On The g-centroidal Problem in Special Classes of Perfect Graphs

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ABSTRACT. In this paper we prove some basic properties of the g-centroid of a graph defined through g-convexity. We also prove that finding the g-centroid of a graph is NP-hard by reducing the problem of finding the maximum clique size of G to the g-centroidal problem. We give an $O(n^2)$ algorithm for finding the g-centroid for maximal outer planar graphs, an $O(m + nlogn)$ time algorithm for split graphs and an $O(m^2)$ algorithm for ptolemaic graphs. For split graphs and ptolemaic graphs we show that the g-centroid is in fact a complete subgraph.

Introduction

In this paper we study some of the structural and algorithmic properties of the g-centroid of a graph defined through g-convexity. We first show that finding the g-centroid of a graph is NP-hard by reducing the problem of finding the maximum clique of a graph to the problem of finding the g-centroid of a graph. However, on special classes of graphs such as split graphs, maximal outer planar graphs and Ptolemaic graphs, we present polynomial time algorithms for finding the g-centroid.

Several concepts of convexity in graphs have been studied in the literature modelled on lines similar to the concepts in topology. Convex sets play an important role in facility location theory (finding vertices simultaneously close to a family of vertices), dynamic search in graphs (meant for optimal self adjusting algorithms for information retrieval) and models for

ARS COMBINATORIA 50(1998), pp. 267-278

measuring dissimilarities. For a detailed survey on convexity in graphs, see Duchet [3].

By a graph we mean an undirected graph without loops and multiple edges. A set $S \subseteq V$ is **geodesic convex** (g-convex for short) if for every $u, v \in S$, all the vertices on any $u - v$ shortest path also belong to S. g-convexity has been studied by several authors. See for example Feldman-Hogassen [5], Mulder [12], and Nieminen [14].

The concept of g-centroid is similar to the notion of branch weight centroid for trees. For the definitions not mentioned here, one can refer the book by Buckley and Harary [1].

Definition 1: *Let $G = (V, E)$ be a connected graph. For $v \in V$, define the weight $w(v) = \max \{| S | : S$ is a g-convex set in G not containing $v\}$. Let $gc(G) = \min \{w(v) : v \in V\}$. Then $gc(G)$ is called the **g-centroidal number** of G and the vertices v for which $w(v) = gc(G)$ are called the **g-centroidal vertices**. The **g-centroid** $C_g(G)$ is the set of all g-centroidal vertices. For $v \in V$, we denote by S_v any g-convex set of G not containing v for which $w(v) =| S_v |$.*

Section 1

In this section we give some basic properties of the g-centroid of a graph.

Proposition 1. *For any connected graph G, $C_g(G)$ is a g-convex set of G and $< C_g(G) >$ is connected.*

Proof: To prove the convexity of $C_gG)$, suppose there exists a pair of vertices u, v in $C_g(G)$ such that some vertex p on a $u - v$ geodesic P is not in $C_g(G)$. Then $w(p) > gc(G)$. Consider an S_p with $| S_p |= w(p)$. If $u \notin S_p$, then $gc(G) = w(u) \geq| S_p |= w(p)$, a contradiction. Thus u and similarly v are in S_p. Since S_p is a g-convex set, $P \subseteq S_p$, a contradiction, establishing the claim.

Connectivity of $< C_g(G) >$ follows from the convexity of $C_g(G)$ and by the connectivity of G □

We now give a generalization of the well-known result of Jordan [10] on the centroid of a tree.

Proposition 2. *For a connected graph G, $C_g(G)$ lies in a block of G.*

Proof: Let G be a separable graph and $u, v \in C_g(G)$. If possible let $u \in B_1$ and $v \in B_2$, where B_1 and B_2 are different blocks of G. Let P be a $u - v$ geodesic and p be any cut vertex on P. By the convexity of $C_g(G)$, $p \in C_g(G)$. Consider an S_p with $| S_p |= gc(G)$. Since $p \notin S_p$, S_p lies in some component A of $G - p$. Let $A' = A \cup \{p\}$. Then $| A' |>| S_p |$. Since for any $x, y \in A'$, every $x - y$ geodesic lies in $< A' >$ (because p is a cutvertex), A' is a g-convex set.

Now A' can contain atmost one of u or v say u. For the other vertex v, $w(v) \geq | A' | > | S_p | = gc(G)$, contradicting v is a centroidal vertex. Thus u and v lie in the same block for every u, v in $C_g(G)$ □

Proposition 3. *Let $G = (V, E)$ be a connected graph. For a vertex v of G, if x, y are two vertices in $N_k(v) \cap S_v$ (where $N_k(v) = \{ y : d(v, y) = k\}$) then $d(x, y) < 2k$.*

Corollary 1. *If for a vertex v of G, $N_1(v) \cap S_v$ is non empty, then it forms a clique.*

The proof of Proposition 3 and the corollary 1 follows immediately from the definition of $N_k(v)$ and by the convexity of S_v. Note that for a vertex v of G, $N_1(v) \cap S_v$ can be empty.

Section 2

In this section we study the intractability of finding the g-centroid, the centroidal number $gc(G)$ and weight of a vertex of a graph.

We now extend definition 1 to disconnected graphs. Let $G = G_1 \cup G_2 \cup \cdots \cup G_k$. For $v \in V(G_i)$, define $w(v \mid G) = w(v \mid G_i)$ (where $w(v \mid H)$ denotes the weight of v with respect to the graph H) and $C_g(G) = \cup_{i=1}^{k} C_g(G_i)$.

By a clique we mean a vertex subset of G inducing a complete subgraph of G. A clique M of G is a **maximum clique** if $| M | \geq$ cardinality of any other clique of G.

Lemma 1. *Let G be a graph and u,v be any two vertices not in G. Let G' be the graph obtained by joining u and v to all the vertices of G. Then*

(i) *If G has a unique maximum clique M then $C_g(G') = M$.*

(ii) *If $M_1, M_2, \cdots, M_r$ are the maximum cliques of G and if $\cap_{i=1}^{r} M_i$ is empty then $C_g(G') = V(G')$ otherwise $C_g(G') = \cap_{i=1}^{r} M_i$.*

In all the cases $w(u) = w(v) = \omega(G) + 1$, where $\omega(G)$ is the maximum clique size of G.

Proof: First we claim that if a g-convex set of G' is a proper subset of $V(G')$ then it is complete. Let $S \subset V(G')$ be a g-convex set of G'. If $u, v \in S$, then clearly $S = V(G')$ If S has two non adjacent vertices x and y, different from u, v then x, u, y and x, v, y are geodesics joining x and y in G'. Thus $u, v \in S$ and hence $S = V(G')$, a contradiction. This establishes the claim.

(i) Let M be the maximum clique of G. Then every vertex other than those in M has weight $\omega(G) + 1$ and every vertex in M has weight $\omega(G)$. Thus $C_g(G') = M$.

(ii) Let $M_1, M_2, \cdots, M_r$ be the maximum cliques of G and let $L = \cap_{i=1}^{r} M$. Suppose that L is empty, then for each vertex x we can find a maximum clique not containing x and hence $C_g(G') = V(G')$. If L is non empty then each vertex not in L has weight $\omega(G) + 1$ and every vertex in L has weight $\omega(G)$. Thus $C_g(G') = L$ □

We now prove that the problem of finding the g-centroid of a graph is NP-hard.

Proposition 4. *Finding the g-centroid of a graph is NP-hard.*

Proof: Let G be a graph whose maximum clique size $\omega(G)$ has to be found. Let $H_i = G \cup K_i$ for $1 \leq i \leq n$ (where n = $| V(G) |$). Let G_{i-1} be got by joining two new vertices to H_i and joining them to all the existing vertices of H_i . Let $G' = G_0 \cup G_1 \cup \cdots \cup G_{n-1}$. By the definition of g-centroid for disconnected graphs, $C_g(G') = \cup_{i=0}^{n-1} C_g(G_i)$. Then by Lemma 1, $\omega(G) = i$ if and only if i is the least integer such that $C_g(G_i) = K_{i+1}$. Thus finding the g-centroid of a graph is NP-hard □

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. The graph $G_3 = G_1 \vee G_2$ is defined as follows

$$V(G_3) = V(G_1) \cup V(G_2)$$
$$E(G_3) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}$$

Proposition 5. *Finding the g-centroidal number gc(G) of a graph G is NP-hard.*

Proof: Let G be a graph whose maximum clique size $\omega(G)$ has to be found. Let $G' = K_1 \vee G$. It is easy to see that $gc(G) = \omega(G)$. Thus finding $gc(G)$ of a graph G is NP-hard □

Proposition F.*inding the weight of a vertex is NP-hard.*

Proof: Let G be any instance and u be a vertex not in G. Let $G'= \{u\} \vee G$, then $w(u) = \omega(G)$, proving the proposition □

Section 3

In this section we give a polynomial time algorithm for finding the g-centroid for maximal outer planar graphs, split graphs and Ptolemaic graphs. For trees the g-centroid coincides with the usual branch weight centroid. A nice linear time algorithm for finding the branch weight centroid is due to Kang and Ault [11]. In this section we consider only connected graphs.

Subsection 3.1

In this subsection we prove some results on chordal graphs which we use subsequently. A graph G is a chordal graph (also called triangulated graph) if G has no induced cycle of length 4 or more.

Proposition 7. *Let G be a chordal graph and $u \in V(G)$. For each $x \in N_i(u)$, $< A_i(x) > = < \{y \in N_{i-1}(u) : xy \in E(G)\} >$ is complete.*

Proof: Consider the set $I(x,u) = \{\ w\ :\ w$ lies on a $x - u$ geodesic $\}$. Since G is connected, $< I(x,u) >$ is a chordal graph by itself. Suppose $y(\neq x,u) \in I(x,u)$ and $d(u,y) = r$, then we can find $a \in N_{r-1}(u) \cap I(x,u)$ and $b \in N_{r+1}(u) \in I(x,u)$ such that $ay, by \in E(G)$. Thus y cannot be a simplicial vertex of $I(x,u)$ and hence only u and x are the simplicial vertices of $< I(x,u) >$, establishing the proposition □

Lemma 2. *Let G be a chordal graph and $x,y \in N_i(u)$. If x and y are adjacent, then $A_i(x) \subset A_i(y)$ or $A_i(y) \subset A_i(x)$ or $A_i(x) = A_i(y)$.*

Proof: Easy □

Proposition 8. *Let G be a chordal graph and $u \in V(G)$. Let S be any maximal g-convex set not containing u; then $S \cap N_1(u)$ is non empty.*

Proof: Let S be a maximal g-convex set not containing u such that $S \cap N_1(u)$ is empty. Let $x \in N_i(u)$ be a closest vertex to u in S. Let M be a minimal separator for x and u. Since G is chordal, M induces a complete graph. It is clear that $A_i(x) \subset M$. Suppose M is disjoint from S, then S will be completely contained in a component C of $G - M$. Thus $C \cup M$ is a g-convex set properly containing S, a contradiction. Let $M' = M \cap N_i(u)$. By lemma 2, $A_i(M') = \{\ y \in N_{i-1}(u)$: there is an $z \in M'$ and $yz \in E(G)\}$ forms a clique of $N_{i-1}(u)$, separates u and x and is disjoint from S, a contradiction. Thus $S \cap N_1(u)$ is non empty for all maximal g-convex sets not containing u □

Subsection 3.2

In this subsection we give a polynomial time algorithm for finding the g-centroid for maximal outer planar graphs.

A graph is outerplanar if it can be drawn in the plane with all the vertices in the exterior face; it is maximal outer planar (mop for short) if no edge can be added without destroying its outer planar property.

All mops can be constructed according to the following recursive rule (See, for instance Proskurowski [15])

(i) The triangle K_3 is a mop.

(ii) A mop with n+1 vertices can be obtained from a mop M on n vertices (n > 3) by adding a new vertex and joining it to two consecutive vertices on the hamilton cycle of M.

We list some of the properties of mops as a proposition which can be proved easily.

Proposition 9.

(i) *Mops are triangulated*

(ii) *Any two non adjacent vertices in a mop can be separated by a pair of adjacent vertices.*

(iii) *A Mop has a unique hamilton cycle*

The following characterization for outer planar graphs is due to Chartrand and Harary [2]

Proposition 10. *A graph is outer planar if and only if it contains no subgraph homeomorphic from K_4 or $K_{2,3}$.*

We quite often use the above proposition for proving our results on mops.

Definition 2: *Let $u \in V(G)$ and $y \in N_i(u)$. We say y is a* **successor** *of x with respect to u if $x \in N_{i-1}(u)$ and $xy \in E(G)$ and x is called a* **parent** *or a* **predecessor** *of y.*

Let $u \in V(G)$. Consider the successor relation with respect to u on G. Let $y \in N_j(u)$. We say y is a **descendant** of $x \in N_i(u)$ if $i < j$ and there is a sequence of vertices $y = y_0, y_1, \cdots, y_{j-i} = x$ such that y_k is a successor of y_{k-1} for $0 \leq i < j - i$.

Proposition 11. *Let G be a mop and let $u \in V(G)$. Let $u_1, u_2, \cdots, u_r$ be the vertices of G which are adjacent to u, then $< N[u] >\cong P\vee \{u\}$ where P is the path on r vertices.*

Proof: Consider the hamilton cycle H of G starting and ending with u. Let u_{i_1} be the first vertex of H occurring in $N_1(u)$ and u_{i_2} be the second vertex. Consider the portion H' of H from u_{i_1} to u_{i_2} not containing the vertex u. $uu_{i_1} + H' + u_{i_2}u$ is a cycle of length four or more. (=3 only when u_{i_1} and u_{i_2} are consecutive vertices of H). By the chordal property of G, $u_{i_1}u_{i_2} \in E(G)$. Proceeding like this we get a sequence of vertices P: $u_{i_1}, u_{i_2}, \cdots, u_{i_r}$ of $N_1(u)$ occurring in H in that order such that $u_{i_{j-1}}u_{i_j} \in E(G)$. Suppose that there is an edge between u_i, u_j and u_i, u_j are not consecutive in the sequence P. Let w be any vertex in between u_i and u_j in P. Then clearly G has a homeomorph of K_4, a contradiction, establishing the proposition □

Consider G and the successor relation with respect to u. Let $x, y \in N_1(u)$ such that $xy \in E(G)$. Suppose $G' = G$ - $\{x, y\}$ is disconnected and C be a component of G' not containing u. Then every vertex of C is either a descendant of x or y or both. In this case the hamilton cycle enters C through x and exits through y after passing through all the vertices of C. It is easy to see that if x and y are consecutive in the hamilton cycle then G' will be connected. We denote the component of G' not containing u together with x and y as $C(x, y)$ and by $H(x, y)$ the set of all vertices which are either descendants of x or y or both.

Lemma 3. *Let $H(y/C(x, y))$ be the set of all descendants of y in $C(x, y)$. Then $H(y/C(x, y))$ is a g-convex set.*

Proof: We prove this using induction. Suppose that $H(y/C(x, y))$ contains y and first level of descendants ($H(y/C(x, y))$ has k-levels of descendants if $H(y/C(x,y)) = \{ z$: z is a descendant of y in $C(x, y)$ and $d(z, y) \leq k \}$). Let a,b be a pair of vertices in $H(y/C(x, y))$ whose geodesic passes through a vertex r not in $H(y/C(x, y))$. Let $r \neq x$ then it is easy to see that G has a homeomorph of $K_{2,3}$, a contradiction. Assume that if $H(y/C(x, y))$ has k-levels of descendants then it is a g-convex set.

Let $H(y/C(x, y))$ have k+1 level of descendants in $C(x, y)$. Let $x_1, x_2, \cdots$, x_n and $y_1, y_2, \cdots, y_n$ be the successors of x and y respectively as they occur in the order in the hamilton cycle H of G. Let $C'(y_i, y_{i+1})$ be the component of G - $\{ y_i, y_{i+1} \}$ not containing y together with y_i and y_{i+1}. Next we show $x_n = y_1$. Trace the hamilton cycle of G from u. Without loss of generality assume that x_n comes before y_1. Let H' be the portion of the hamilton cycle from x_n to y_1 not passing through u. Then $xx_n + H' + y_1y + yx$ is a cycle of length four or more. Since G is a chordal graph we have $x_n = y_1$.

It is easy to show that $H(y/C(x, y)) = \{ y \} \cup (\cup_{i=1}^{m-1} C'(y_i, y_{i+1}) \cup H(y/C(x, y))$. By the induction hypothesis and by a trivial observation we can show that $H(y/C(x, y))$ is a g-convex set □

Proposition 12. *Let G be a mop and $u \in V(G)$. Consider the successor relation defined with respect to u. Then $H(x, y)$ is a maximum g-convex set not containing u whose intersection with $N_1(u)$ is $\{ x,y \}$.*

Proof: $H(x, y) = A \cup C(x, y) \cup B$ where $A = H(x/C(x_0, x))$ and $B = H(y/C(y, y_0))$ where x_0, $y_0 \in N_1(u)$ such that x_0x, $yy_0 \in E(G)$. Note that A or B or both may be empty. The convexity of $H(x, y)$ follows from the previous lemma and by an obvious observation. The other part of the proposition follows from the construction of $H(x, y)$ □

Proposition 13. *Let G be a mop and $u \in V(G)$. Let S be any maximal g-convex set not containing u, then $S = H(x, y)$ for some $x, y \in N_1(u)$.*

Proof: Since mops are triangulated, by Proposition 8, $S \cap N_1(u)$ is non empty. We can easily show that the vertices of $S \cap N_1(u)$ form a complete subgraph of $< N_1(u) >$. Suppose $S \cap N_1(u) = \{ x \}$. In this case we claim that S contains only the descendants of x. Suppose that $a \in S$ and an $y (\neq x) \in N_1(u)$ such that a is a descendant of y. Then $d(a,y) < d(a,x)$ and hence the $a - x$ geodesic will contain y, a contradiction. Let $y \in N_1(u)$ such that $xy \in E(G)$. It is easy to see that $S \subset H(x,y)$, contradicting the maximality of S, establishing $| S \cap N_1(u) | > 1$. Since $N_1(u)$ has no clique of size 3 or more we have $| S \cap N_1(u) | = 2$. Let $S \cap N_1(u) = \{ x, y \}$. Then by the maximality of S and $H(x,y)$, $S = H(x,y)$ □

Now we are ready to give a polynomial time algorithm for finding the g-centroid for a mop. Assume that MBFS(v) is a procedure which gives the successor relation with respect to v. It is easy to see that MBFS will take $O(n+m)$ time for each vertex v.

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Procedure GCENTMOP(G)
Begin
        For each vertex u ∈ V(G) do
             Execute MBFS to get the successor relation.
             For each adjacent vertices x,y in N1(u)
             find H(x,y)
             w(u) = max { | H(x,y) | }
        end for
        output the least weighted vertices.
end.
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Complexity Analysis: For each vertex u, MBFS(u) will take $O(n+m)$ time. Finding $H(x,y)$ will take $O(n+m)$ time for each pair of vertices x, y. For each u there will be $d(u)$ ($d(u)$ is the degree of the vertex u) adjacent pairs x, y in $N(u)$. Thus finding the weight of a vertex will take $O((n+m)d(u))$ time. Hence finding the g-centroid of a mop takes $O(m^2)$ time. Since $m \leq 3n - 6$, our algorithm indeed takes only $O(n^2)$ time.

Theorem 1. *If G is a maximal outer planar graph, then its g-centroid can be found in $O(n^2)$ time* □

Subsection 3.3

In this subsection we give a polynomial time algorithm for finding the g-centroid for a split graph. An undirected graph $G = (V,E)$ is defined to be a split graph if there is a partition $V = S \cup K$ of its vertex set into a **stable set** S and a **complete set** K. There is no restriction on the number of edges between S and K.

Theorem 2. *Let G be a split graph with stable set S and a complete set K as a partition of V with $|S| = \alpha(G)$ and $|K| = \omega(G)$ then $S \cup K$ is unique.* □

Theorem 3. *Let $G = (V,E)$ be a simple graph with degree sequence $d_1 \geq d_2 \geq \cdots \geq d_n$. Let $m = \max\{ i : d_i \geq i - 1 \}$. Then G is split if and only if $\sum_{i=1}^{m} d_i = m(m-1) + \sum_{m+1}^{n} d_i$. Further in this case $\omega(G) = m$.* □

For detailed proofs of the above theorems refer Golumbic [7]. From these two theorems if G is a split graph one can find $\omega(G)$ in $O(nlogn)$ time. If $|K| = m$ then $S \cup K$ is unique, otherwise we have to 'switch' a vertex of degree m-1 from S to K. After a switching $S \cup K$ will be unique. We now present our algorithm.

Procedure **GCENTSPLIT(G)**
begin
    For each $u \in K$ do
        $w(u) = n$ -1 - $|N(u) \cap S|$
    output the least weighted vertices.
end.

It is easy to see that any vertex in S will have weight n-1, therefore it is enough to compute the weights of the vertices in K. For any $k \in K$, $S_k = V$ - $\{k\}$ - $\{ x: x \in N_1(u) \cap S\}$. The algorithm given above will take $O(m + nlogn)$ time.

Note. It is easy to see that if G is a split graph then the g-centroid induces a complete graph as $C_g(G) \subseteq K$.

Theorem 4. *If G is a spilt graph then the g-centroid can be found in $O(m + nlogn)$ time.* □

Subsection 3.4

In this subsection we give a polynomial time algorithm for finding the g-centroid for Ptolemaic graphs. A **Ptolemaic graph** is a connected graph such that for every four vertices v_1, v_2, v_3, v_4, the following inequality is satisfied.

$$d(v_1, v_2) * d(v_3, v_4) \leq d(v_1, v_3) * d(v_2, v_4) + d(v_1, v_4) * d(v_2, v_3)$$

A graph is **distance hereditary** if every two vertices have the same distance in every connected induced subgraph containing both. Howorka [9] proved that ptolemaic graphs are precisely chordal distance hereditary graphs. The following theorem is due to Howorka [8].

Theorem 5. *Given a graph $G = (V,E)$ the following statements are equivalent.*

(i) *G is distance hereditary.*

(ii) *Every cycle in G with five or more vertices has two crossing chords.*

(iii) *Every induced path in G is a geodesic path.* □

We quite often use the second equivalent definition of the above theorem for proving our results.

Proposition 14. *Let G be a Ptolemaic graph. Let M be any maximal clique of $N_1(u)$. Let $x \in N_2(u)$ such that $A_2(x) \cap M \neq \emptyset$ then $A_2(x) \subseteq M$. (that is for any $x \in N_2(u)$, $A_2(x)$ will be contained in a maximal clique of $N_1(u)$)*

Proof: Suppose that there exists a vertex x of $N_2(u)$ such that $A_2(x)$ is partially contained in a maximal clique M of $N_1(u)$. Let $M' = M \cap A_2(x)$. Let $p \in A_2(x) \setminus M'$, $q \in M'$ and $r \in M \setminus M'$. Then $C : x, p, u, r, q$ forms a cycle of length five with qp and qu as its chords. Since G is a Ptolemaic graph, it must have two intersecting chords. Hence $pr \in E(G)$. Since r is any arbitrary vertex of $M \setminus M'$, p is adjacent to all the vertices in M, and hence $\{p\} \cup M$ is a clique of $N_1(u)$, contradicting the maximality of M. Thus $A_2(x) \subseteq M$ □

Proposition 15. *Let M be a maximal clique of $N_1(u)$. Let $H(M) = \{ y : y$ is a descendant of some $x \in M \}$. Then $H(M)$ is a g-convex set of G.*

Proof: Follows from the above Proposition and Lemma 2 □

Proposition 16. *Let G be a Ptolemaic graph and $u \in V(G)$. Let S be any maximal g-convex set not containing u then $S = H(M)$ for some maximal clique M of $N_1(u)$.*

Proof: Let S be any maximal g-convex set not containing u. By Proposition 8, $S \cap N_1(u)$ is a non empty clique of $N_1(u)$. Let $B = S \cap N_2(u)$. It is easy to see that $A_2(B)$ is a clique of $N_1(u)$. Suppose that $A_2(B)$ is not a maximal clique of $N_1(u)$. Let M be any maximal clique of $N_1(u)$ containing $A_2(B)$. Then $S \subset H(M)$, contradicting the maximality of S. Thus $A_2(B)$ is a maximal clique of $N_1(u)$ and $S = H(A_2(B))$ □

Proposition 17. *If G is a ptolemaic graph then $C_g(G)$ is a complete subgraph of G.*

Proof: Assuming the contrary, let $x, y \in C_g(G)$ be at a distance 2 apart. Let $x - z - y$ be a distance path joining x and y in G. It is easy to see that S_z cannot contain both x and y. Without loss of generality assume that $x \notin S_z$. Observe that if G is a ptolemaic graph, for each vertex u, $\{u\} \cup S_u$ is also a g-convex set. Thus $w(x) \geq | S_z \cup \{z\} | > w(z)$. This is a contradiction. Hence $C_g(G)$ is a complete subgraph of G □

We now give a polynomial time algorithm.

Procedure **GCENTPTOLEMAIC(G)**
begin
    For each $u \in V(G)$ do
        Execute MBFS(u) to get the successor
        relation with respect to u.
        For each maximal clique M of $N_1(u)$ find $H(M)$.
        $w(u) = \max \{| H(M) |\}$
    endfor
    Output the least weighted vertices.
end.

Complexity Analysis For each vertex u, MBFS(u) will take $O(n+m)$ time. Let $m(N(u))$ be the number of edges in $< N(u) >$. Then finding all the maximal cliques of $N_1(u)$ will take $O(d(u) + m(N(u)))$ time and there will be $O(d(u))$ maximal cliques. Finding $H(M)$ for each M will take $O(n+m)$ time. Thus fining $w(u)$ will take $O(n + d(u)m)$ time and hence finding the g-centroid will take $O(m^2)$ time.

Theorem 6. *If G is a ptolemaic graph then the g-centroid can be found in $O(m^2)$ time* □

Conclusion

It is well known that

maximal outer planar ⊂ *chordal* ⊂ *perfect*
split ⊂ *chordal* ⊂ *perfect*
ptolemaic ⊂ *distance heriditary* ⊂ *parity* ⊂ *perfect*

We have given polynomial time algorithm to find the g-centroid for maximal outer planar, split and ptolemaic graphs. It will be quite interesting to narrowdown the gap between P and NP along these hierarchies of graphs. Specially, the complexity status of the problem on distance heriditary graphs will be an interesting open problem. Improving the complexity of existing polynomial time algorithm, investigating on other classes of perfect graphs such as interval graphs, permutation graphs cocomparability graphs etc. are other open problems in this direction.

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