V_L| - 1)\,.$$

Taking $X = V_L$, the theorem is proved. □

2.2 Tutte's Theorem as a Corollary of Nash-Williams' Theorem

The necessity is clear; we will prove the sufficiency by induction on $n = |V(G)|$, the result being obvious for $n = 2$.

Let P be a partition of $V(G)$, say $V_1 \cup \ldots \cup V_r$, with $|P| = r \geq 2$ and such that $|E(P)| - k\,(|P| - 1)$ is as small as possible. Denote this optimal difference by d; by hypothesis $d \geq 0$. In order to derive a contradiction, suppose G does not contain k edge-disjoint spanning trees. We consider two cases.

Case 1. $|V_i| \geq 2$, for some i.

We begin by proving two lemmas.

Lemma 1. *$\langle V_i\rangle$ contains k edge-disjoint spanning trees.*

Proof of Lemma 1: Otherwise by induction there exists a partition P_i of V_i with $|E(P_i)| \leq k\,(|P_i| - 1) - 1$. Consider the refinement P' of P obtained by replacing V_i in P by the sets of P_i. Then $|P'| = |P| + |P_i| - 1 \geq 2$, while $|E(P')| = |E(P)| + |E(P_i)| \leq k(|P| - 1) + d + k(|P_i| - 1) - 1 = k(|P'| - 1) + (d - 1)$. Thus, $|E(P')| - k(|P'| - 1) < d$, contradicting the optimality of P. This proves Lemma 1.

Let G' denote the graph which results when V_i is contracted to a single vertex v (keeping all resulting parallel edges, but eliminating any loops).

Lemma 2. *G' contains k edge-disjoint spanning trees.*

Proof of Lemma 2: Otherwise by induction $V(G')$ has a partition P' with $|E(P')| \leq k(|P'|-1)-1$. Construct the partition Q of $V(G)$ by deleting v from its partition set in P', and adding to this partition set the vertices in V_i. We find $|Q| = |P'| \geq 2$ while $|E(Q)| = |E(P')| \leq k(|P'|-1)-1 = k(|Q|-1)-1$, contradicting the optimality of P since $d \geq 0$. This proves Lemma 2.

By these two lemmas, G' (resp, $\langle V_i \rangle$) contains k edge-disjoint spanning trees $T_1', \ldots, T_k'$ (resp, $T_1'', \ldots, T_k''$). But then G itself would contain k edge-disjoint spanning trees $T_1, \ldots, T_k$, a contradiction, where T_j is obtained from T_j' by replacing v in T_j' by the tree T_j'' as shown in Figure 1. This completes Case 1.

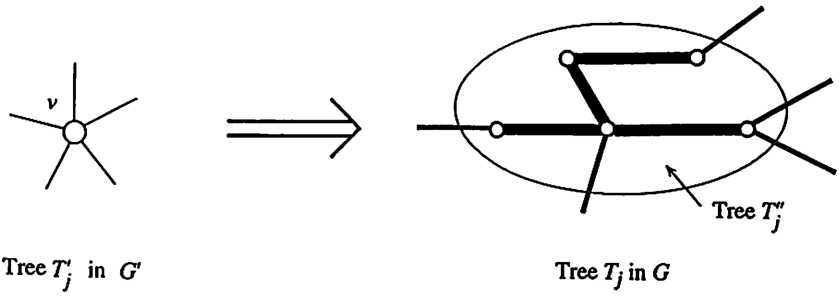

Figure 1

Case 2. $|V_i| = 1$, for all i.

Then, $|E(G)| = k(n-1)+d$ with $d \geq 0$. Delete d edges from G arbitrarily to obtain a subgraph G' with exactly $k(n-1)$ edges. Clearly, G' does not contain k edge-disjoint spanning trees (else G would); equivalently, G' cannot be edge-partitioned into k forests. By Nash-Williams' theorem, there exists a subset $S \subset V(G') = V(G)$ with $|E_{G'}\langle S \rangle| \geq k(|S|-1)+1$. Consider the partition Q of $V(G)$ consisting of the set S together with $n-|S|$ singletons. Then $|Q| = n-|S|+1 \geq 2$, while $|E(Q)| = |E(G)| - |E_G\langle S \rangle| \leq |E(G)| - |E_{G'}\langle S \rangle| \leq k(n-1)+d-(k(|S|-1)+1) = k(|Q|-1)+d-1$, again contradicting the optimality of P.

This completes the proof of Tutte's theorem. □

3 A Short Proof of Tutte's Theorem

We now give a short proof of Tutte's theorem which does not assume Nash-Williams' theorem, and then show how Nash-Williams' theorem can be easily derived as a corollary.

3.1 Proof of Tutte's Theorem

The necessity is clear and thus we prove only the sufficiency.

Suppose G does not contain k edge-disjoint spanning trees and is edge maximal in this regard. Then G contains k edge-disjoint spanning forests $F_1, F_2, \ldots, F_k$, where F_1 has two components and $F_2, \ldots, F_k$ have one component each. We now describe a simple algorithm to label certain edges in $F_1 \cup \ldots \cup F_k$.

algorithm {labelling edges in $F_1 \cup \ldots \cup F_k$}
initially all edges in $F_1 \cup \ldots \cup F_k$ are unlabelled;
$F_{i,1} \leftarrow F_i$, for all i; $r \leftarrow 1$;
while there is an unlabelled edge in some $F_{i,r}$ joining different
components in some $F_{j,r}, j \neq i$
begin {another round of labelling edges}
label all such edges with the label r;
$F_{i,r+1} \leftarrow F_{i,r} - \{$ edges in $F_{i,r}$ labelled $r\}$, for $i = 1, 2, \ldots, k$;
$r \leftarrow r + 1$
end;
end. {labelling edges in $F_1 \cup \ldots \cup F_k$} □

Suppose the labelling algorithm concludes with final forests $F_{1,f}, F_{2,f}, \ldots, F_{k,f}$, where $F_{i,f}$ has exactly c_i components. Then there are exactly $c_1 - 2$ labelled edges in F_1, and $c_j - 1$ labelled edges in F_j, for $j = 2, 3, \ldots, k$. Define an equivalence relation $\sim$ on $V(G)$ by $v \sim w$ if and only if v and w belong to the same component of $F_{j,f}$ for every $j = 1, 2, \ldots, k$. Let the partition P of $V(G)$ consist of the equivalence classes of vertices under $\sim$. It is immediate that $|P| \geq \max\{c_1, \ldots, c_k\}$. In a moment, we will prove the following lemma.

Lemma. *$|E(P)|$ is precisely the set of labelled edges.*

Assuming the truth of the Lemma, we find $|E(P)| = (c_1 - 2) + (c_2 - 1) + \ldots + (c_k - 1) = (c_1 + \ldots + c_k) - (k + 1) \leq k \max\{c_1 + \ldots + c_k\} - (k + 1) \leq k|P| - (k+1) = k(|P| - 1) - 1 < k(|P| - 1)$. This contradicts the hypothesis, and thus Tutte's theorem would be proved. It remains only to prove the Lemma.

Proof of the Lemma: It is clear from the definition of P that every labelled edge belongs to $E(P)$. On the other hand, suppose some unlabelled edge $e_o = v_o w_o$ belongs to $E(P)$. By the definition of P, e_o must join different components in some forest $F_{i_o,f}$. If e_o belonged to $F_1 \cup \ldots \cup F_k$, then of course e_o would have been labelled by the labelling algorithm, and hence we may assume $e_o \notin F_1 \cup \ldots \cup F_k$.

We now describe an edge-exchange procedure which begins by adding

e_o to F_{i_o} and ultimately produces k edge-disjoint spanning trees in G, a contradiction.

Consider the path P_o in F_{i_o} joining the endvertices of e_o. P_o must contain a labelled edge (otherwise v_o, w_o would not be in different components of $F_{i_o,f}$). Select as e_1 the labelled edge on P_o with smallest label $\ell(e_1)$, and do the edge exchange $F_{i_o} \leftarrow F_{i_o} + e_o - e_1$. Move $e_1 = v_1w_1$ into the forest F_{i_1} which "caused" e_1 to be labelled $\ell(e_1)$ during the labelling algorithm (because e_1 joined two different components of $F_{i_1,\ell(e_1)}$). The path P_1 in F_{i_1} joining v_1 and w_1 must contain at least one labelled edge. Select as e_2 the labelled edge on P_1 with smallest label $\ell(e_2)$. Of course, $\ell(e_1) > \ell(e_2)$, since for e_1 to be labelled $\ell(e_1)$ in $F_{i_1,\ell(e_1)}$, at least one edge on the path in F_{i_1} joining the end vertices of e_1 must have been *previously* labelled. Assuming we could continue this exchange process, we would eventually move an edge e_s with label $\ell(e_s) = 1$ into $F_{i_s} = F_1$. But when e_s is moved into F_1, the forests $F_1, \ldots, F_k$ would be k edge-disjoint spanning trees in G, yielding the contradiction mentioned above.

If this edge-exchange procedure ever gets blocked, say for the first time when we attempt to move e_j into the current F_{i_j}, it would be because the path then in F_{i_j} joining the end vertices of e_j contains no labelled edge with label less than $\ell(e_j)$. We now complete the proof of the Lemma by showing this never happens.

Claim. Consider the path P_j in the original forest F_{i_j} joining the endvertices v_j, w_j of e_j. Let e' denote the edge on P_j with smallest label $\ell(e')$ (so $\ell(e') < \ell(e_j)$). Then e' will still be on the path in F_{i_j} joining v_j and w_j at the time we move e_j into F_{i_j}.

Proof of the Claim: Otherwise, consider the first edge exchange $F_{i_j} \leftarrow F_{i_j} + e_k - e_{k+1} (k < j)$ after which the path in F_{i_j} joining v_j and w_j no longer contains e'. Consider the situation in F_{i_j} just before this exchange occurs. Let P' denote the path then in F_{i_j} which joins v_j and w_j. The path in F_{i_j} joining the endvertices v_k, w_k of e_k consists, without loss of generality (see Figure 2), of a path from v_k to some $x \in P'$, a subpath $P'[x,y]$ of P' containing e_{k+1}, and a path from $y \in P'$ back to w_k. There are just two possibilities now.

Case 1. $e' \notin P'[x,y]$.

Then the path in $F_{i_j} + e_k - e_{k+1}$ joining v_j and w_j still would contain e', a contradiction.

Case 2. $e' \in P'[x,y]$.

Then $P'[x,y]$ contains both the labelled edges e_{k+1} and e'. Since we selected e_{k+1} rather than e' to exchange out of F_{i_j}, it must be that $\ell(e_{k+1}) \leq \ell(e') < \ell(e_j)$. But since we moved e_{k+1} before e_j, and the exchange process does not fail until we attempt to move e_j, it must also be that $\ell(e_{k+1}) > \ell(e_j)$, a contradiction.

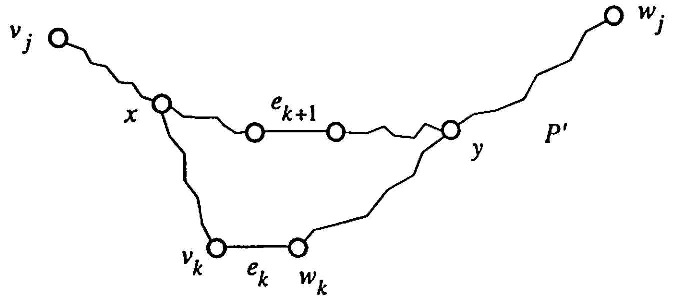

Figure 2

This completes the proof of the Claim, and thereby the proof of Tutte's theorem. □

3.2 Nash-Williams' Theorem as a Corollary of Tutte's Theorem

The necessity is clear; we prove the sufficiency by induction on $n \geq 2$, the result being trivial for $n = 2$.

Let $S \subseteq V(G)$ with $|S| \geq 2$ be such that the *deficiency* $d(S) \doteq k(|S| - 1) - |E\langle S\rangle|$ is as small as possible. By hypothesis, $d(S) \geq 0$. Suppose, trying for a contradiction, that G cannot be edge-partitioned into k forests. We consider two cases.

Case 1. $S \neq V(G)$.

Let G' denote the graph which results when S is contracted to a single vertex v_S (keeping all resulting parallel edges, but eliminating any loops). We first establish the following lemma.

Lemma. *G' can be edge-partitioned into k forests $F'_1, F'_2, \ldots, F'_k$.*

Proof of the Lemma: Since $|S| \geq 2$, we have $|V(G')| < n$. So, by induction, if G' cannot be so partitioned, there exists a nonempty $S' \subseteq V(G')$ with $|E_{G'}\langle S'\rangle| \geq k(|S'| - 1) + 1$. We may assume $v_S \in S'$, or else $S' \subseteq V(G)$ would satisfy $|E_G\langle S'\rangle| \geq k(|S'| - 1) + 1$ and we would be done.

Consider the set $T \subseteq V(G)$ given by $T = (S' - v_S) \cup S$. Note that $|T| = |S| + |S'| - 1 \geq 2$. We have

$$\begin{aligned} |E_G\langle T\rangle| &= |E_{G'}\langle S'\rangle| + |E_G\langle S\rangle| \\ &\geq k(|S'| - 1) + 1 + k(|S| - 1) - d(S) \\ &= k(|T| - 1) - (d(S) - 1) \end{aligned}$$

So $k(|T| - 1) - |E_G\langle T\rangle| < d(S)$, contradicting the optimality of S. This proves the Lemma.

We also have by induction that $\langle S \rangle$ can itself be edge-partitioned into k forests $F_1'', \ldots, F_k''$ (since the subgraph of $\langle S \rangle$ induced by any $X \subseteq S$ is, of course, an induced subgraph of G itself). But then G can be partitioned into k forests $F_1, \ldots, F_k$, where we obtain F_j from F_j' in G' by replacing v_S in F_j' by F_j'' as indicated earlier in Figure 1. This contradiction completes Case 1.

Case 2. $S = V(G)$.

We have $|E(G)| = k(n-1)-d$, where $d = d(S)$. Arbitrarily add d edges to G to get G', with $|E(G)| = k(n-1)$. Clearly, G' cannot be edge-partitioned into k forests (else G could be), and so G' does not contain k edge-disjoint spanning trees. By Tutte's theorem, there exists a partition P of $V(G)$, say $V_1 \cup \ldots \cup V_r$, with $|E(P)| \leq k(|P|-1)-1$. If $|E\langle V_j \rangle| \leq k(|V_j|-1)$ for all j, we would have

$$\begin{aligned} |E(G')| &= |E(P)| + \sum_{i=1}^{r} |E\langle V_i \rangle| \\ &\leq k(|P|-1) - 1 + \sum_{i=1}^{r} k(|V_i|-1) \\ &= k(n-1)-1, \end{aligned}$$

contradicting $|E(G')| = k(n-1)$. It follows that $|E\langle V_j \rangle| \geq k(|V_j|-1) + 1$ for some j. Taking $X = V_j$, the proof of Nash-Williams' theorem is complete. □

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